Insights into coherent structure formation arising from application of statistical state dynamics

Brian Farrell

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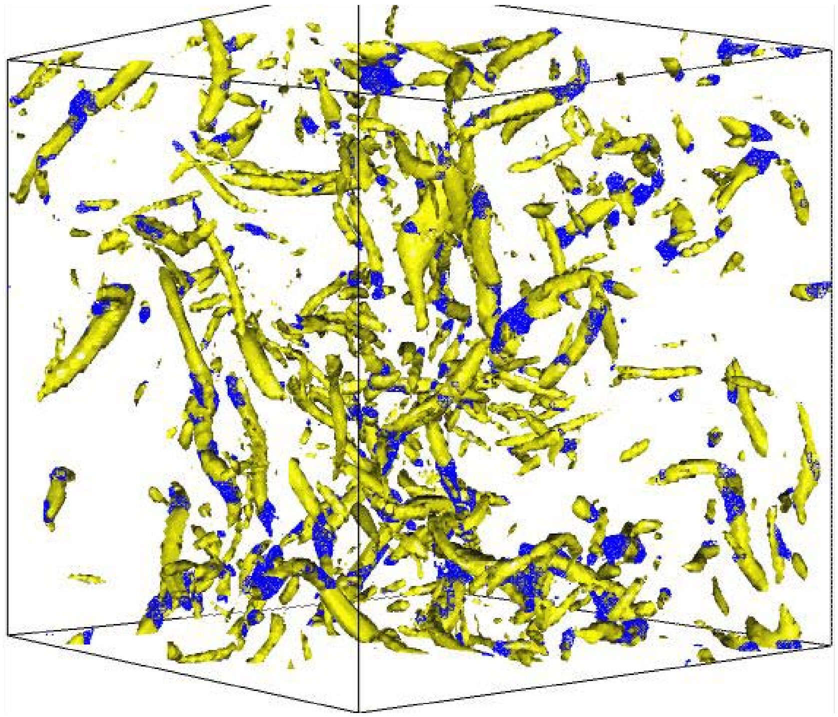
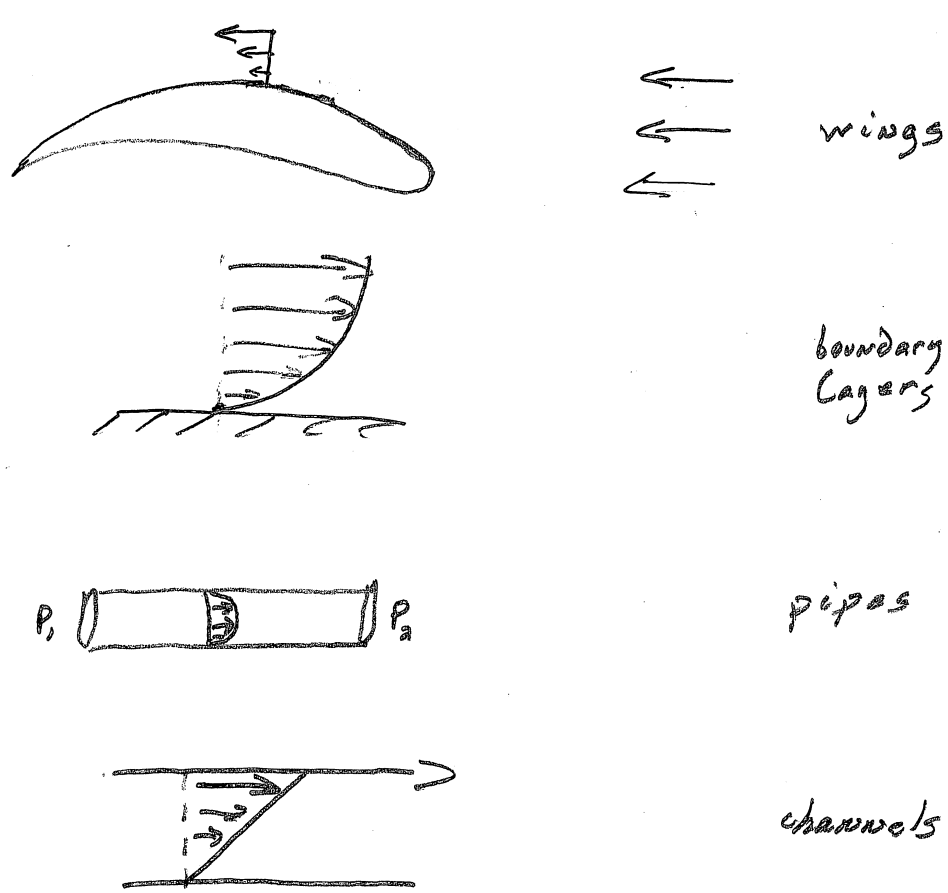
thanks to: Petros Ioannou, Nikos Bakas, Tim DelSole, Dennice Gayme,
Nikos Constantinou, Marios Nikolaidis

I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.

