Combinatorial optimization algorithms for lines and interfaces in random media

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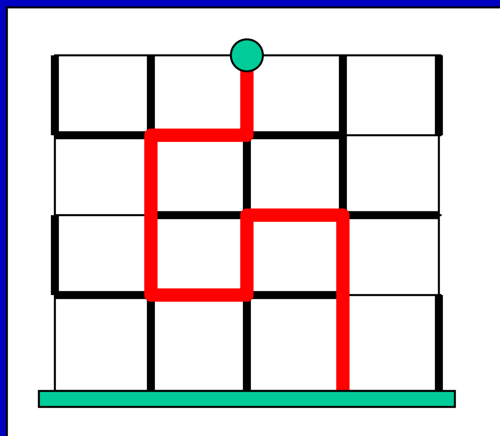
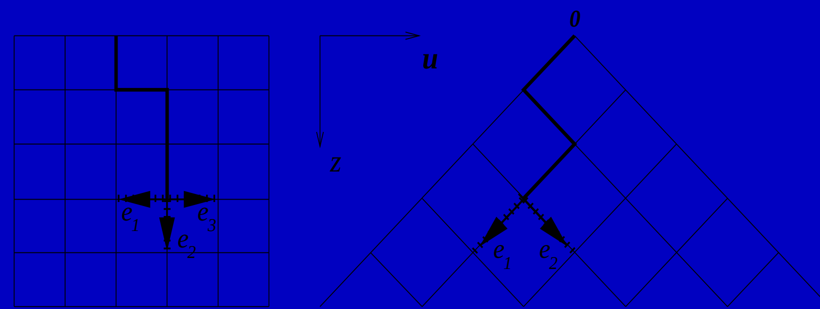
Stochastic Geometry and Field Theory:
From Growth Phenomena to Disordered Systems, 7.8.-15.12.06 KITP UCSB

(Directed) Polymers in a Random environment



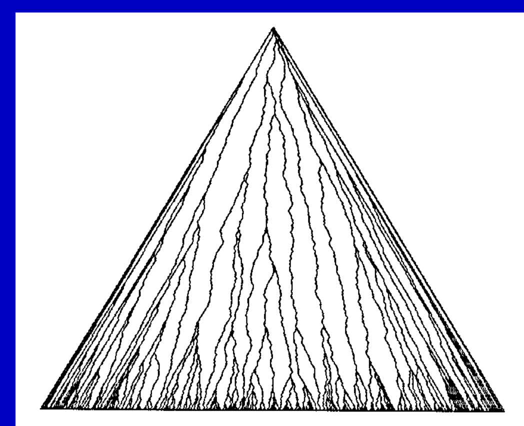


Non-directed directed

Random bond energies: $e_i \in [0,1]$

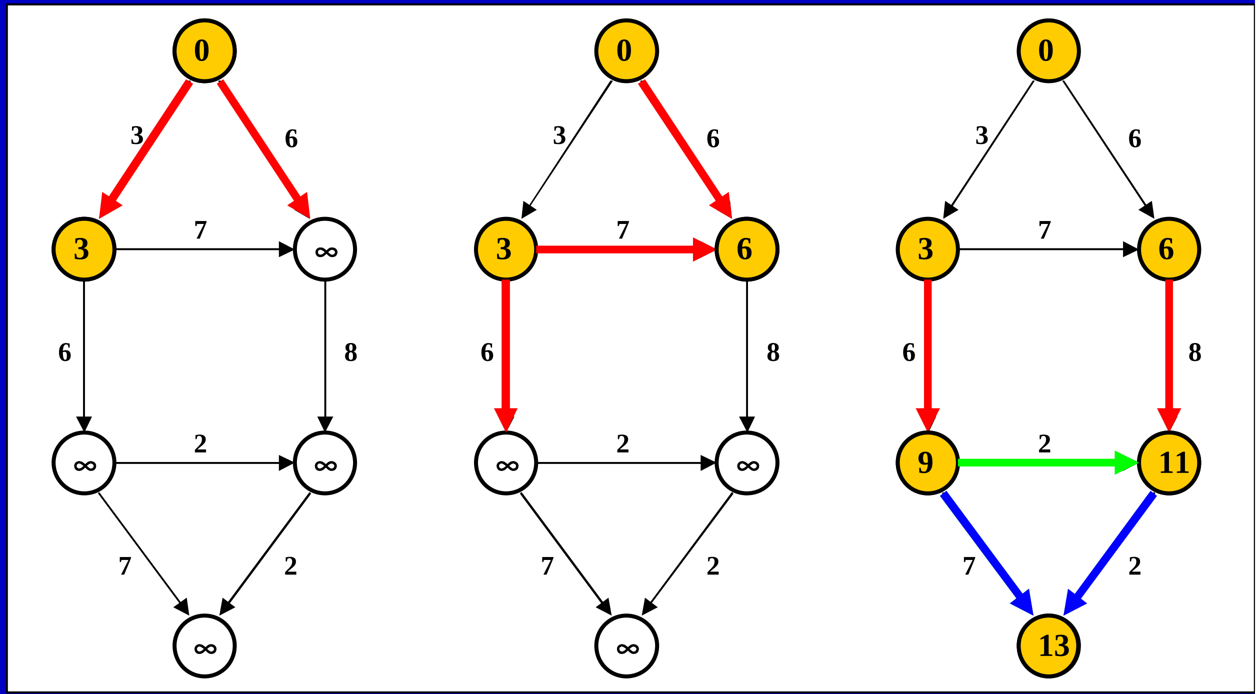
Total energy: $E = \sum_{i \in \, path} e_i$

Find ground state – i.e. optimal path
From top node to a bottom nodes:
Dijkstras algorithm



Dijkstras algorithm
for shortest paths in general graphs

Start node: s
Minimal distance (energy) from s to j: d(j)
Predecessor of j: pred(j)



algorithm Dijkstra
begin
S:={s}, S‘=N\{s};
d(s):=0, pred(s):=0;
while |S|<|N| **do**
begin
choose (i,j):
$d(j):=\min_{k,m}\{d(k)+c_{km} | k \text{ in } S, m \text{ in } S'\}$;
S‘=S‘\{j}; S=S+{j};
pred(j):=i;
end
end

Performance $O(N^2)$,
with heap reshuffling $O(N \log(N))$

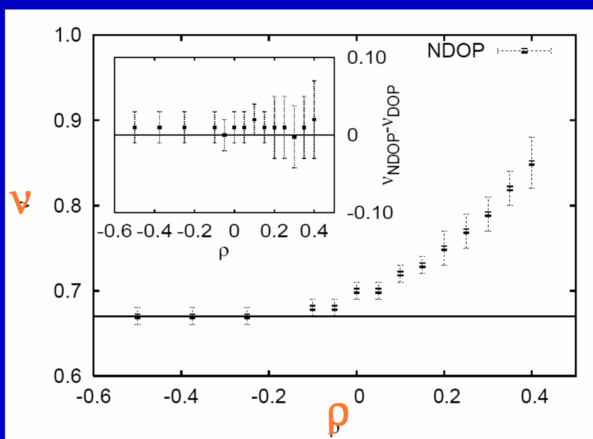
Optimal paths with correlated disorder

Isotropically correlated disorder: $\langle e_i e_{i+r} \rangle \sim r^{2\rho-1}$

