Wave function multifractality at Anderson localization transitions

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Mesoscopic Physics

Key ingredients:

- wave-like electron propagation, quantum interference
- disorder
- Coulomb interaction between electrons $\longrightarrow$ switched off in this talk

Non-interacting mesoscopics: electron in a random potential:

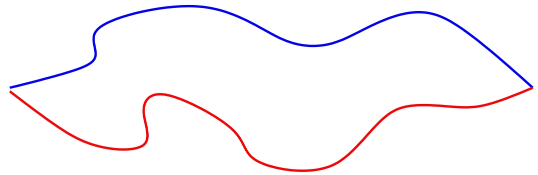
$$H = \frac{\hat{p}^2}{2m} + U(r), \qquad \langle U(r) U(r') \rangle = W(r-r') \qquad \text{e.g. white noise} = \frac{1}{2\pi\nu\tau}\delta(r-r')$$

disorder $\longrightarrow$ ensemble $\longrightarrow$ statistical treatment: mesoscopic fluctuations

Classical analogue: Electromagnetic/acoustic waves in random media

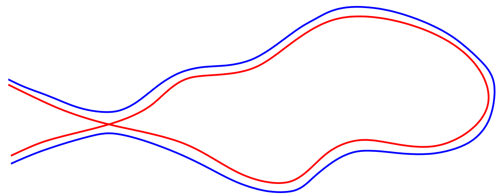
Quantum interference in disordered mesoscopic conductors

- **weak-localization correction to resistivity**



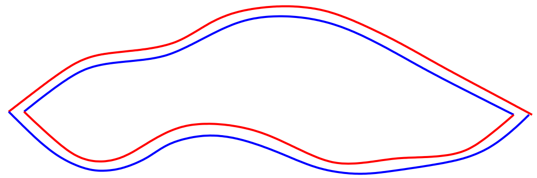
$$G \sim |\sum_i A_i|^2 = \sum |A_i|^2 + \sum_{i \neq j} A_i^* A_j$$

generically $\langle A_i^* A_j \rangle \simeq 0$



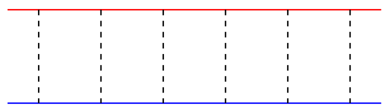
but: time-reversed paths: $A_i = A_{i'}$

- **mesoscopic conductance fluctuations**



$$\langle (\delta G)^2 \rangle \sim \langle (\sum_{i \neq j} A_i^* A_j)^2 \rangle$$
$$\sim \sum_{i \neq j} \langle |A_i|^2 \rangle \langle |A_j|^2 \rangle$$

Calculus: Green's functions (retarded and advanced), diagrammatics

Key objects: diffusons & cooperons  $\Pi_D(q, \omega) \sim \dfrac{1}{Dq^2 - i\omega}$

Weak localization: $\Delta G / G \propto \int d^d q / Dq^2 = \Pi_D(\mathrm{r}, \mathrm{r})$ return probability

Conductance fluctuations: $\langle (\delta G)^2 \rangle / G^2 \propto \int d^d q / (Dq^2)^2 = \int d\mathrm{r} d\mathrm{r}' \Pi_D^2(\mathrm{r}, \mathrm{r}')$

Wave function statistics

Fyodorov, ADM 92; ... Review: ADM, Phys. Rep. 2000

distribution function $\mathcal{P}(|\psi^2(\mathrm{r})|)$, correlation functions $\langle|\psi^2(\mathrm{r}_1)\psi^2(\mathrm{r}_2)|\rangle$ etc.

perturbative (diagrammatic) approach not sufficient

Field-theoretical method: σ-model   Wegner 79, Efetov 82 (SUSY)

$$S[Q] \propto \int \mathrm{d}^d r\, \mathrm{Str}[-D(\nabla Q(\mathrm{r}))^2 - 2i\omega\Lambda Q(\mathrm{r})] \qquad Q^2(\mathrm{r}) = 1$$

$Q \in \{\text{sphere} \times \text{hyperboloid}\}$ “dressed” by Grassmannian variables

σ-model contains all the diffuson-cooperon diagrammatics + much more (strong localization; Anderson transition & RG; non-perturbative effects)

- zero mode ($Q = \text{const}$) $\longrightarrow$ RMT distribution $\mathcal{P}(|\psi^2(\mathrm{r})|)$
  $\psi(r)$ – uncorrelated Gaussian random variables
- diffusive modes $[\Pi_D(\mathrm{r}_1, \mathrm{r}_2)]$ $\longrightarrow$ deviations from RMT, long-range spatial correlations

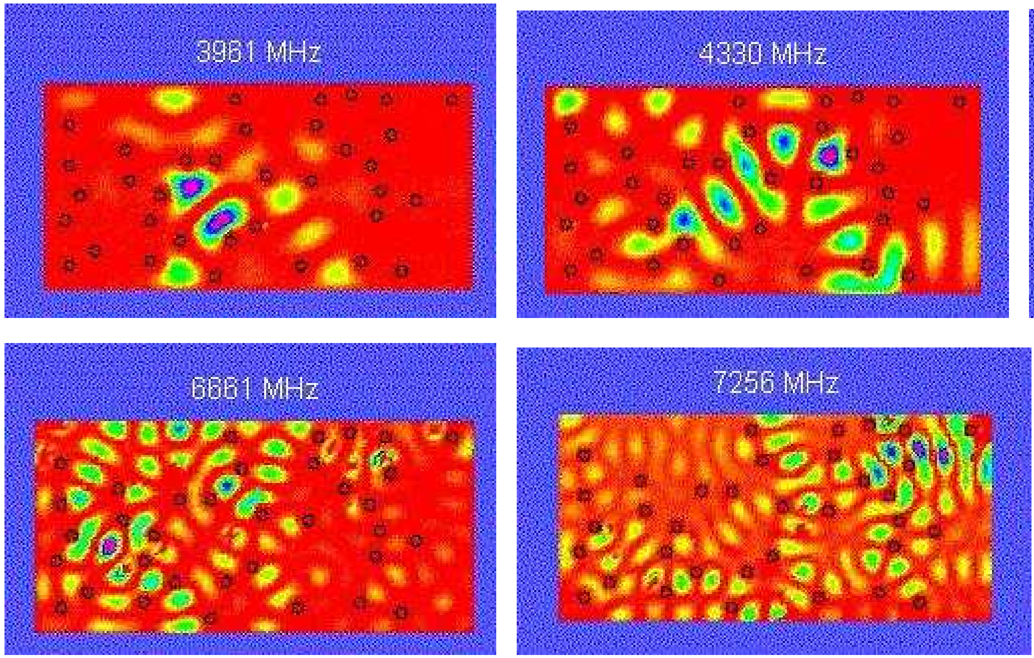
parameter $g = \dfrac{D/L^2}{\Delta} \equiv \dfrac{\text{Thouless energy}}{\text{level spacing}} = \dfrac{G}{e^2/h}$ dimensionless conductance

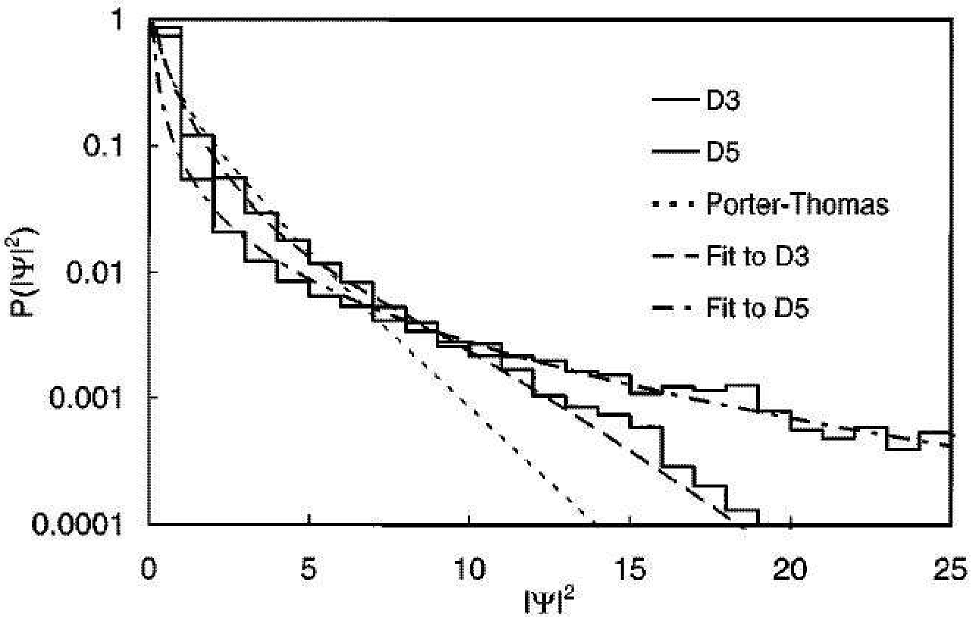
$g \gg 1$: metal $g \ll 1$: strong localization