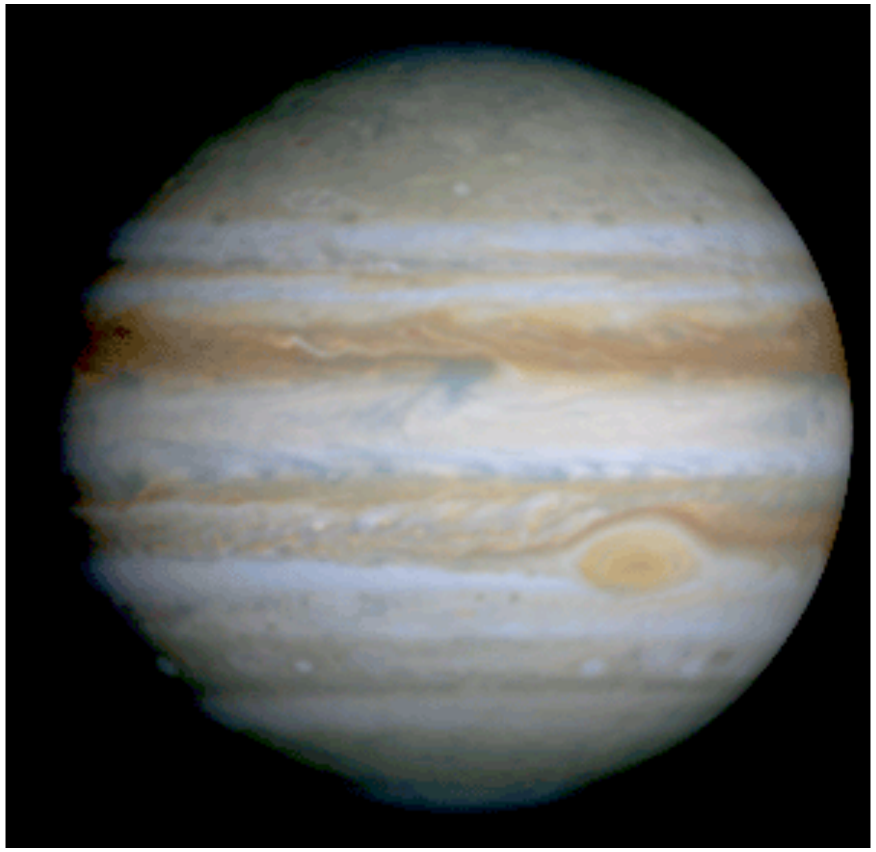
Horace Lamb (1932)



- Concentrating on isotropic homogeneous turbulence was a strategic error in the history of the science of fluid dynamics.
- The reason is IHT is essentially, intrinsically and irreducibly nonlinear so it is difficult to make analytical progress.

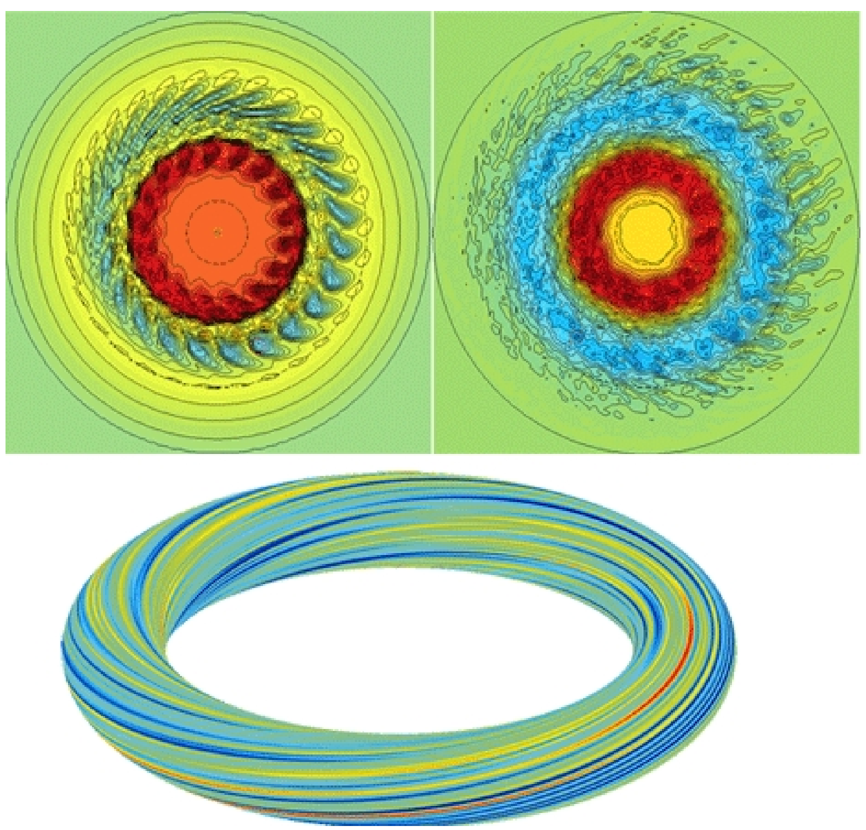


- Remarkably, the turbulent systems of primary practical interest are essentially linear and admit analytic solution.
- The reason for this linearity is that turbulence in pipes and channels, in the atmospheric boundary layer, baroclinic turbulence in the Earth's midlatitude jets, turbulent jets in the gaseous planets, drift wave turbulence in magnetic confinement devices etc. are characterized by strong shear.
- Linearizing about this shear results in a quasilinear dynamics that allows analytic solution for the statistical equilibrium turbulent state.