Ovehangs relevant? $\rightarrow$ non-directed lattice

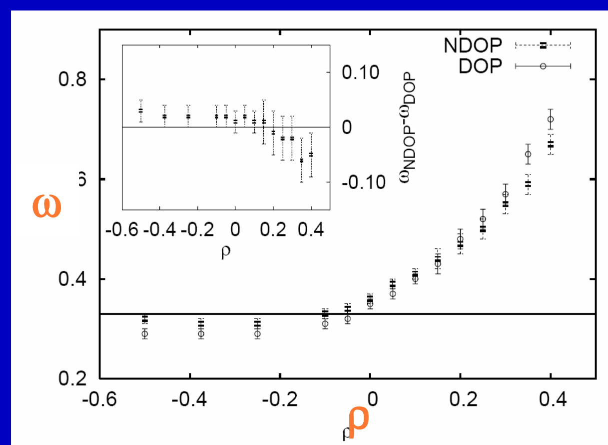
Roughness:

$D(L) = \langle x^2 \rangle - \langle x \rangle^2 \sim L^{\nu}$



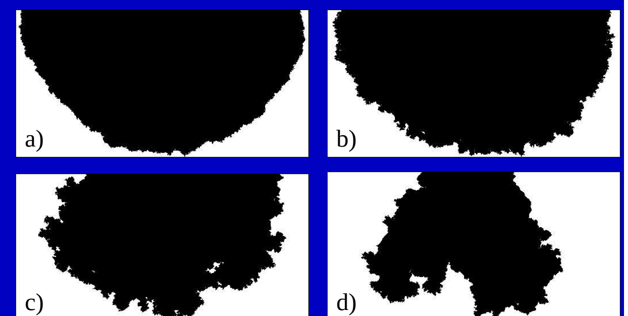
Energy fluctuations:

$\Delta E(L) = \langle E^2 \rangle - \langle E \rangle^2 \sim L^{\omega}$



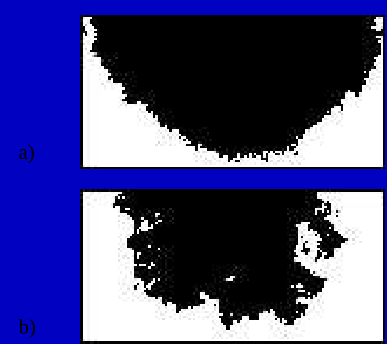
Optimal paths with $E<E_0$

2d ρ<0



ρ=0.4

3d



[Schorr, HR, EPJB 33, 347 (03)]

From one line to many lines

Continuum model for N interacting elastic lines in a random potential

$$H = \sum_{i=1}^{N} \int_0^H dz \left\{ \frac{\gamma}{2} \left[\frac{dr_i}{dz} \right]^2 + V_{rand}[r_i(z), z] + \sum_{j(\neq i)} V_{\text{int}}[r_i(z) - r_j(z)] \right\}$$

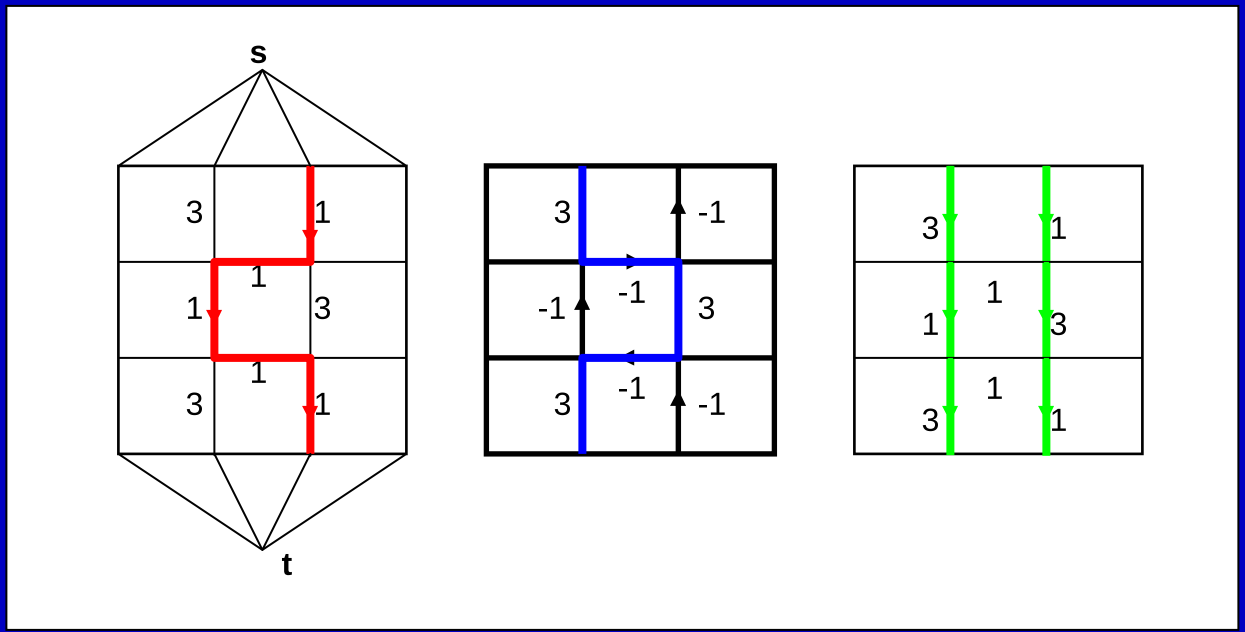
Strong disorder: $V_{\text{rand}} >> \gamma$; V_{int} short ranged, hard core

$$\Rightarrow \quad H = \sum_{i(bond)} e_i n_i \qquad n_i = 0,1 \qquad e_i \in [0,1]$$

Ground state of N-line problem:
Minimum Cost Flow problem

From one line to many lines

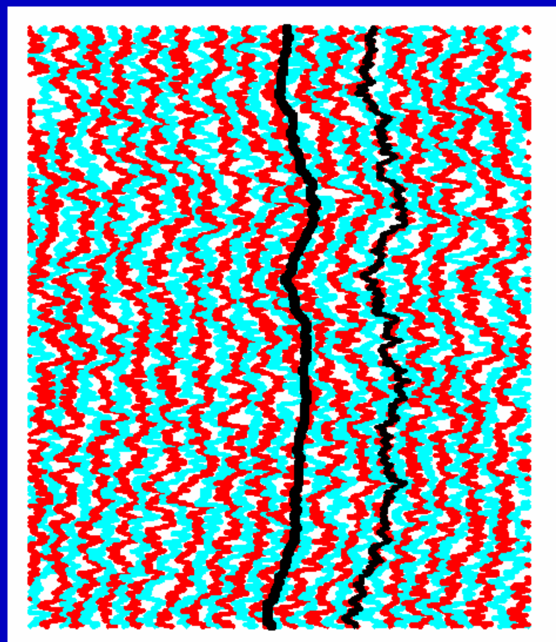
(with hard-core interactions)