quasi-1D: $g = \xi/L$, ξ – localization length

Experiment:
Wave function statistics in disordered microwave billiards

Kudrolli, Kidambi, Sridhar, PRL 95

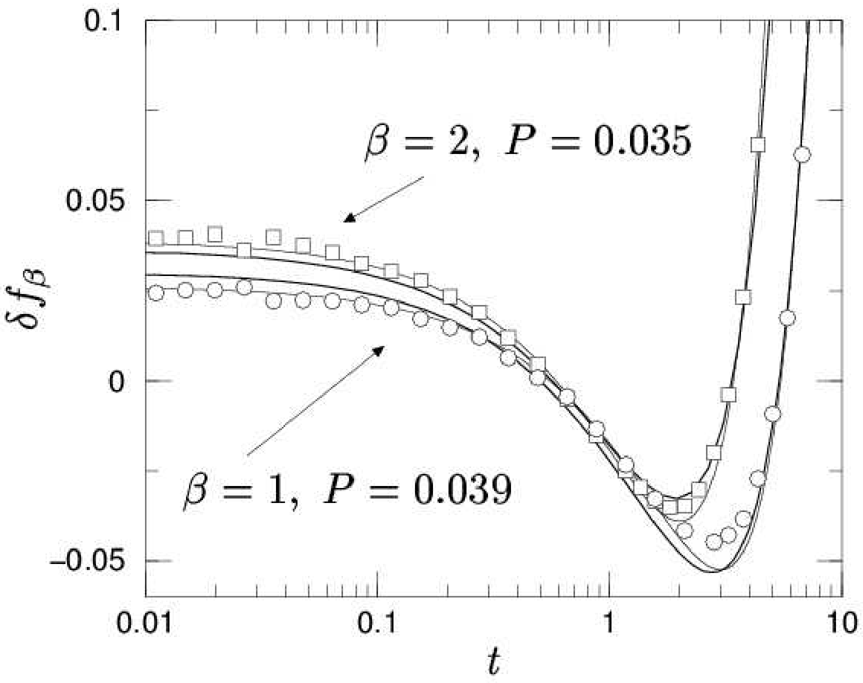


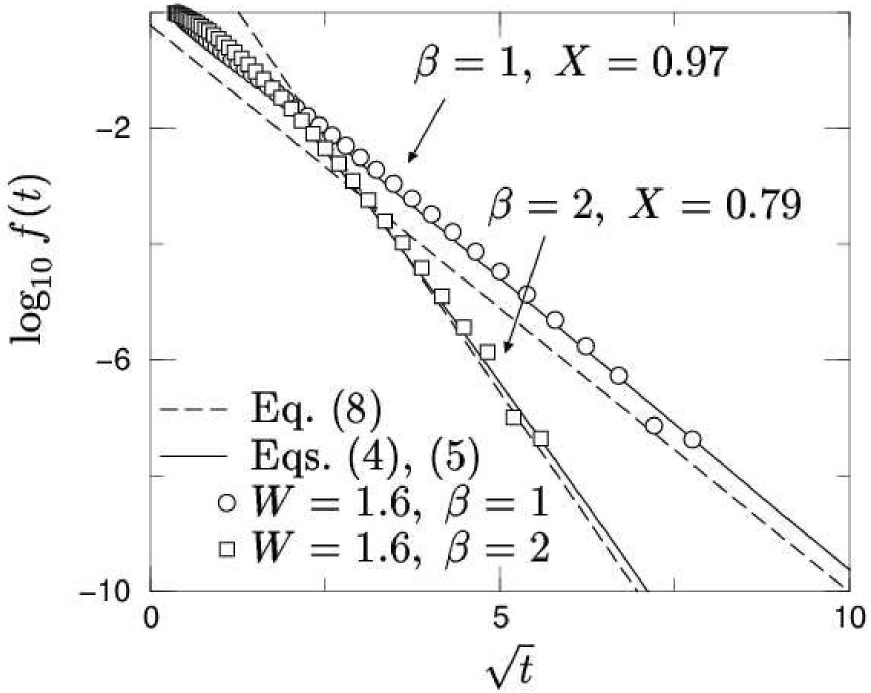


Wave function statistics: numerical verification

numerics: Uski, Mehlig, Römer, Schreiber 01

quasi-1d, metallic regime, distribution of $t = V|\psi^2(r)|$





"body": $1/g$ corrections to RMT

"tail" $\propto \exp(-\ldots\sqrt{t})$

Physics of the slowly decaying "tail": anomalously localized states

Anomalously localized states: imaging

numerics: Uski, Mehlig, Schreiber '02

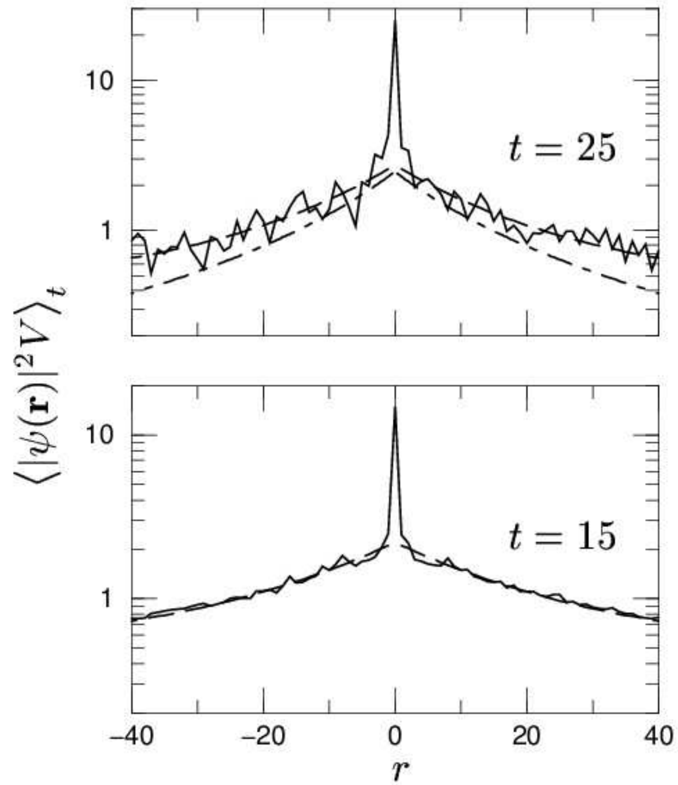
spatial structure

of an anomalously localized state (ALS):

$$\langle|\psi(\mathbf{r})|^2\delta(\mathrm{V}|\psi(0)|^2 - \mathrm{t})\rangle$$

with t atypically large

ADM '97

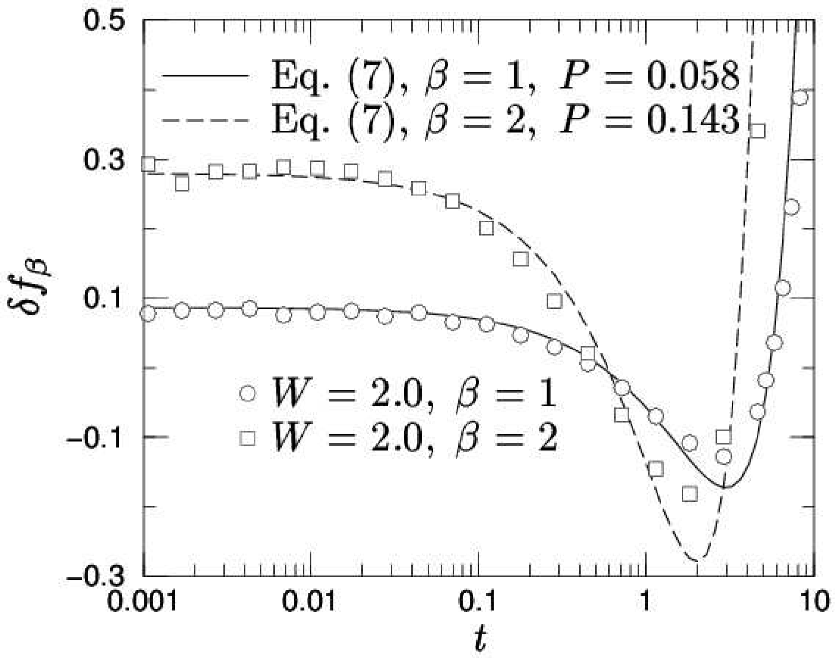


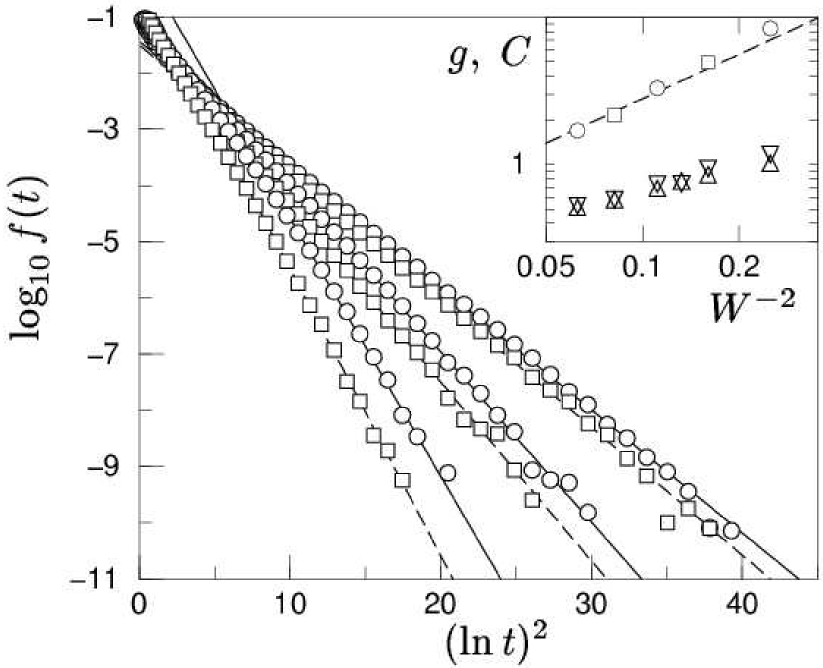
ALS determine asymptotic behavior of distributions of various quantities (wave function amplitude, local and global density of states, relaxation time,. . .)