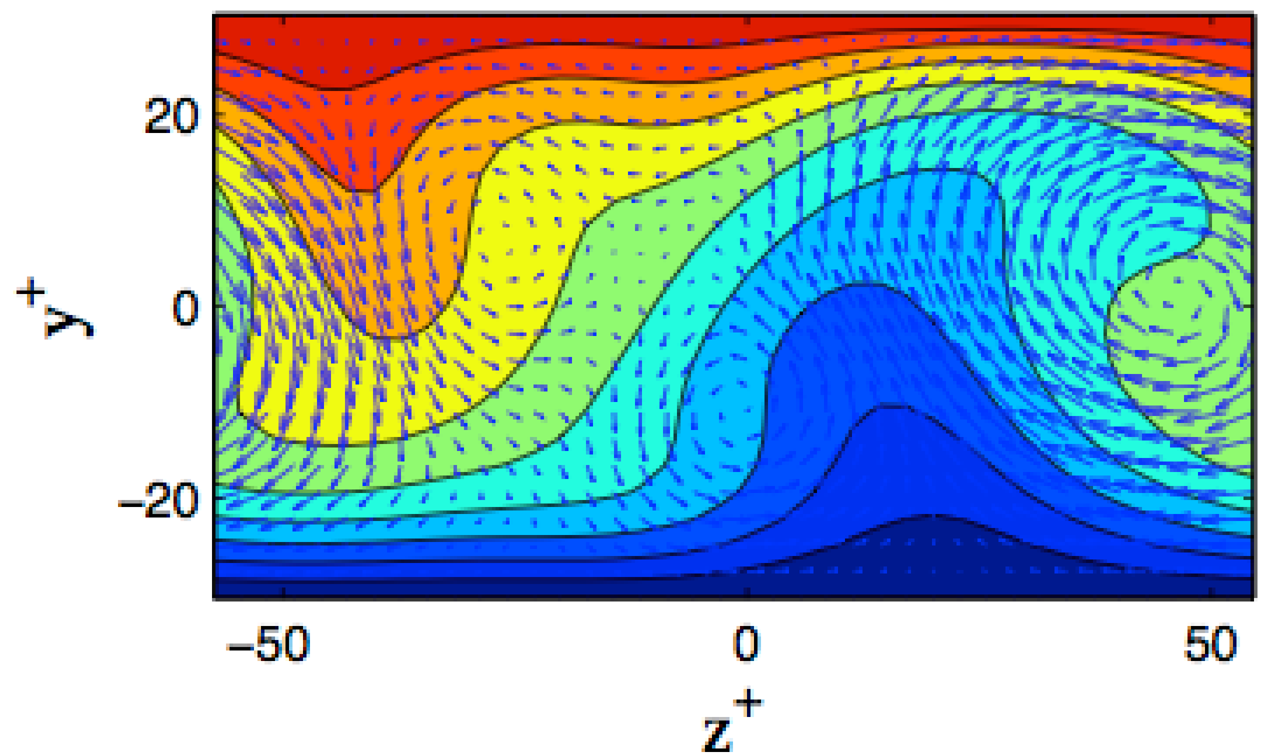


Jovian jets

Earth’s polar front jet

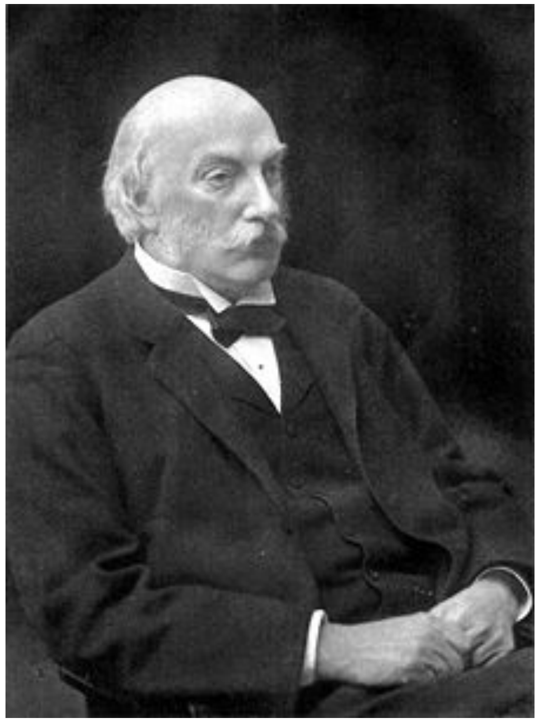


jets in tokamaks



jets in shear flow turbulence

Perturbation Dynamics and Turbulence Theory



In *The Theory of Sound* (1877) John William Strutt, 3rd Baron Rayleigh advanced the method of normal mode stability analysis.

- If the turbulence problems of interest are essentially linear why was this not immediately recognized?
- The reason is that solutions for the turbulent state were sought as eigenmodes of stationary states in the linearized N-S equations.
- However, this application of Rayleigh's method fails to identify the entire manifold of modes that are responsible for establishing and sustaining turbulence.
- In fact, the structures that sustain and regulate turbulence do arise as instabilities, but not in the linearized N-S equations.

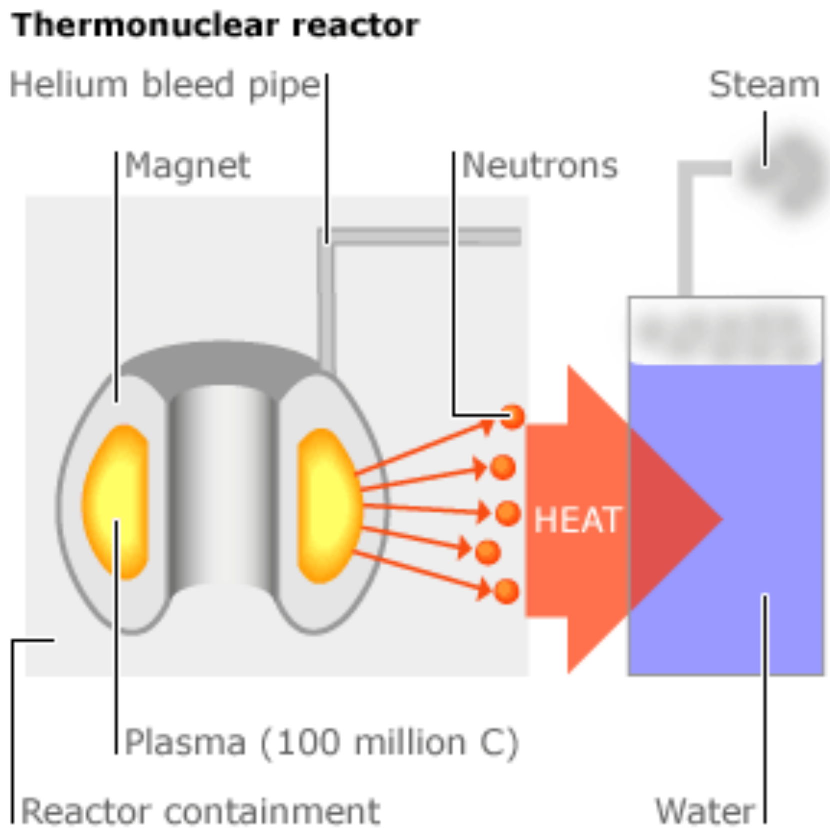
Statistical State Dynamics (SSD) underlies turbulence

The turbulence problem has analytical solution only when the N-S equations are expressed using statistical variables (means and moments of velocity) rather than the state variables (velocity).