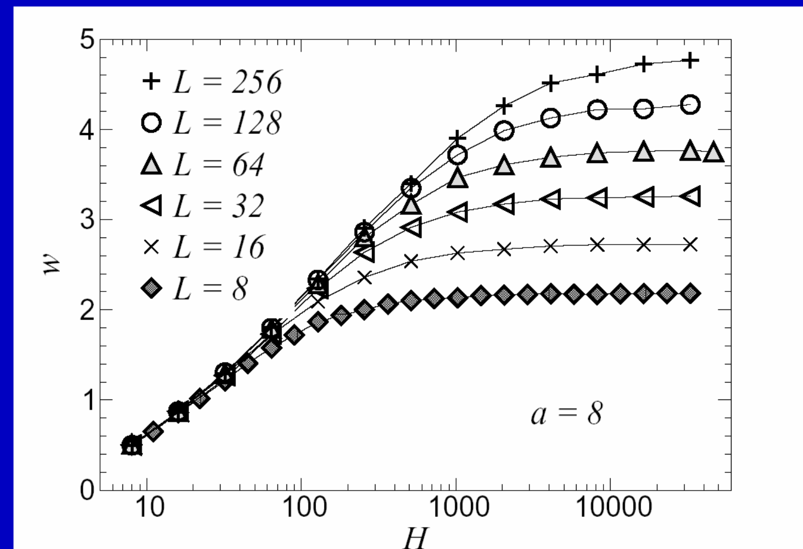
Successive shortest path algorithm

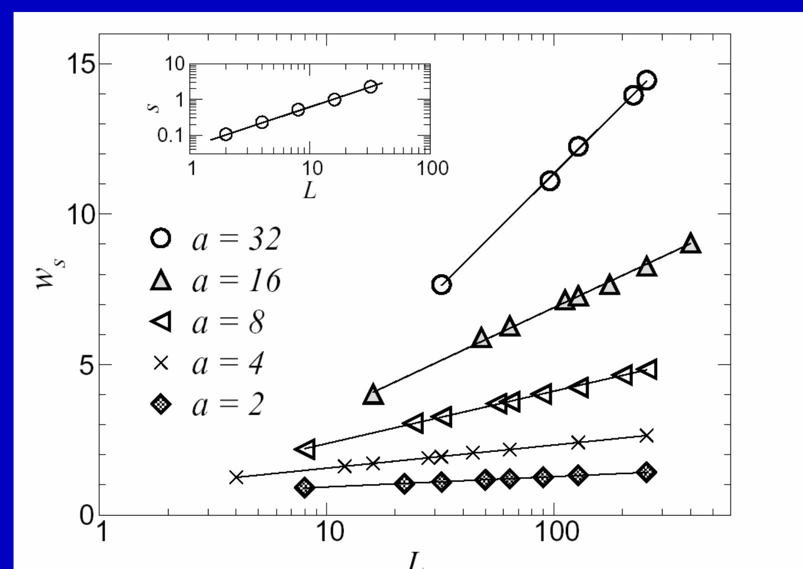
Lines in 2d - Roughness

H



L





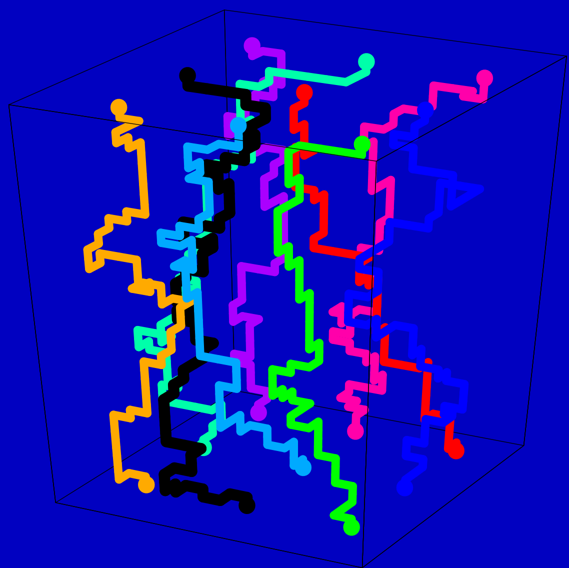
Roughness $$w^2 = \langle x_i^2 \rangle - \langle x_i \rangle^2$$

For $H \to \infty$

roughness saturates $w \to w_s(L)$

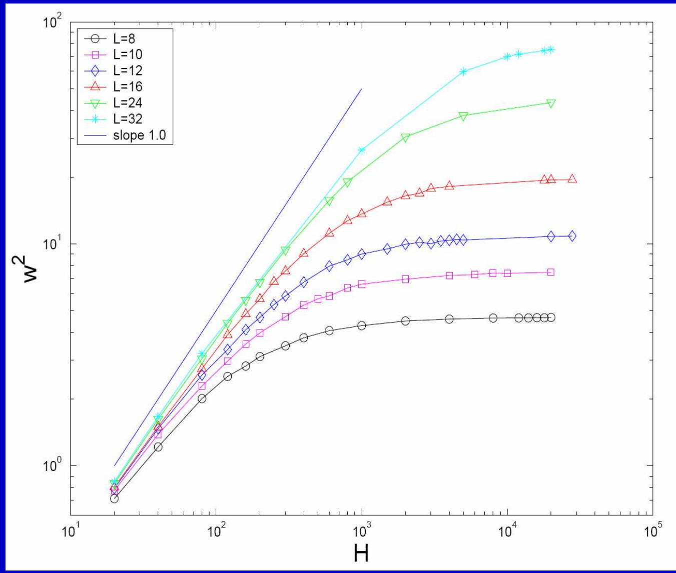
$w_s(L) \sim \ln(L)$ means **„super-rough“**