Altshuler, Kravtsov, Lerner, Fyodorov, ADM, Muzykantskii, Khmelnitskii, Falko, Efetov. . .

Wave function statistics: numerical verification. 2D.

numerics: Uski, Mehlig, Römer, Schreiber '01





"body" of the distribution:

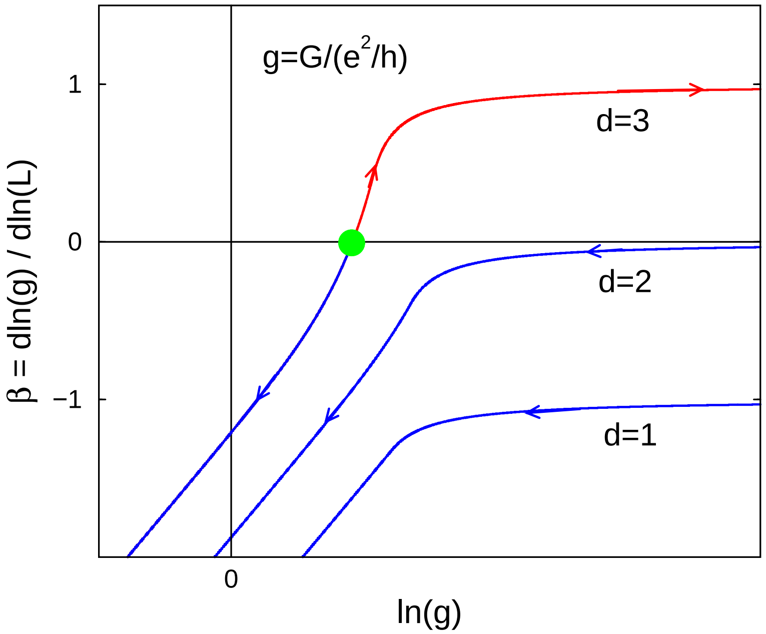
$(1/g)\ln(L/l)$ corrections

"tail" $\propto \exp(-\ldots \ln^2 t)$

Falko, Efetov '95

$\longrightarrow$ precursors of Anderson criticality

Anderson transition

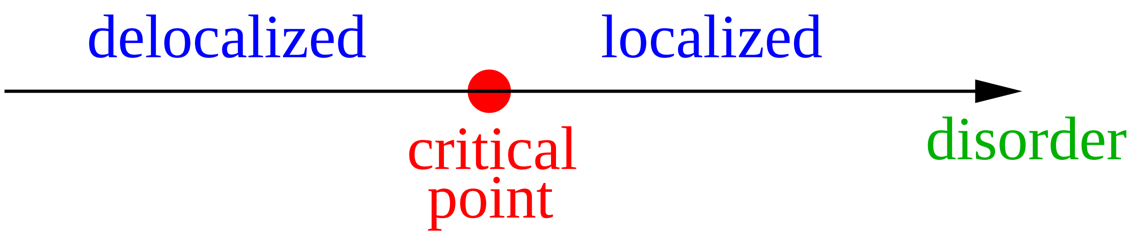


Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

Modern approach:
RG for field theory (σ-model)

quasi-1d: **metallic → localized crossover with decreasing $g = \xi/L$**

$d > 2$: Anderson metal-insulator transition (sometimes also in $d = 2$)



Continuous phase transition with highly unconventional properties!

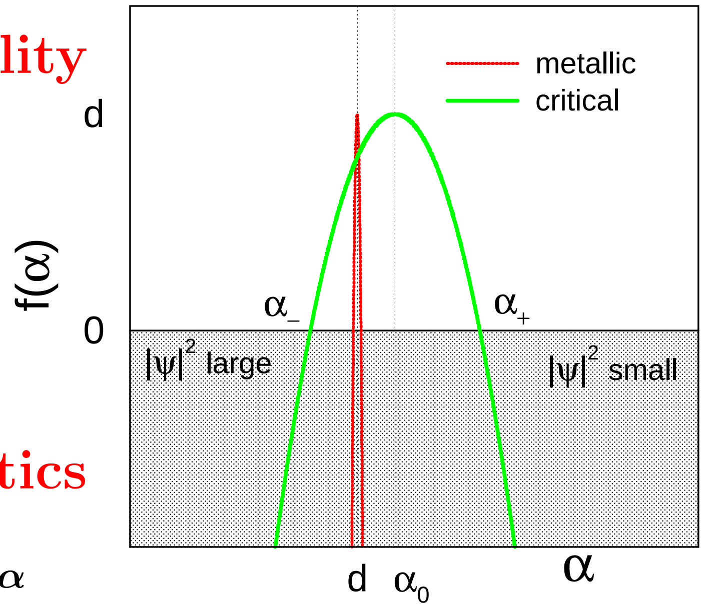
Multifractality at the Anderson transition

$P_q = \int d^d r |\psi(\mathrm{r})|^{2q}$ inverse participation ratio

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

$\tau_q = d(q-1) + \Delta_q \equiv D_q(q-1)$ multifractality

normal anomalous



$\tau_q \longrightarrow$ Legendre transformation

$\longrightarrow$ singularity spectrum $f(\alpha)$

$\mathcal{P}(\ln|\psi^2|) \sim L^{-d+f(\ln|\psi^2|/\ln L)}$ wave function statistics

$L^{f(\alpha)}$ – measure of the set of points where $|\psi|^2 \sim L^{-\alpha}$

Multifractality is characteristic for a variety of complex systems:

turbulence, strange attractors, diffusion-limited aggregation, ...

Peculiarity: Statistical ensemble $\longrightarrow$ $f(\alpha)$ may become negative

Multifractality and the field theory

Δ_q – scaling dimensions of operators $\mathcal{O}^{(q)} \sim (Q\Lambda)^q$

$d = 2 + \epsilon$: $\Delta_q = -q(q-1)\epsilon + O(\epsilon^4)$ Wegner '80

- Infinitely many operators with negative scaling dimensions
- $\Delta_1 = 0 \longleftrightarrow \langle Q \rangle = \Lambda$ naive order parameter uncritical

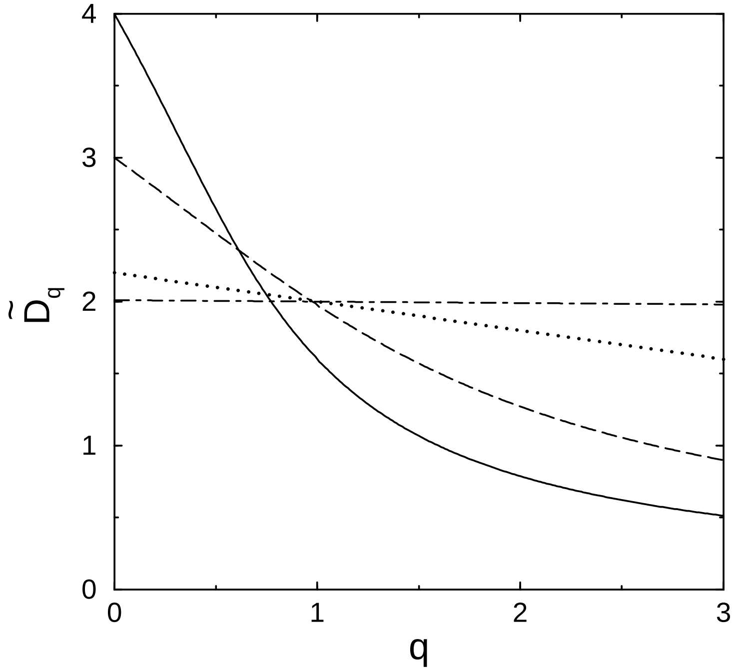
Transition described by an order parameter function $F(Q)$