The approach to studying turbulence based on statistical state dynamics is reviewed here:

F & Ioannou, (2019): Statistical State Dynamics: A New Perspective on Turbulence in Shear Flow. *Zonal Jets Phenomenology, Genesis, and Physics*. Ed. Boris Galpirin and Peter L. Read. Cambridge University Press 2019, 380-400.

ITER and the Jet Stream

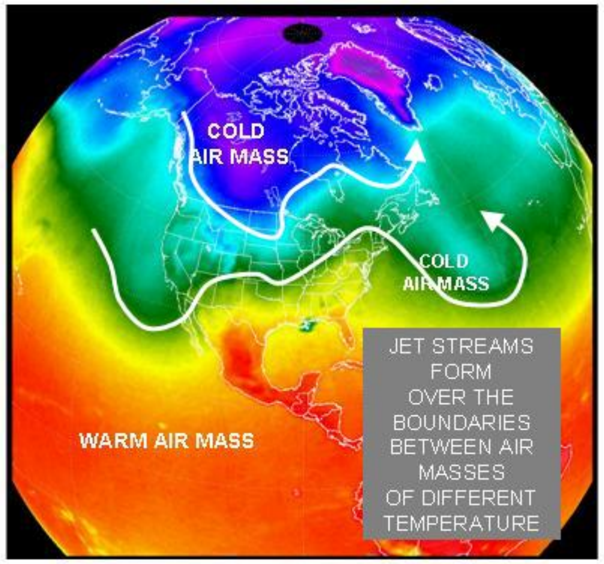


The Lorentz force:

$$\vec{F} = q\vec{V} \times \vec{B}$$

contains the Tokamak plasma.

F & Ioannou, 2009: A Stochastic Structural Stability Theory model of the drift wave-zonal flow system., *Physics of Plasmas,* 16, 112903



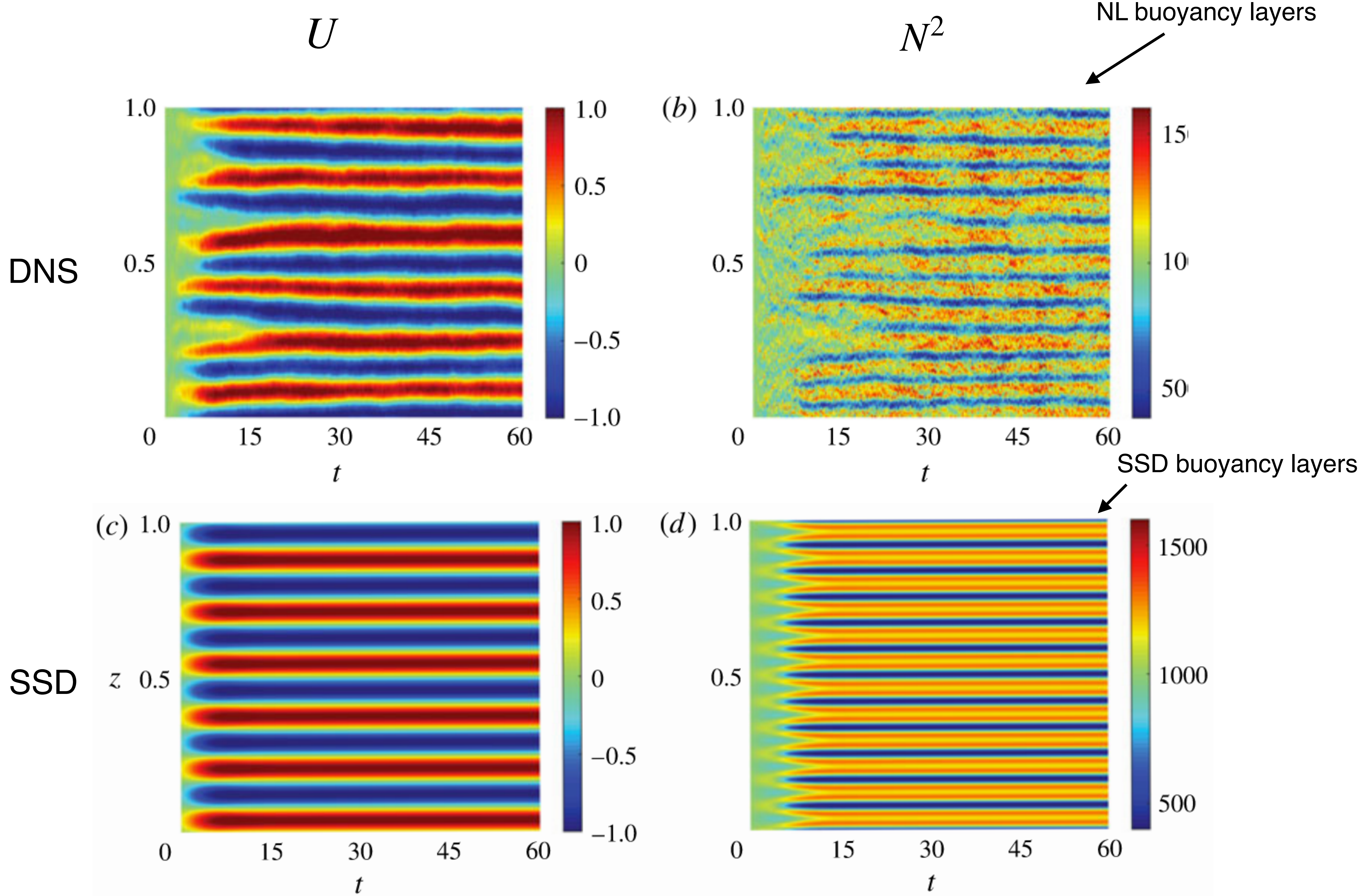
The Coriolis force:

$$\vec{F} = 2\vec{V} \times \vec{\Omega}$$

supports the Earth's P-E thermal contrast.

F & Ioannou, (2009): A Theory of Baroclinic Turbulence. *J. Atmos. Sci.* , *66*, 2445-2454 . J. Atmos. Sci.

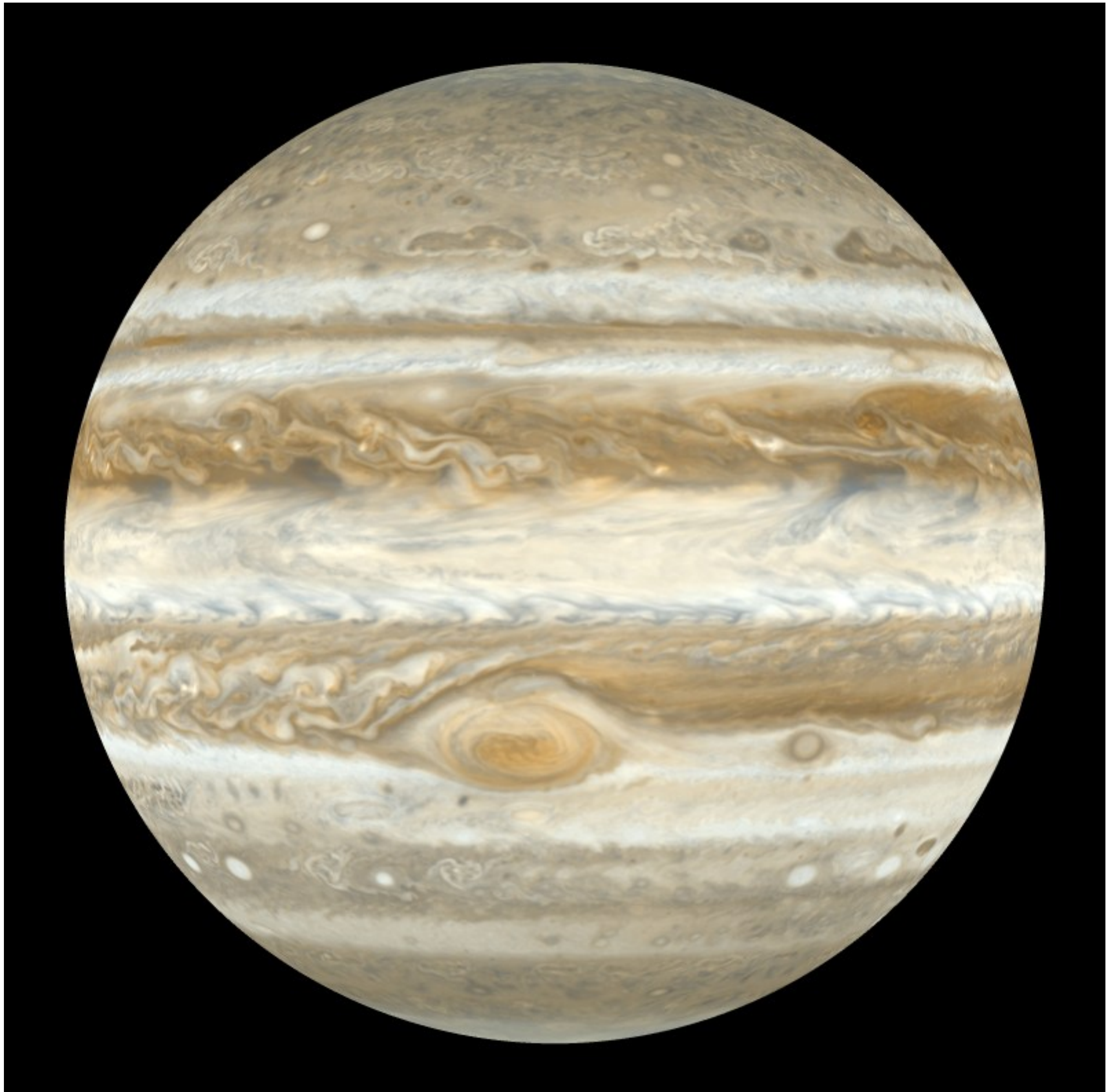
Layering in Stratified Flow



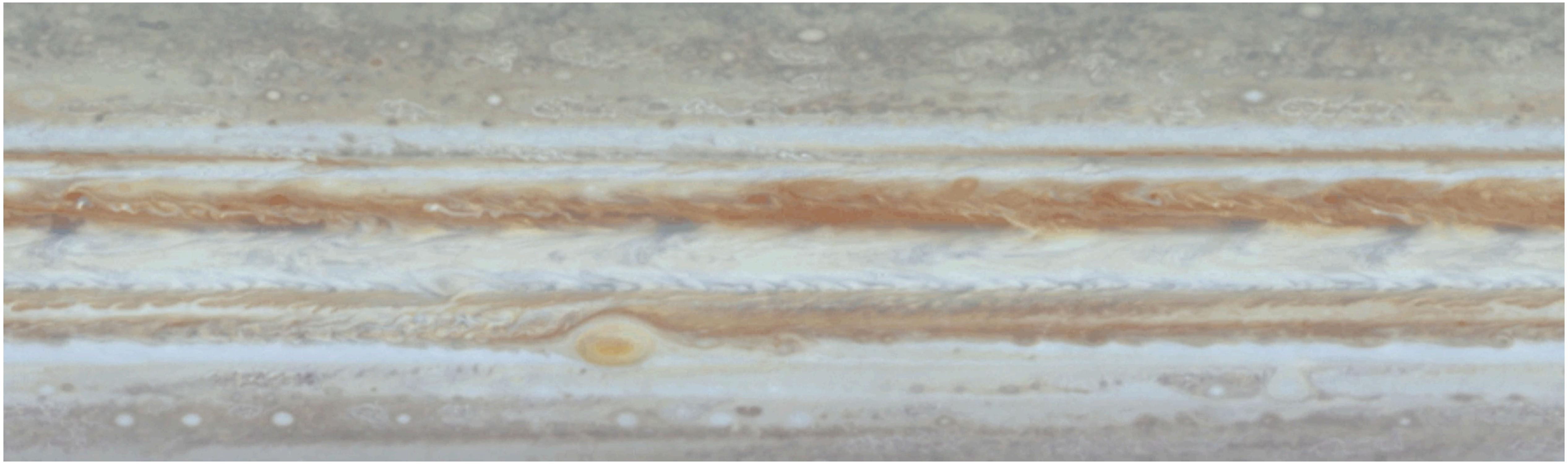
Fitzgerald, J. G., & F (2018). Statistical state dynamics of vertically sheared horizontal flows in two-dimensional Stratified turbulence. Journal of Fluid Dynamics , 854, 544-590.

Some background has been given on the problem of the emergence of zonal winds in the gaseous planets so we will build on that for our example.