Lines in 3d - Roughness



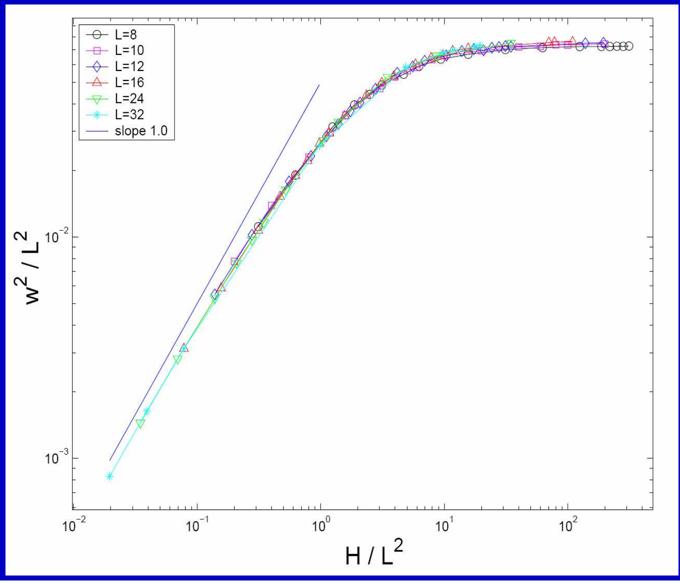
$$w^2 = \overline{\langle x_i^2 \rangle - \langle x_i \rangle^2}$$

Line density: ρ=0.2 ρ=N/L²



FSS:

$$w = L \cdot \tilde{w}(H / L^2)$$



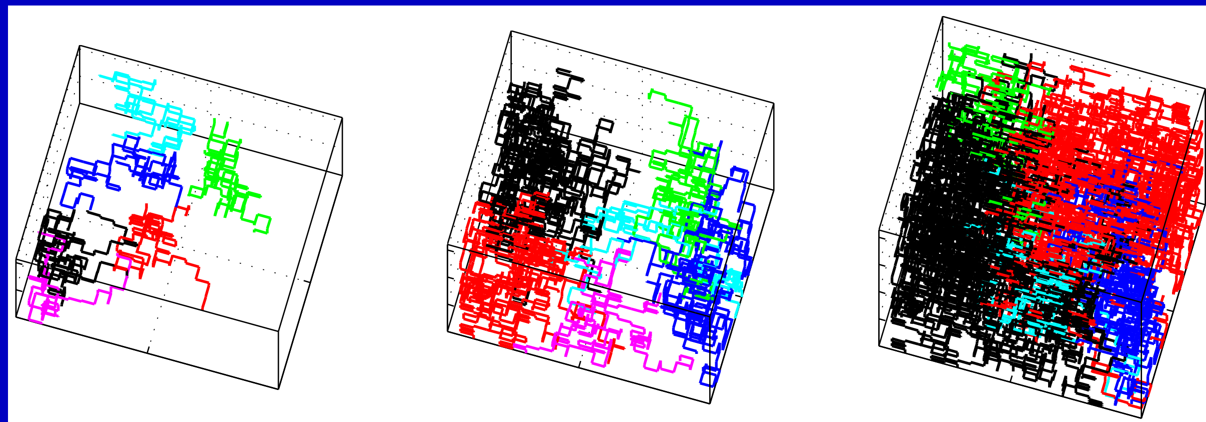
For L $\to\infty$: w~$H^{1/2}$ $\Leftrightarrow$ Random walk behavior

Saturation roughness (H$\to\infty$): w~L

($\neq$ elastic media!)

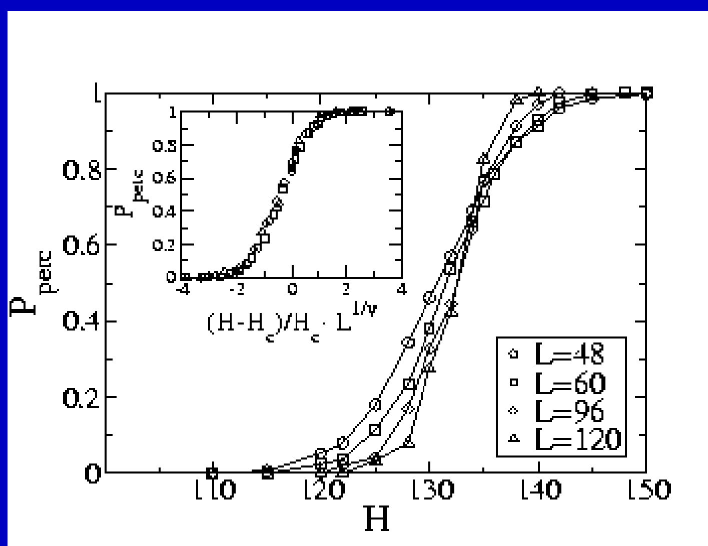
[Petäjä, Lee, HR, Alava: JSTAT P10010 (2004)]

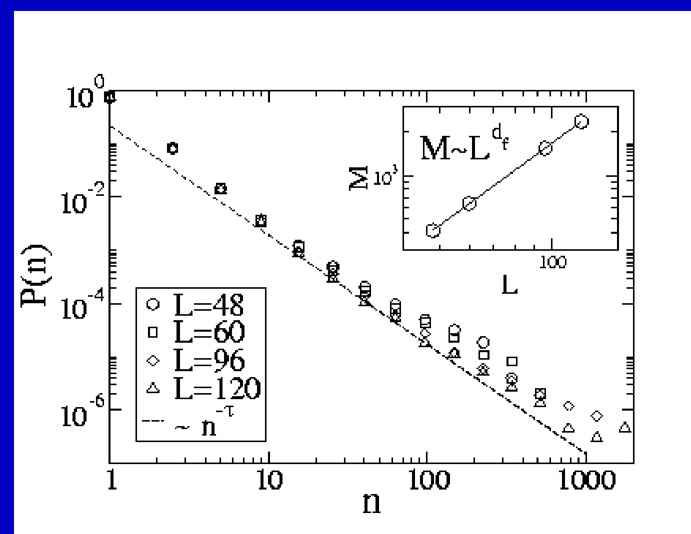
Entanglement transition of elastic lines