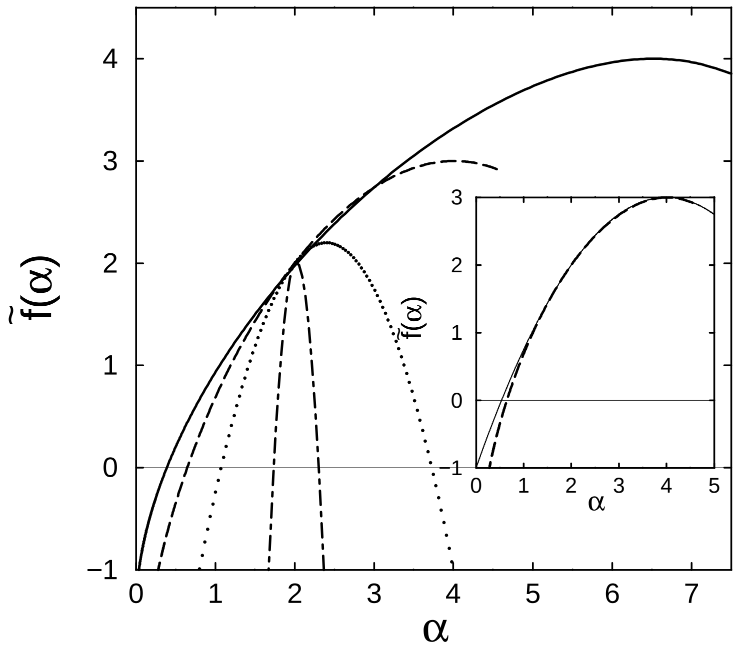
Zirnbauer 86, Efetov 87

$\longleftrightarrow$ distribution of local Green functions and wave function amplitudes

ADM, Fyodorov '91

Dimensionality dependence of multifractality





Analytics ($2+\epsilon$, one-loop) and numerics

$$\tau_q = (q-1)d - q(q-1)\epsilon + O(\epsilon^4)$$

$$f(\alpha) = d - (d+\epsilon-\alpha)^2/4\epsilon + O(\epsilon^4)$$

$d = 4$ (full)
$d = 3$ (dashed)
$d = 2+\epsilon,\quad \epsilon = 0.2$ (dotted)
$d = 2+\epsilon,\quad \epsilon = 0.01$ (dot-dashed)

Inset: $d = 3$ (dashed)
vs. $d = 2+\epsilon,\quad \epsilon = 1$ (full)

Mildenberger, Evers, ADM '02

Power-law random banded matrix model (PRBM)

Anderson transition: dimensionality dependence:

$d = 2 + \epsilon$: weak disorder/coupling $\qquad$ $d \gg 1$: strong disorder/coupling

Evolution from weak to strong coupling – ?

PRBM $\qquad$ ADM, Fyodorov, Dittes, Quezada, Seligman '96

$N \times N$ random matrix $H = H^\dagger$ $\qquad$ $$\langle |H_{ij}|^2 \rangle = \frac{1}{1 + |i-j|^2/b^2}$$

$\longleftrightarrow$ 1D model with $1/r$ long range hopping $\qquad$ $0 < b < \infty$ parameter

Critical for any b $\longrightarrow$ family of critical theories!

$b \gg 1$ analogous to $d = 2 + \epsilon$ $\qquad$ $b \ll 1$ analogous to $d \gg 1$ (?)

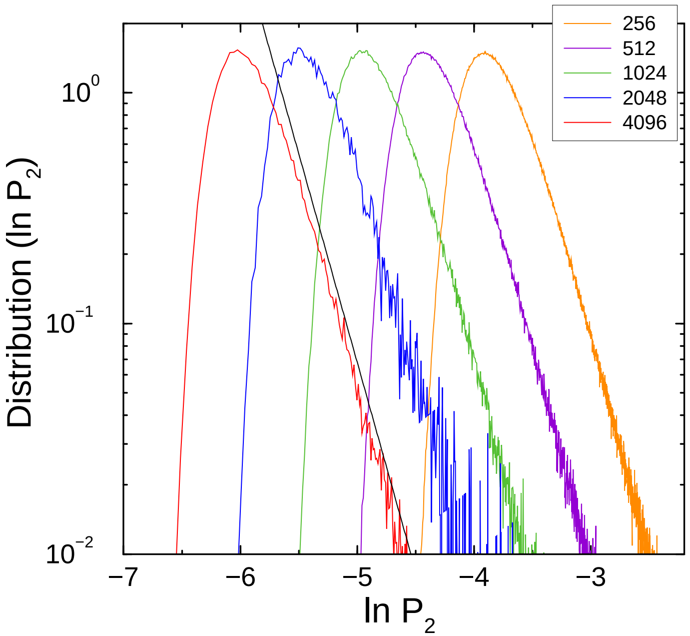
Analytics: $b \gg 1$: σ-model RG

$b \ll 1$: real space RG

Numerics: efficient in a broad range of b

Evers, ADM '01

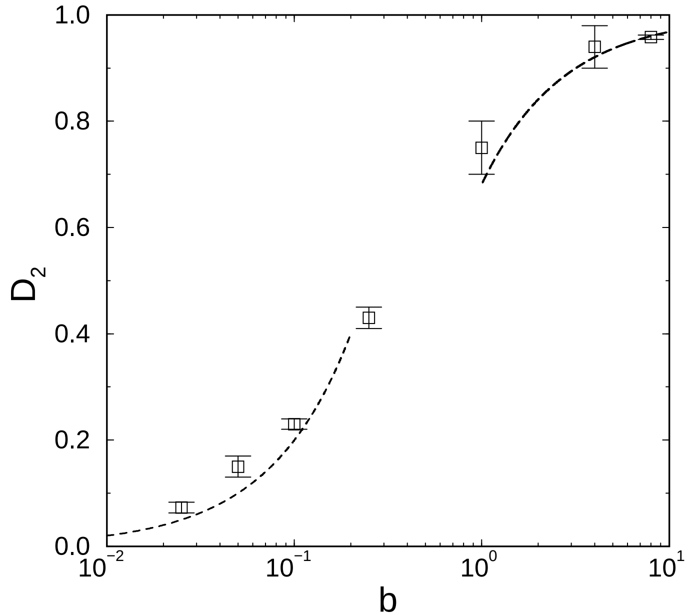
Power-law random banded matrix model

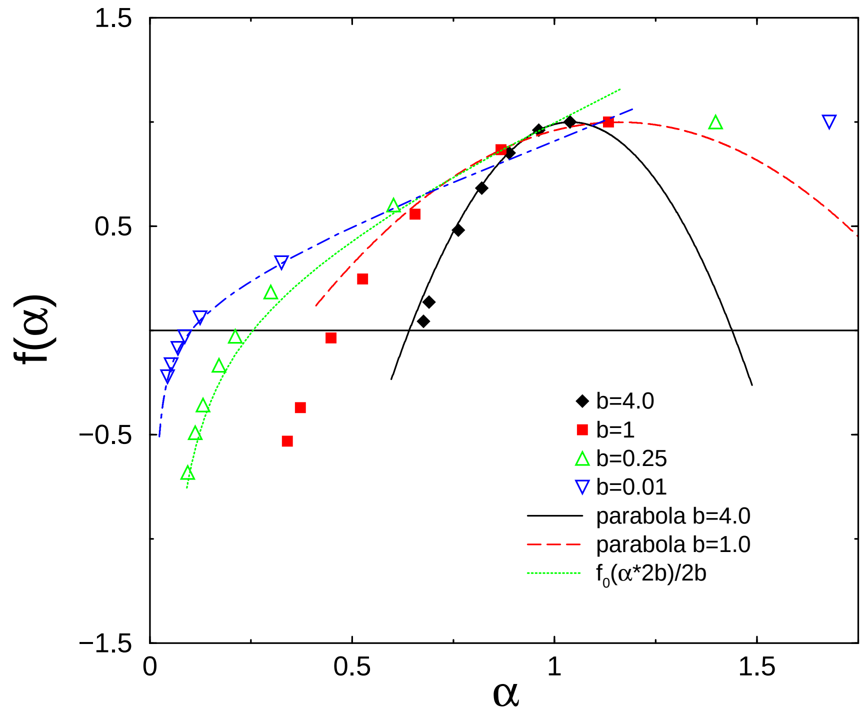


evolution of $\mathcal{P}(\ln P_2)$ with N for $b = 1$

- scale invariance
- fractal dimension $D_2 \simeq 0.75$

fractal dimension $D_2(b)$ $\longrightarrow$





Multifractality spectrum $f(\alpha)$

for $b = 4.0,\ \ 1.0,\ \ 0.25$ and 0.01

Lines: analytics for $b \gg 1$ and $b \ll 1$

Symbols: numerics

Evers, ADM '01

Relations between multifractal exponents

Non-linear σ-model

$\longrightarrow$ distribution of local Green function $\quad G_R(r,r)/\pi\langle\rho\rangle = u - i\tilde{\rho}$

$$P(u,\tilde{\rho}) = \frac{1}{2\pi\tilde{\rho}} P_0\left((u^2+\tilde{\rho}^2+1)/2\tilde{\rho}\right)$$

$\longrightarrow$ symmetry of LDOS distribution: $\quad \mathcal{P}_\rho(\tilde{\rho}) = \tilde{\rho}^{-3}\mathcal{P}_\rho(\tilde{\rho}^{-1})$

ADM, Fyodorov '94 $\quad (\beta = 2)$

Recently:

more complete derivation $(\beta = 1, 2, 4)$ via relation to a scattering problem:
system with a channel attached at a point r

S-matrix $\quad S = (1 - iK)/(1 + iK) \qquad K = \frac{1}{2}V^\dagger G_R(r,r)V = u - i\tilde{\rho}$

distribution $\;P(S)\;$ invariant with respect to the phase θ of $S = \sqrt{r}e^{i\theta}$.