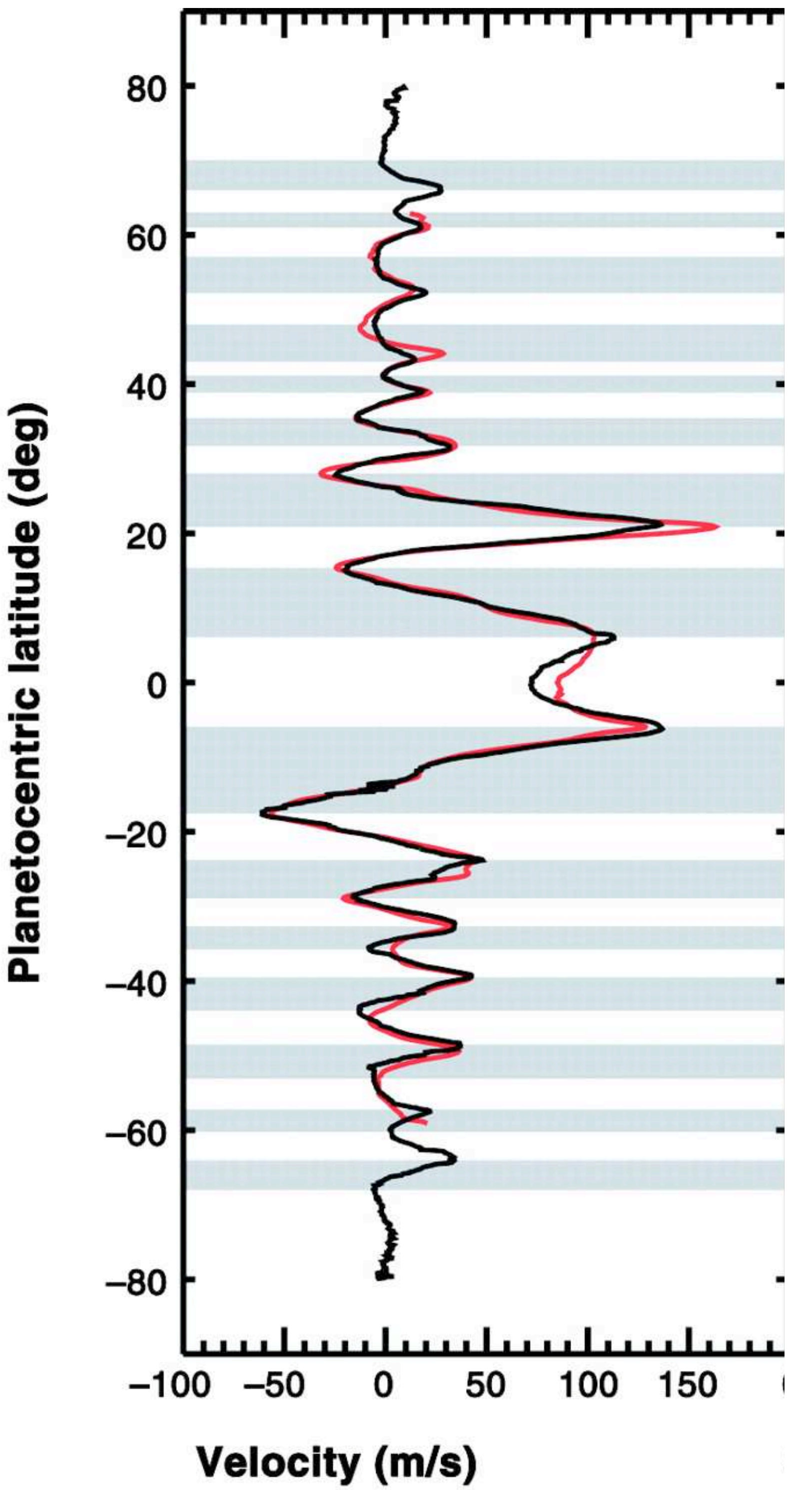
Jupiter’s Winds are Emergent Jets



Jupiter's Winds are Emergent Jets



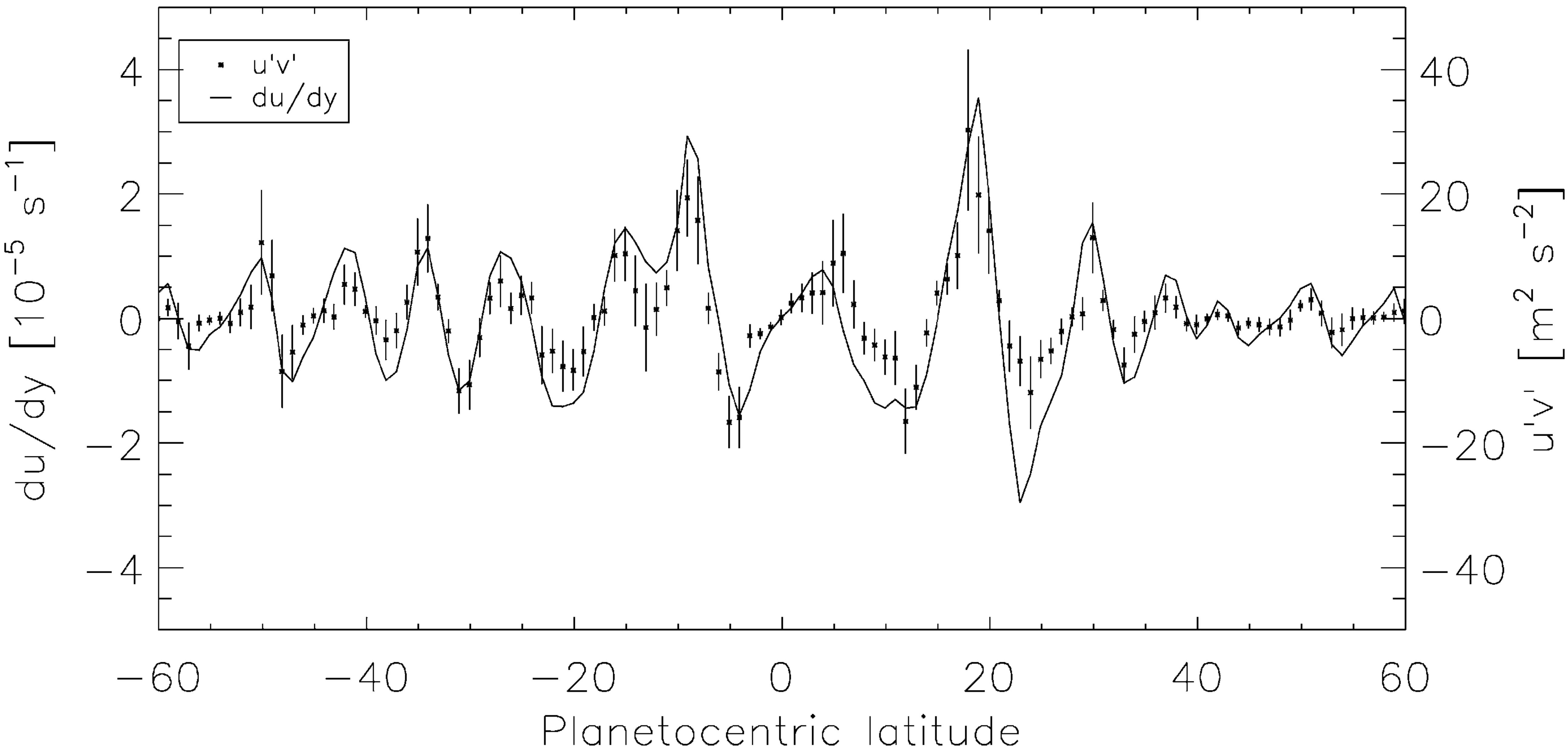
Jupiter's Winds are Steady



Winds from Cassini (1997; black); Voyager 2 (1979; red)

(Porco et al., 2003; Limaye, 1986)

Jupiter's Winds are Eddy Driven



Momentum flux and shear are correlated with coefficient 0.86.

(Salyk et al., 2006)

- midlatitude beta-plane with x, y the zonal and the meridional coordinates.
- streamfunction, ψ, is the state variable with $u = -\partial_y \psi$ and $v = \partial_x \psi$.
- total vorticity is $q + 2\Omega + \beta y$, with $q = \nabla^2 \psi$ the relative and $2\Omega + \beta y$ the planetary component.
- dynamics is governed by the barotropic vorticity equation:

$$\partial_t q + u \partial_x q + v \partial_y q + \beta v = -rq - \nu_4 \Delta^2 q + \sqrt{\epsilon} F \ .$$

Form the Zonal Mean/Perturbation Form of the Vorticity Equation:

perturbation nonlinearity

stochastic excitation

$$\partial_t U = \overline{v'q'} - rU$$

$$\partial_t q' = \underbrace{-U\partial_x q' + (U_{yy} - \beta)\partial_x \psi' - rq' - \nu_4 \nabla^2 q'}_{\text{Linear perturbation operator} \equiv A} + F_e + \epsilon^{1/2} F$$

Drop the nonlinearity in the perturbation vorticity equation to form the quasi-linear (QL) vorticity equation:

perturbation nonlinearity (ignored)

stochastic excitation

$$\partial_t U = \overline{v'q'} - rU$$

$$\partial_t q' = -U\partial_x q' + (U_{yy} - \beta)\partial_x \psi' - rq' - \nu_4 \nabla^2 q' + \cancel{N_e} + \epsilon^{1/2} F$$

Linear perturbation operator $\equiv A(U)$

Note two properties of the QL vorticity equation:

- Rewriting the NL vorticity equation in mean/perturbation form provides the opportunity to isolate, retain or eliminate, mechanisms in the vorticity equation.
- Eliminating the perturbation nonlinearity by forming the QL equation eliminates both the mechanism of the arrested cascade as well as the mechanism of wave breaking i.e. the “surf zone” jet formation mechanism.