H =64 H =96 H=128

Conventional 2d percolation transition





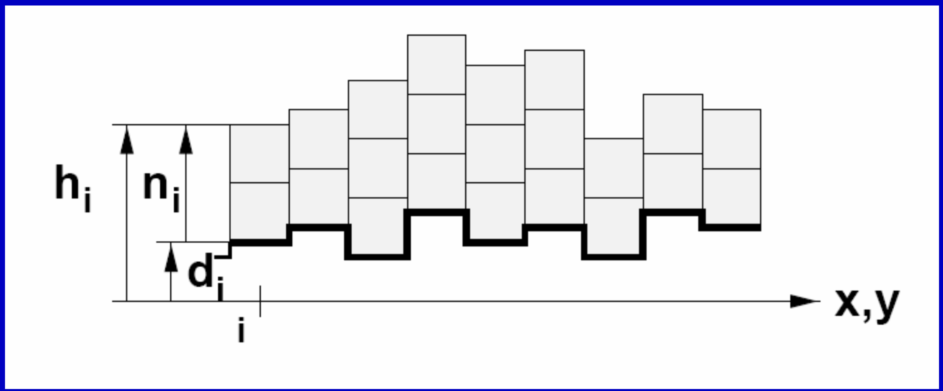
υ=4/3

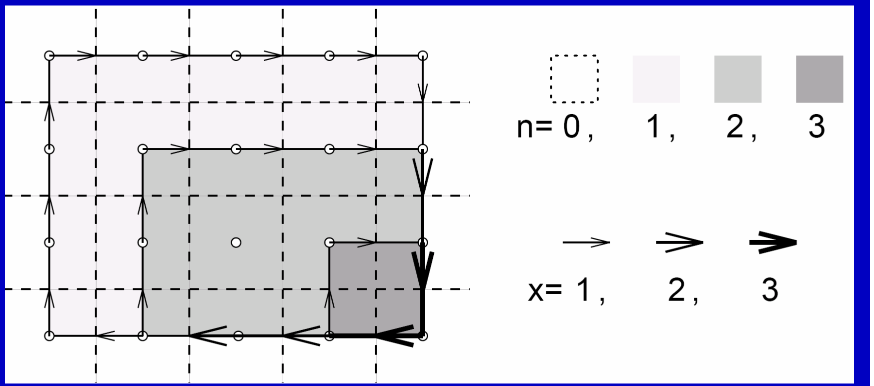
d_f=1,896

τ=2,055

[Petäjä, Alava, HR: EPL 66, 778 (04)]

The SOS model on a random substrate





Height profile ↔ Flow configuration

$$H = \sum_{(ij)} (h_i - h_j)^2 , \quad h_i = n_i + d_i , \quad d_i \in [0,1]$$

Ground state (T=0):

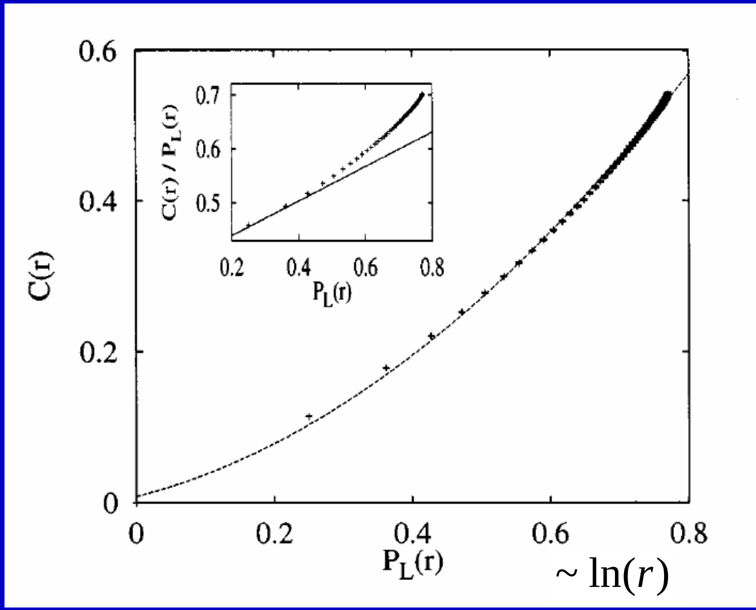
In 1d: h_i- h_{i+r} performs random walk

$C(r) = [(h_i - h_{i+r})^2] \sim r$

In 2d: Ground state superrough,

$C(r) \sim \log^2(r)$

stays superrough at temp. $0<T<T_g$



[HR, Blasum PRB 55, R7394 (1997)]

2d Universality classes: Interacting lines = elastic medium

N interacting elastic lines in a 2d random environment

$$H = \sum_{i=1}^{N} \int_0^H dz \left\{ \frac{\gamma}{2} \left[\frac{dr_i}{dz} \right]^2 + V_{rand}[r_i(z), z] + \sum_{j(\neq i)} V_{\text{int}}[r_i(z) - r_j(z)] \right\}$$

$\Leftrightarrow$ 2d Sine-Gordon model with random phase shifts

$$H = \int d^2 r [(\nabla \phi(r))^2 - \lambda \cos(\phi(r) - \theta(r))]; \quad \phi(r) \in [0, 2\pi[$$

$$\theta(r) \in [0, 2\pi[\quad random$$

$\Leftrightarrow$ SOS-model on a disordered substrate

$$H = \sum_{\langle ij \rangle} (n_i + d_i - n_j - d_j); \quad n_i \in \{0, \pm 1, \pm 2, \ldots\}$$

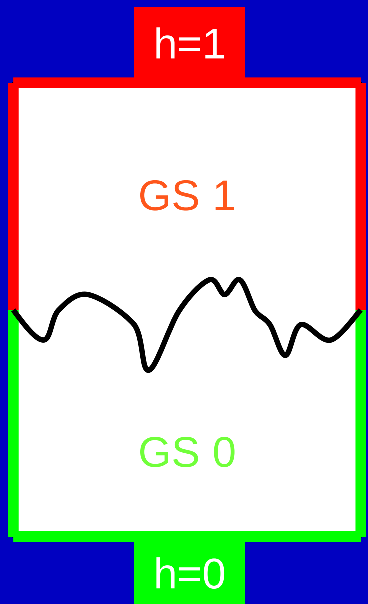
$$d \in [0, 1[\quad random$$

$T>T_g$: Rough phase, $\langle (n_i - n_{i+r})^2 \rangle \sim \ln r$ $\quad T_g = 2/\pi$

$T<T_g$: Super-rough phase, $\langle (n_i - n_{i+r})^2 \rangle \sim \ln^2 r$

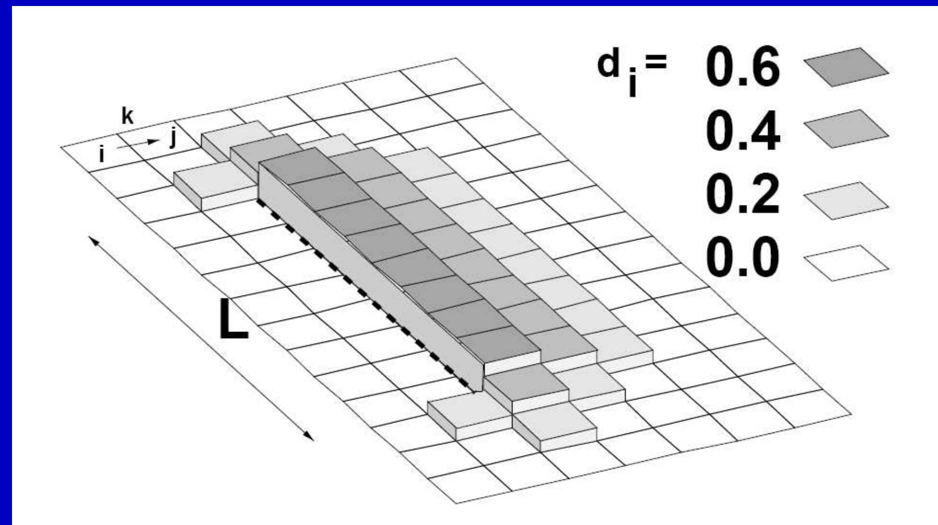
Domain walls …. and dislocations:

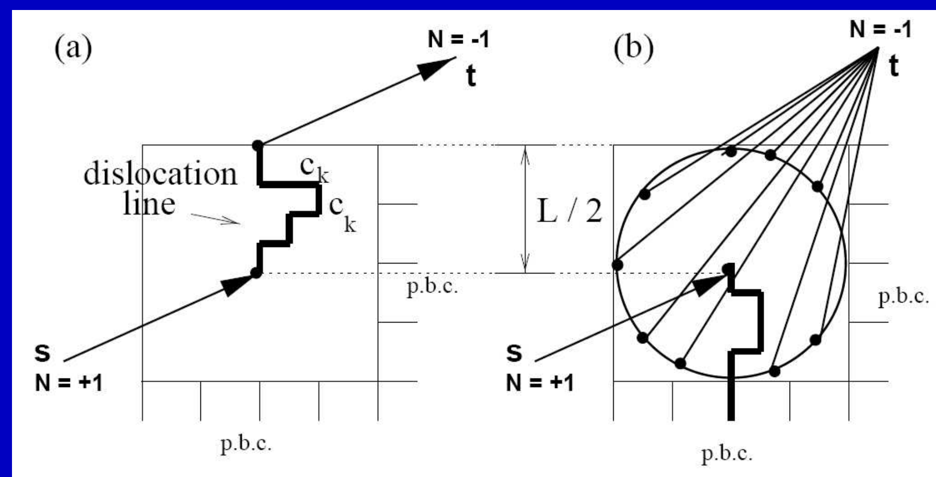
(random SOS model)



$D_f = 1.32 \pm 0.02$

[HR, Blasum PRB 55, R7394 (1997)]





$$[\Delta E]_{\mathrm{dis}} \sim \begin{cases} \ln(L) & \text{fixed defect pair} \\ -0.27(7) \times \ln^{3/2}(L) & \text{partially optimized} \\ -0.73(8) \times \ln^{3/2}(L) & \text{completely optimized} \end{cases}$$

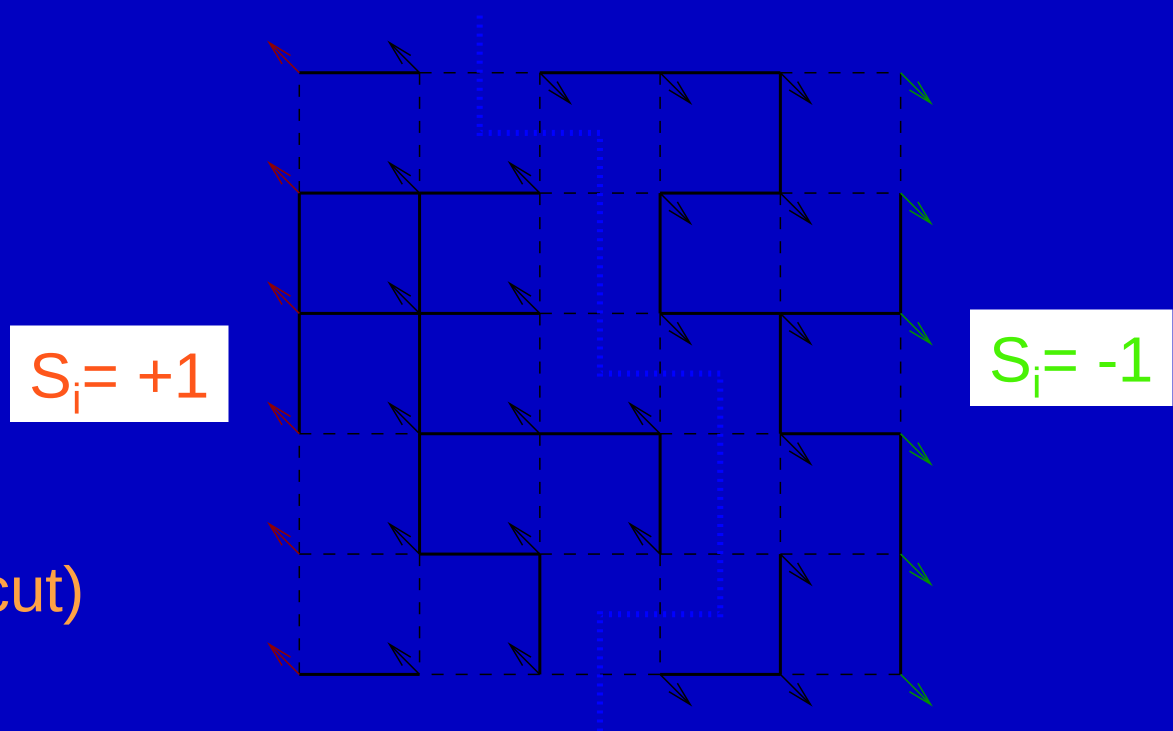
[Pfeiffer, HR JPA 33, 2489 (00)]

Another combinatorial optimization problem: Interfaces in random bond Ising ferromagnets

$$H = -\sum_i J_{ij} S_i S_j \qquad J_{ij} \geq 0, \quad S_i = \pm 1$$

Find for given random bonds J_{ij} the ground state configuration $\{S_i\}$ with fixed +/- b.c.

$\Leftrightarrow$ Find interface (cut) with minimum energy

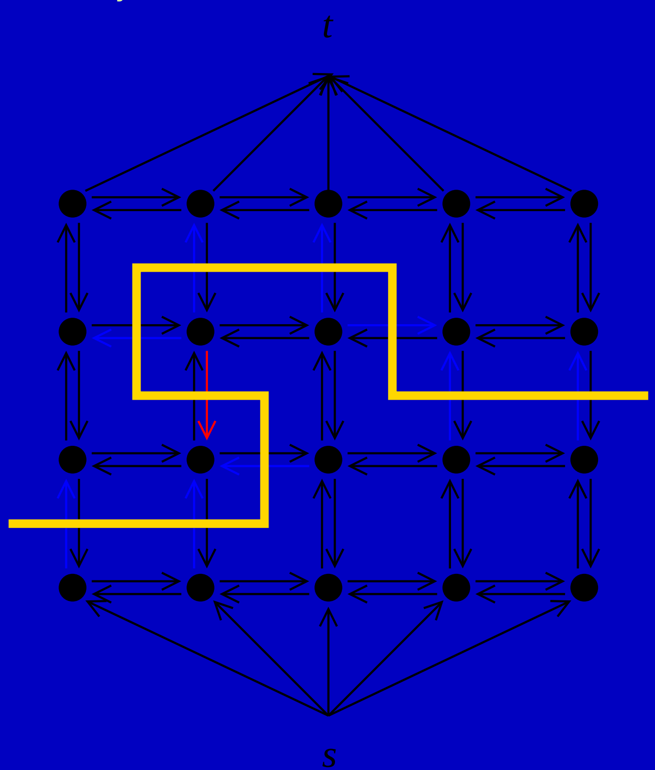


Min-Cut-Max-Flow Problem

network $G(V,A)$, arcs (Bonds) $(i,j)\in A$ have capacity $u_{ij}>0$,
flow $0<=n_{ij}<=u_{ij}$ fulfills mass balance constraint

$$\sum_{\{j|(ji)\in A\}} n_{ji} - \sum_{\{j|(ij)\in A\}} n_{ij} = \begin{cases} v & \text{for } i=s \\ -v & \text{for } i=t \\ 0 & \text{else} \end{cases}$$