Fyodorov, Savin, Sommers '04-05

Relations between multifractal exponents (cont'd)

ADM, Fyodorov, Mildenberger, Evers, cond-mat'06

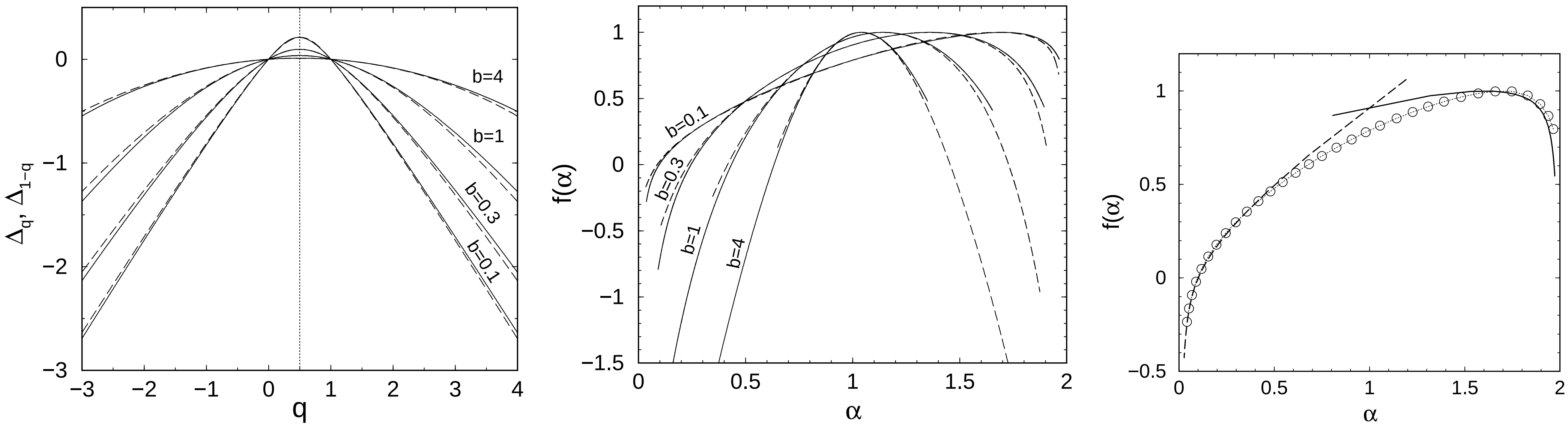
LDOS distribution in σ-model + **universality**

$\longrightarrow$ **exact symmetry of the multifractal spectrum:**

$$\Delta_q = \Delta_{1-q} \qquad\qquad f(2d-\alpha) = f(\alpha) + d - \alpha$$

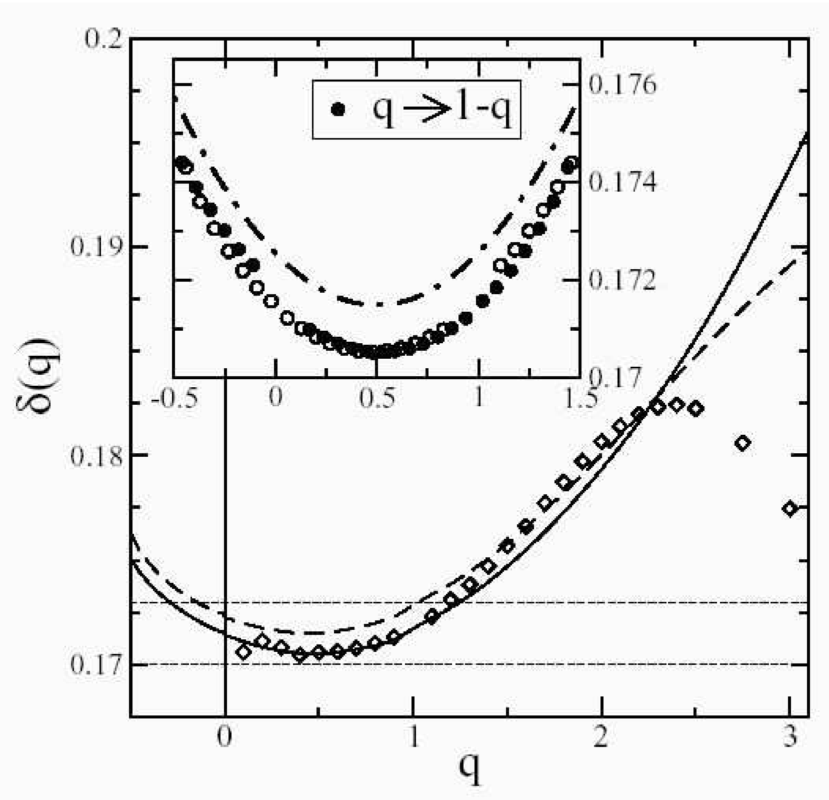
Consequence:

support of the singularity spectrum $f(\alpha)$ **is bounded by the interval** $[0, 2d]$



Relations between multifractal exponents (cont'd): Anderson transition in symplectic class in 2D

Mildenberger, Evers, cond-mat/0608560



Symmetry of the spectrum: $\Delta_q = \Delta_{1-q}$

Non-parabolicity of the spectrum

$$\delta(q) \equiv \frac{\Delta_q}{q(1-q)} \neq \text{const}$$

Conformal invariance

2D $\leftrightarrow$ quasi-1D strip

$$\Lambda_c = 1/\pi\delta_0$$

$\delta_0 \simeq 0.172$, $\Lambda_c \simeq 1.844$

$$\longrightarrow \pi\delta_0\Lambda_c = 0.999 \pm 0.003$$

Related work:

Obuse, Subramaniam, Furusaki, Gruzberg, Ludwig, in preparation

Relations between multifractal exponents (cont'd)

Wigner delay time: $t_W = \partial\theta(E)/\partial E$

Relation between wave function ($\mathcal{P}_y$) and delay time ($\mathcal{P}_W$) distribution in the σ-model:

$$\mathcal{P}_W(\tilde{t}_W) = \tilde{t}_W^{-3}\mathcal{P}_y(\tilde{t}_W^{-1}) \qquad \tilde{t}_W = t_W \Delta/2\pi$$

Ossipov, Fyodorov '05

$+$ universality

$\longrightarrow$ exact relation between multifractal exponents

for closed (wave functions, τ_q) and open (delay times, γ_q) systems

$$\gamma_q = \tau_{1+q}$$

ADM, Fyodorov, Mildenberger, Evers, cond-mat'06

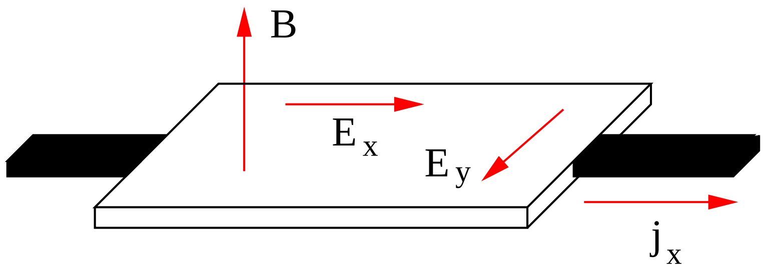
Integer Quantum Hall Transition

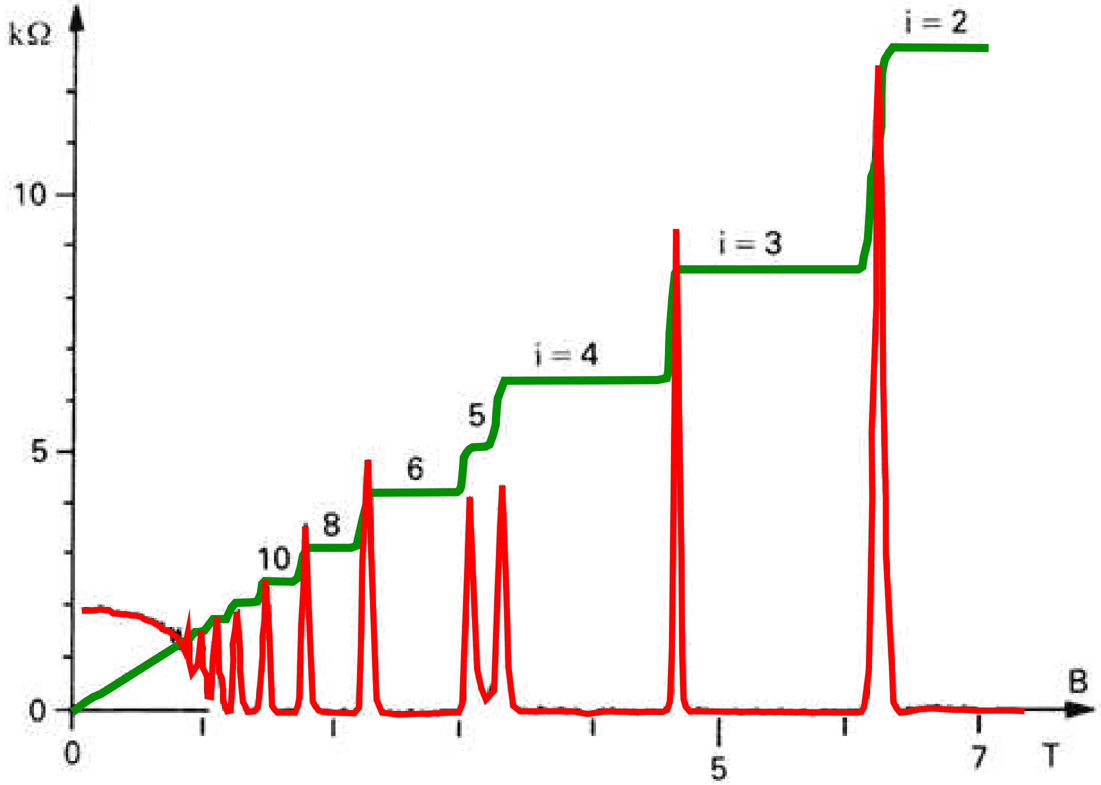
Magnetotransport:

Resistivity tensor:

$\rho_{xx} = E_x / j_x$

$\rho_{yx} = E_y / j_x$ **Hall resistivity**





Quantum Hall plateau transition:

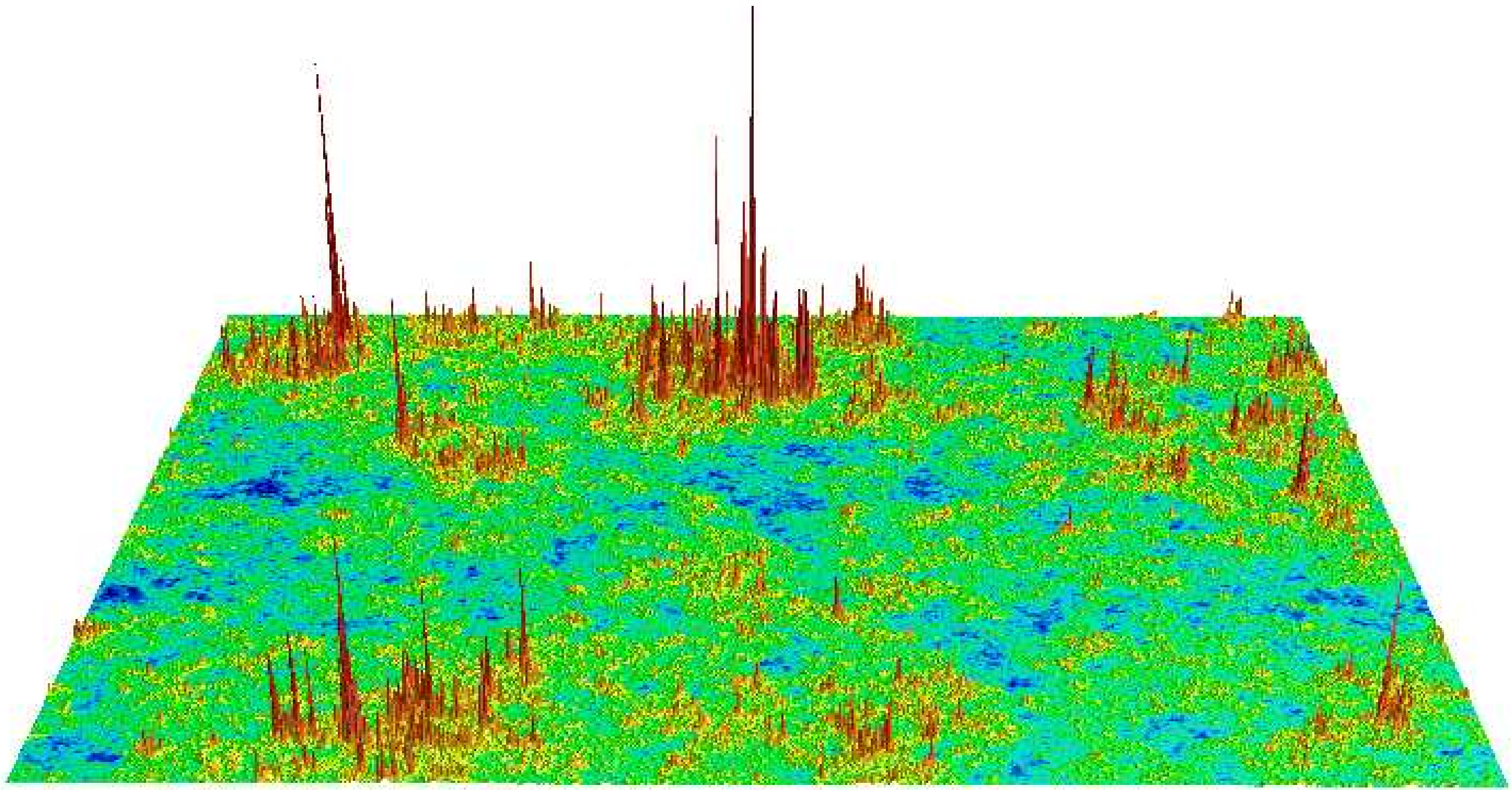
localized — critical point — localized

$\sigma_{xy} = n\frac{e^2}{h}$ critical point $\sigma_{xy} = (n+1)\frac{e^2}{h}$

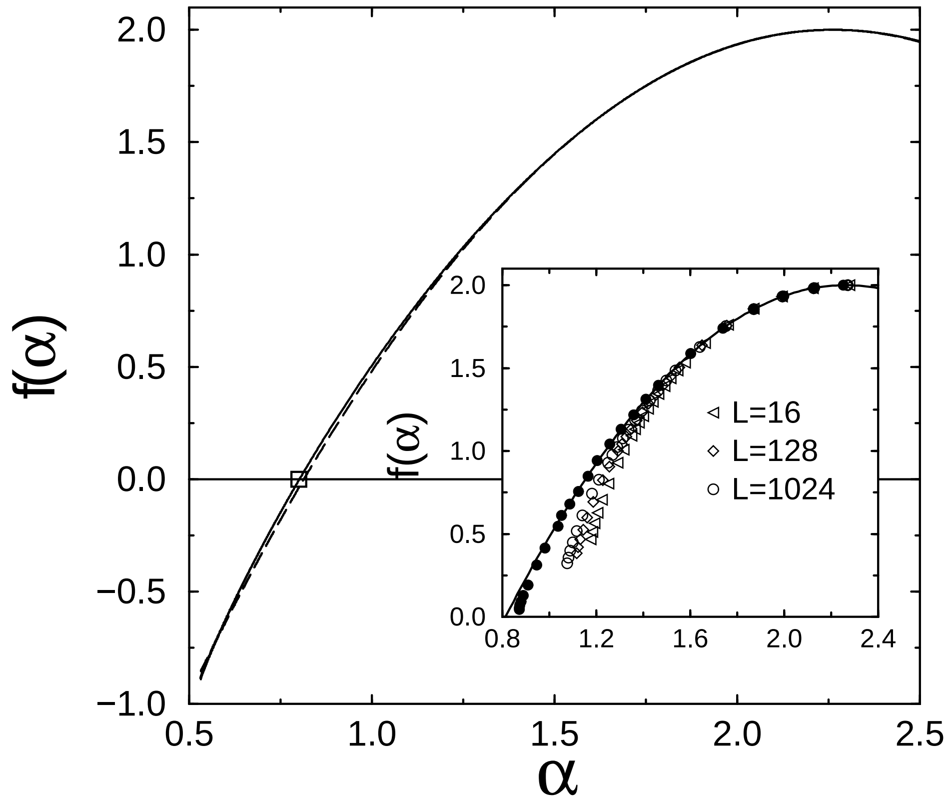
Delocalization in 2D?! **Topological term in σ-model (Pruisken)**