Find the maximum flow n^* with value v from s to t



residual network G(n) with residual capacities

$$r_{ij} = u_{ij} - n_{ij} + n_{ji}$$

n^* maximum flow $\Leftrightarrow$ no directed path $s\rightarrow t$ in $G(n^*)$

s-t cut [S,S‘] is a partition of V in two disjoint sets with $s\in S$, $t\in S' = V\backslash S$,
$(S,S') = \{(i,j) \in A | i \in S, j \in S'\}$; capacity of the s-t-cut $v[S,S'] = \sum_{(i,j)\in(S,S')} u_{ij}$

Min-Cut-Max-Flow-Theorem: $\max_{\{n\}} v = \min_{[S,S']} v[S,S']$ and $r_{ij}^*=0$ along (S,S‘)

Problems that can be mapped on min-cut / max-flow

- Interfaces / wetting in random media
- Random field Ising model (in any dimension)
- Periodic media (flux lines, CDW, etc.) in disordered environments
- Elastic manifolds with periodic potential and disorder

E.g.: Periodic elastic medium + periodic potential

Mapping to an RBIFM interface problem

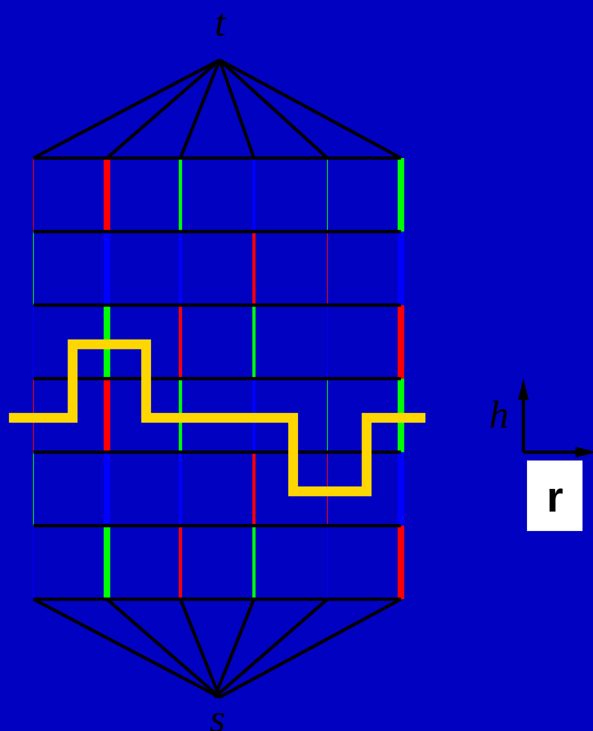
Discrete interface hamiltonian:

$$H = \sum_{(ij)} (h_i - h_j)^2 - \sum_i \eta_i \cos(2\pi h_i / p - \theta_i)$$

Ising model:

$$H = -\sum_{(ij)} J_{ij} S_i S_j$$

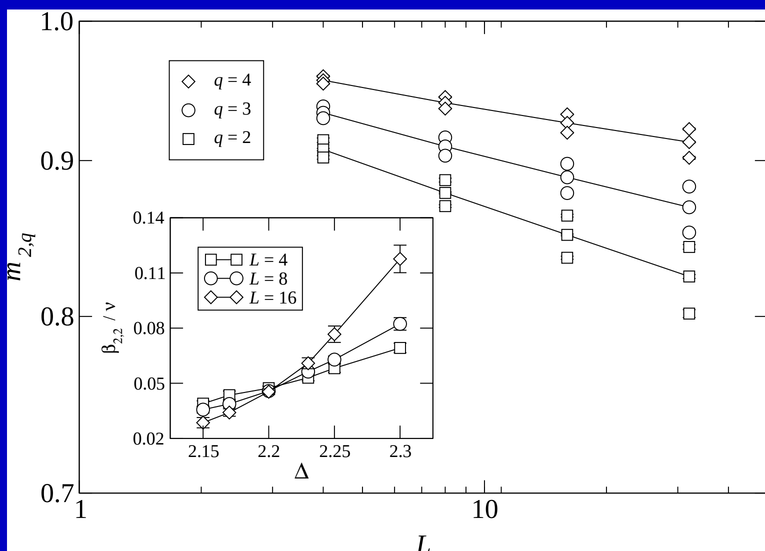
$$J_{h-direction} = \eta(r) \cos(2\pi h / p - \theta(r))$$
$$J_{r-direction} = const. \to \gamma(r,h)$$

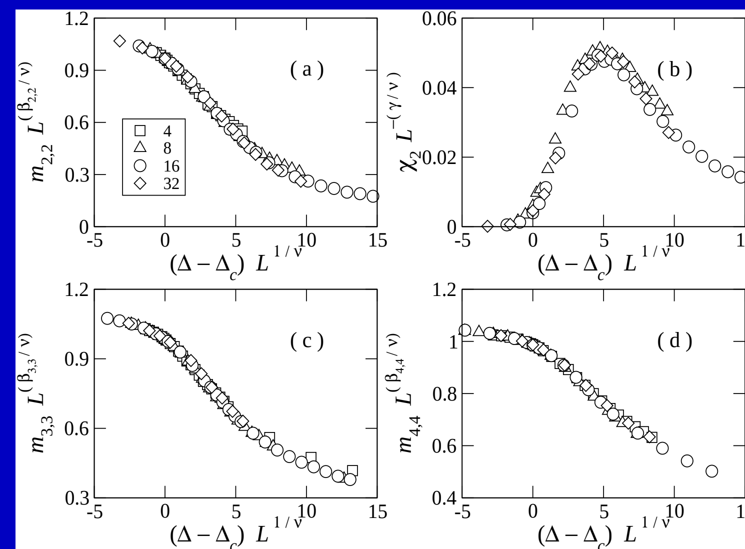


Periodic elastic medium + periodic potential

The roughening transition in 3d:

Order parameter: $m_{p,q}(L,\Delta) = |\langle e^{2\pi i/q} \rangle|$





p	Δ_c	$\beta_{p,2}/\nu$	$\beta_{p,3}/\nu$	$\beta_{p,4}/\nu$	ν
2	2.20	0.046	0.034	0.022	1.25
3	2.48	0.049	0.037	0.024	1.29
4	2.95	0.044	0.033	0.022	1.28

[Noh, HR: PRL 87, 176102 (01)],
[c.f. Emig & Nattermann: PRL 79, 5090 (97)]