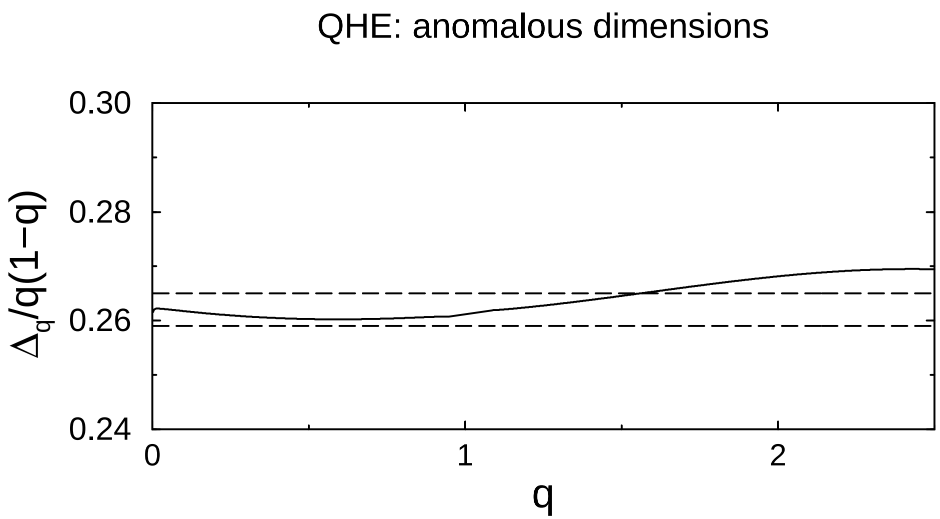
Closely related to existence of edge states

Multifractal wave functions at the Quantum Hall transition



Multifractality at the Quantum Hall critical point





⟶ spectrum is parabolic with a very high (1%) accuracy:

$$f(\alpha) = 2 - \frac{(\alpha-\alpha_0)^2}{4(\alpha_0-2)}, \qquad \Delta_q = (\alpha_0-2)q(1-q) \ \ \text{with} \ \ \alpha_0 - 2 = 0.262 \pm 0.003$$

Evers, Mildenberger, ADM '01

important for identification of the CFT of the Quantum Hall critical point

Disordered electronic systems: Symmetry classification

Altland, Zirnbauer '97

Conventional (Wigner-Dyson) classes

	T	spin rot.	chiral	p-h	symbol
GOE	+	+	−	−	AI
GUE	−	+/−	−	−	A
GSE	+	−	−	−	AII

Chiral classes

	T	spin rot.	chiral	p-h	symbol
ChOE	+	+	+	−	BDI
ChUE	−	+/−	+	−	AIII
ChSE	+	−	+	−	CII

$$\boldsymbol{H} = \begin{pmatrix} \mathbf{0} & \mathrm{t} \\ \mathrm{t}^\dagger & \mathbf{0} \end{pmatrix}$$

Bogoliubov-de Gennes classes

T	spin rot.	chiral	p-h	symbol
+	+	−	+	CI
−	+	−	+	C
+	−	−	+	DIII
−	−	−	+	D

$$\boldsymbol{H} = \begin{pmatrix} \mathrm{h} & \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^* & -\mathrm{h}^T \end{pmatrix}$$

Spin quantum Hall effect

- disordered d-wave superconductor (class C):

charge not conserved but spin conserved

- time-reversal invariance broken: $d_{x^2-y^2} + id_{xy}$ order parameter
  ($\longleftrightarrow$ strong effective magnetic field)

$\longrightarrow$ SQH plateau transition: spin Hall conductivity quantized

Kagalovsky, Horovitz, Avishai, Chalker '99; Senthil, Marston, Fisher '99

Similar to IQH transition but:

- DoS critical
- mapping to percolation: $\rho(E) \sim E^{1/7}$

Gruzberg, Ludwig, Read '99; Beamond, Cardy, Chalker '02

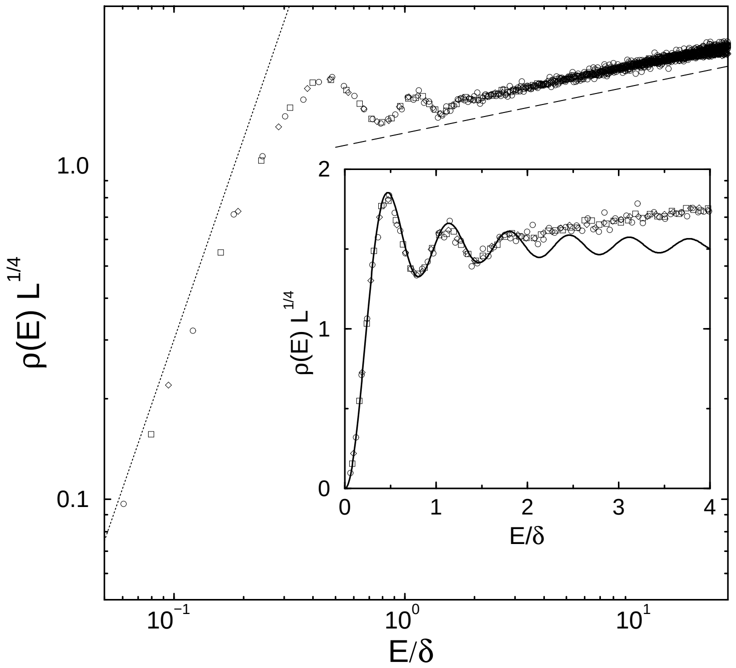
Multifractality at the spin quantum Hall transition

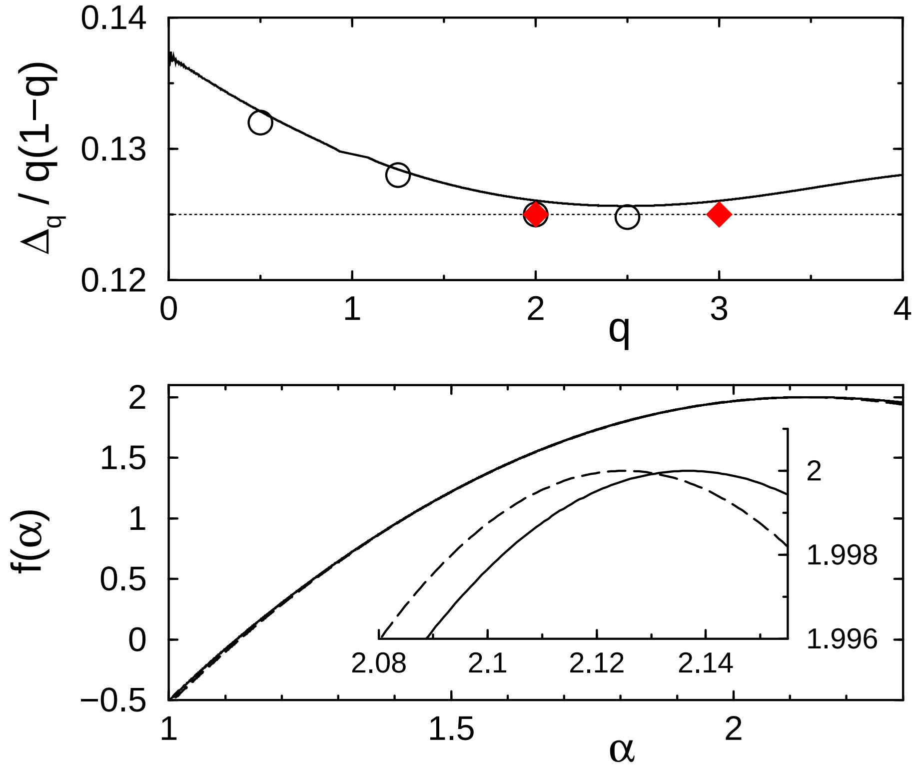
Evers, Mildenberger, ADM '03

- mapping to percolation:

analytical evaluation of lowest multifractal exp's: $\Delta_2 = -1/4, \; \Delta_3 = -3/4$

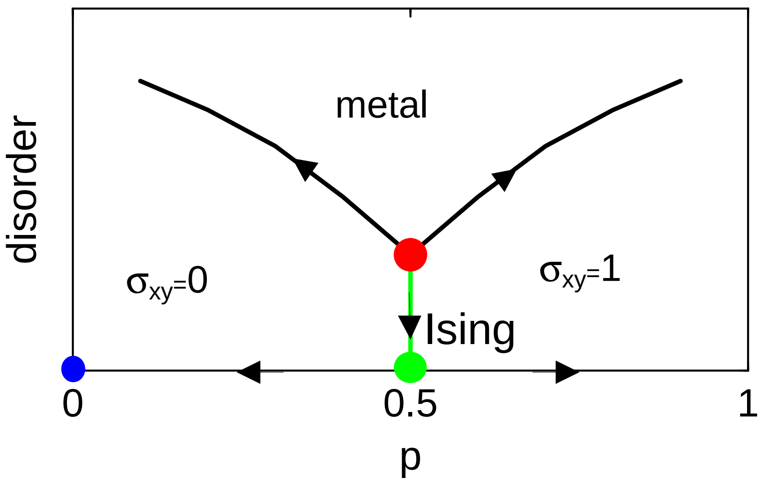
- numerics: $\Delta_q, \quad f(\alpha)$ not parabolic

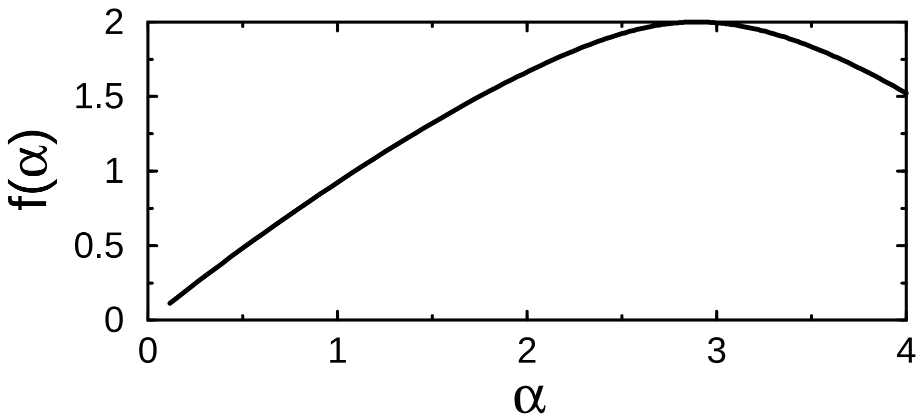




Thermal quantum Hall effect

- disordered p-wave superconductor (class D):
  neither charge nor spin conserved; only energy conservation
  $\longrightarrow$ TQH plateau transition: thermal Hall conductivity quantized
- very rich phase diagram: insulator, quantized Hall, and metallic phases
  $\longrightarrow$ both QH and Anderson (metal-insulator) transitions
  $\longrightarrow$ multicritical point
- Multifractality spectrum: "freezing"? $f(\alpha = 0) = 0 \longrightarrow \tau_{q\geq 1} = 0$
  Critical state has a localized "core" with multifractal "tails"





Mildenberger, Evers, Chalker, ADM, in preparation/progress

Surface multifractality

Subramaniam, Gruzberg, Ludwig, Evers, Mildenberger, ADM, cond-mat/05120401

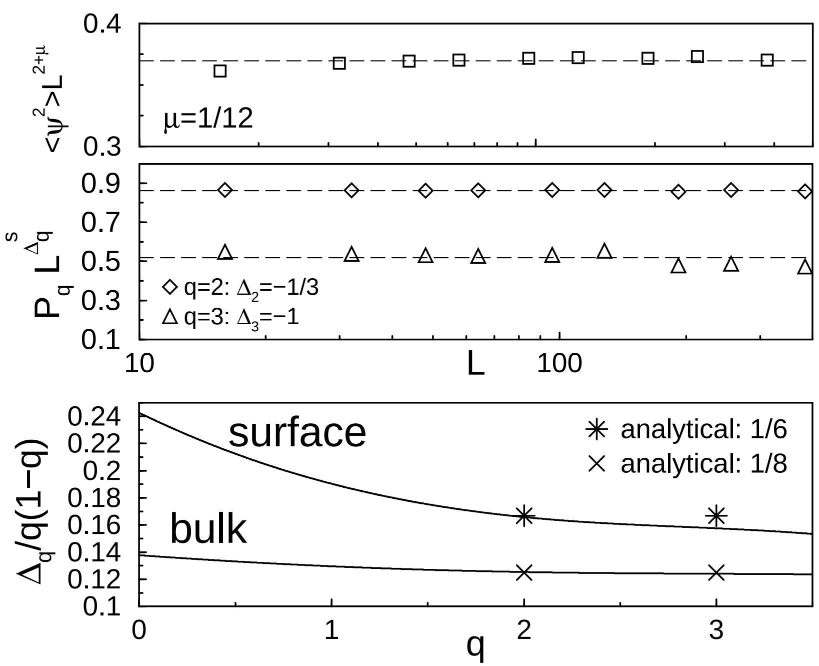
$\boldsymbol{r}$ near the boundary $\longrightarrow$

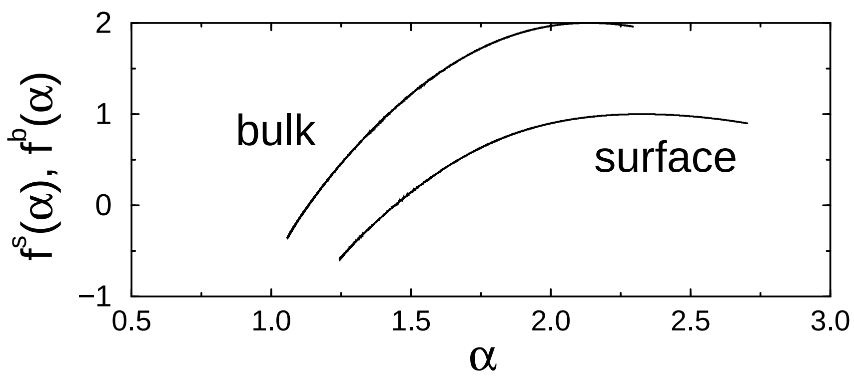
$$L^{d-1}\langle|\psi(\mathrm{r})|^{2q}\rangle \sim L^{-\tau_q^{\mathrm{s}}} \qquad \tau_q^{\mathrm{s}} = d(q-1) + q\mu + 1 + \Delta_q^{\mathrm{s}}$$

τ_q^{s}, Δ_q^{s} – surface multifractal exponents

- SQHE $\longrightarrow$ analytical evaluation for $q = 1, 2, 3$ via mapping to percolation

$\mu = 1/12$ $\qquad$ $\Delta_2^s = -1/3$ (bulk: -1/4) $\qquad$ $\Delta_3^s = -1$ (bulk: -3/4)

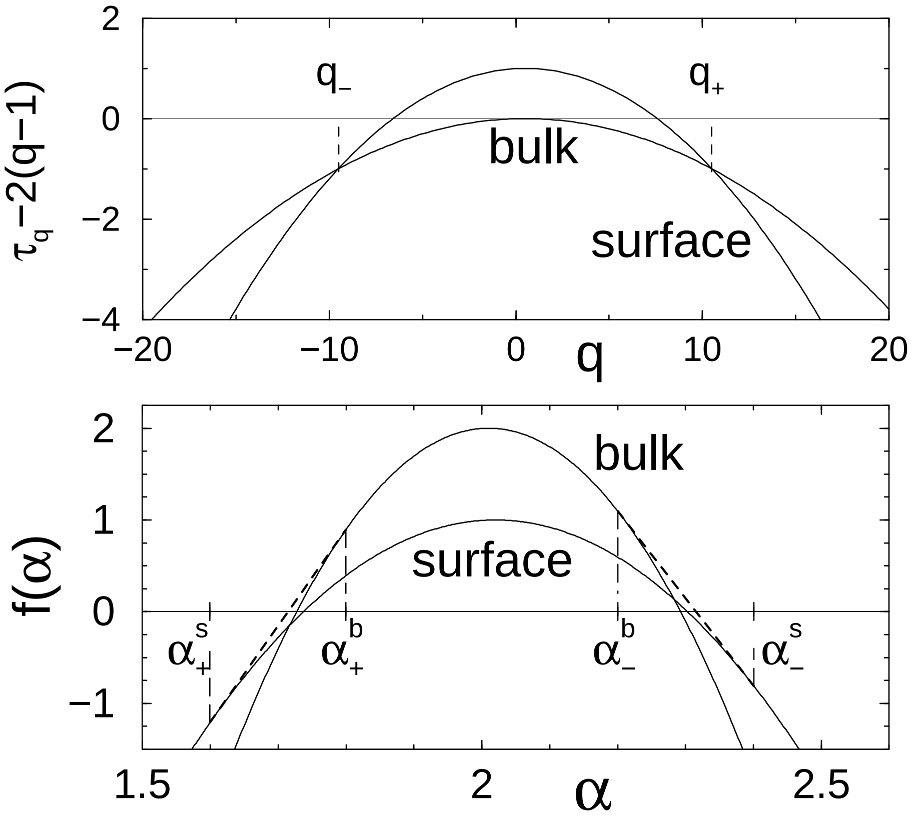




Surface multifractality in 2D or $(2+\epsilon)$D

bulk : $\tau_q^{\mathrm{b}} = 2(q-1) + \gamma q(1-q)$ $\qquad \gamma = (\beta\pi g)^{-1} \ll 1$

surface : $\tau_q^{\mathrm{s}} = 2(q-1) + 1 + 2\gamma q(1-q)$



$$f^{\mathrm{b}}(\alpha) = 2 - (\alpha - 2 - \gamma)^2/4\gamma$$

$$f^{\mathrm{s}}(\alpha) = 1 - (\alpha - 2 - 2\gamma)^2/8\gamma$$

Summary and Outlook

- Anderson localization transition

$\longrightarrow$ wave functions multifractal at criticality

- dimensionality dependence: from weak $(2+\epsilon)$ to strong $(d=3,4,\ldots)$ MF

- PRBM: $1d$ model with $1/r$ random hopping:

family of critical theories from weak $(b \gg 1)$ to strong $(b \ll 1)$ multifractality

- 2d + magnetic field $\longrightarrow$ QH transitions: IQHE, SQHE, TQHE

- exact relations for MF spectra: symmetry $q \leftrightarrow 1-q$; closed $\leftrightarrow$ open

- surface multifractality

Open problems / Current activity:

- Multifractality at the Anderson transition in large d:
  qualitatively similar to PRBM with $b \ll 1$?
- Anderson and quantum Hall transitions in novel universality classes
- Criticality in 2D: strong coupling fixed points. CFT?
- Effect of interaction on multifractality
- Experimental observation of wave function multifractality – ?