E.g.: Fractal properties of Contour Loops

$$H(\{h(\mathbf{x})\}) = \int d^2\mathbf{x}\left[\frac{K}{2}|\nabla h(\mathbf{x})|^2 + V[\mathbf{x}, h(\mathbf{x})]\right],$$

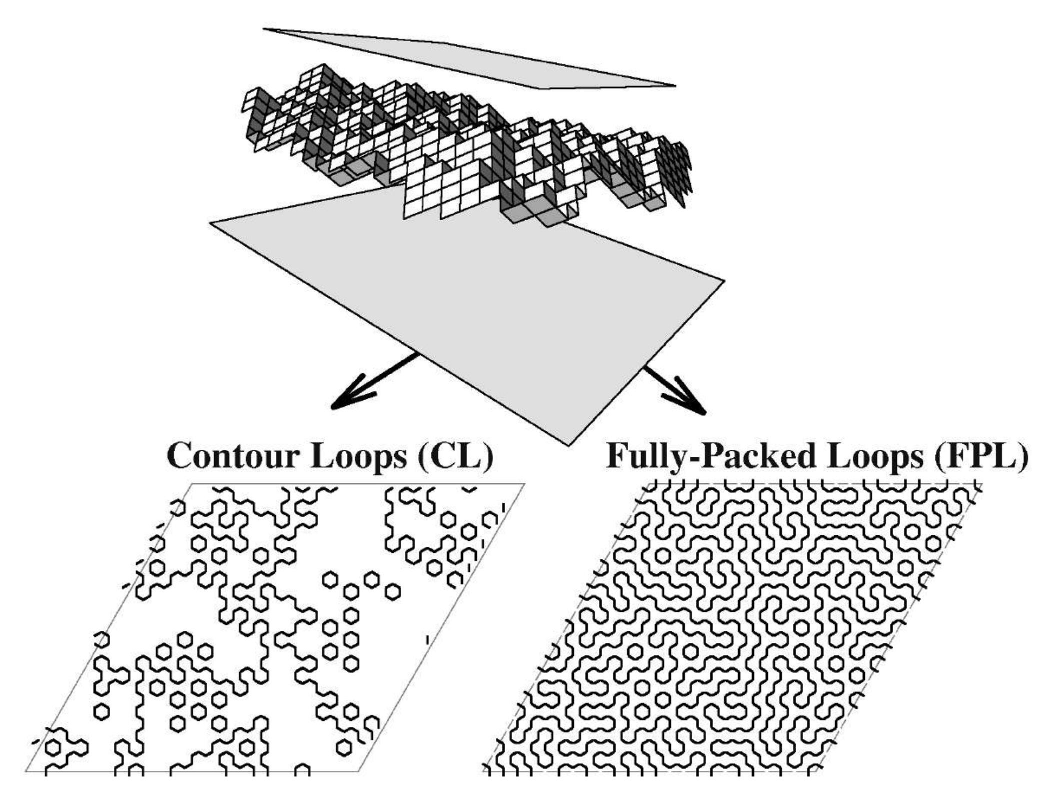


TABLE I. Geometric exponents of both contour loops and fully packed loops. Rational numbers are the proposed exponents.

Random elastic medium contour loops	Fully packed loops
$D = 1.46 \pm 0.01$ (3/2)	$D = 1.75 \pm 0.01$ (7/4)
$\tau = 2.32 \pm 0.01$ (7/3)	$\tau = 2.15 \pm 0.01$ (15/7)
$x_l = 0.50 \pm 0.01$ (1/2)	$x_l = 0.25 \pm 0.01$ (1/4)
$\zeta = 0.08 \pm 0.01$ (0)	$\zeta = 0.00 \pm 0.01$ (0)
Random manifold contour loops	**Fully packed loops**
$D = 1.31 \pm 0.02$ (?)	$D = 1.74 \pm 0.01$ (7/4)
$\tau = 2.19 \pm 0.02$ (?)	$\tau = 2.15 \pm 0.01$ (15/7)
$x_l = 0.49 \pm 0.02$ (1/2)	$x_l = 0.25 \pm 0.01$ (1/4)
$\zeta = 0.40 \pm 0.02$ (?)	$\zeta = 0.01 \pm 0.01$ (0)

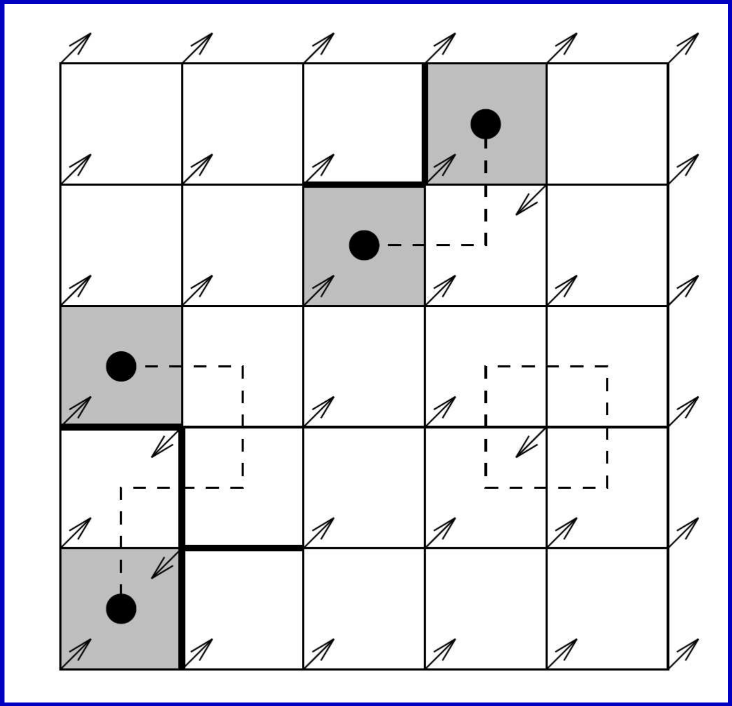
[Zeng et al., PRL 80, 109 (98)]

2d Ising spin glass:
Minimum weigted matching problem

$$H = -\sum_{(ij)} J_{ij}\sigma_i\sigma_j \ ,$$

J_{ij} Gaussian

$$H(\underline{\sigma}) = -C + 2 \sum_{\text{unsatisfied edges}} |J_{ij}| \ .$$



Recent applications: See A. Middletons talk next week on SLE and domain walls in 2d Ising spin glasses…

Further applications of combinatorial optimization methods in Stat-Phys.

- Flux lines with hard core interactions
- Vortex glass with strong screening
- Interfaces, elastic manifolds, periodic media
- Wetting phenomena in random systems
- Random field Ising systems
- Spin glasses (2d polynomial, d>2 NP complete)
- Statistical physics of complexity (K-Sat, vertex cover)
- Random bond Potts model at T_c in the limit $q \to \infty$
- ...