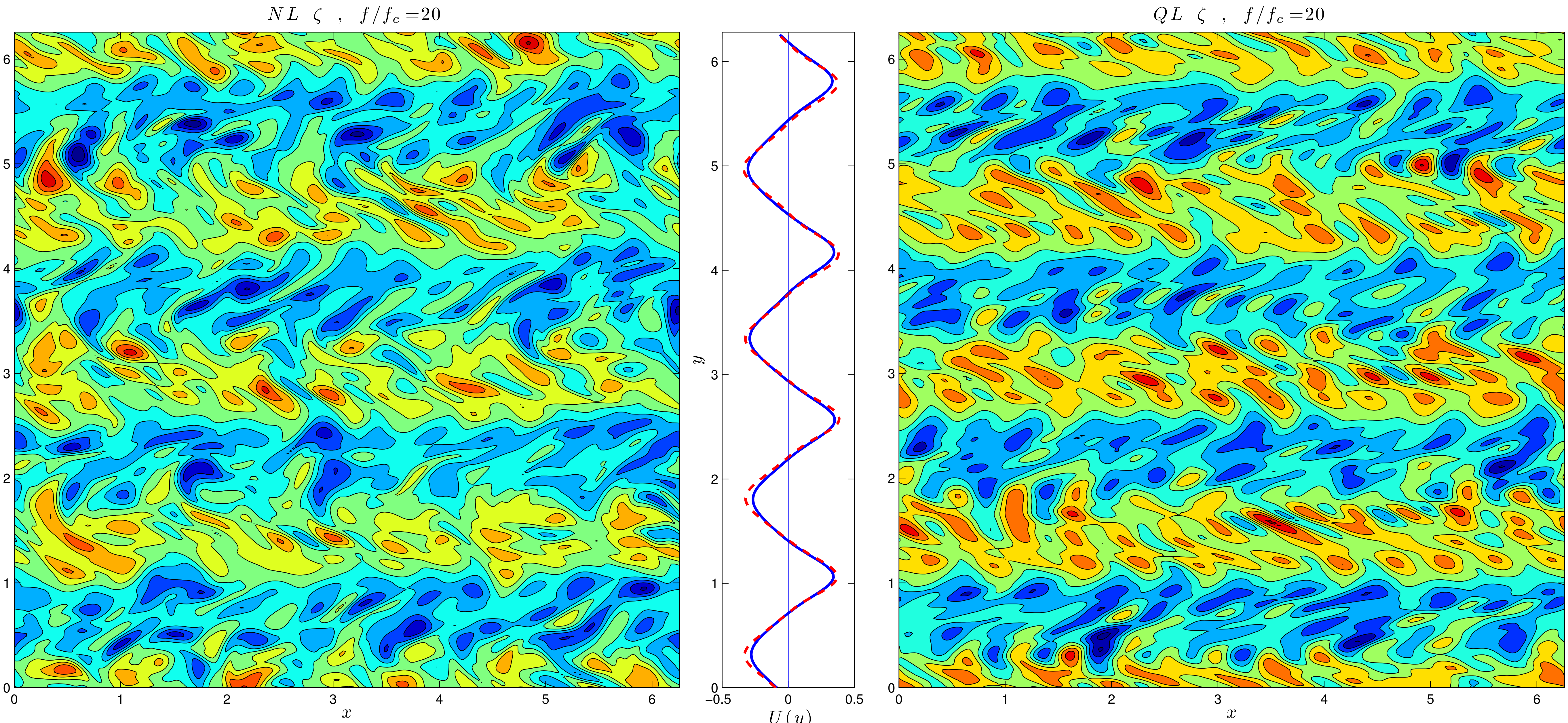
Under stochastic excitation of isotropic turbulence NL and QL dynamics produce indistinguishable jets

(NB: despite QL supporting neither a cascade nor wave breaking)

Zonal jets are the same

NL vorticity

QL vorticity



β=10

From QL dynamics to SSD

- The QL system affords insight into the dynamics of β-plane turbulence but even more insight is obtained by formulating the associated SSD.
- The associated SSD follows from replacing the perturbation dynamics of QL by its associated ensemble mean dynamics.
- Stable fixed point solutions of the SSD for a turbulent system correspond to analytic solution of the turbulence problem for that system.

Obtaining the SSD equation

In the limit of a large number of independent realizations of the excitation acting simultaneously on the mean jet the individual time dependent forcing of the mean flow is replaced by the ensemble mean of these forcings (central limit theorem) and this nonlinear coupled system becomes autonomous:

function of C

$$\frac{dU}{dt} = v'q' - rU$$

zonal mean jet equation

excitation spatial covariance

$$\frac{dC}{dt} = AC + CA + Q$$

ensemble mean perturbation covariance equation

- The SSD governs the dynamics of the statistical moment evolution.
- The SSD has the same dynamical restriction as QL dynamics while in addition it has zero fluctuations. This allows exact, analytical, fixed point solution for the statistical state of a turbulent system to be obtained - which is what is meant by solving a turbulence problem.

The test function example

Use the ensemble mean perturbation covariance to obtain the steady response to a test function imposed on the turbulence:

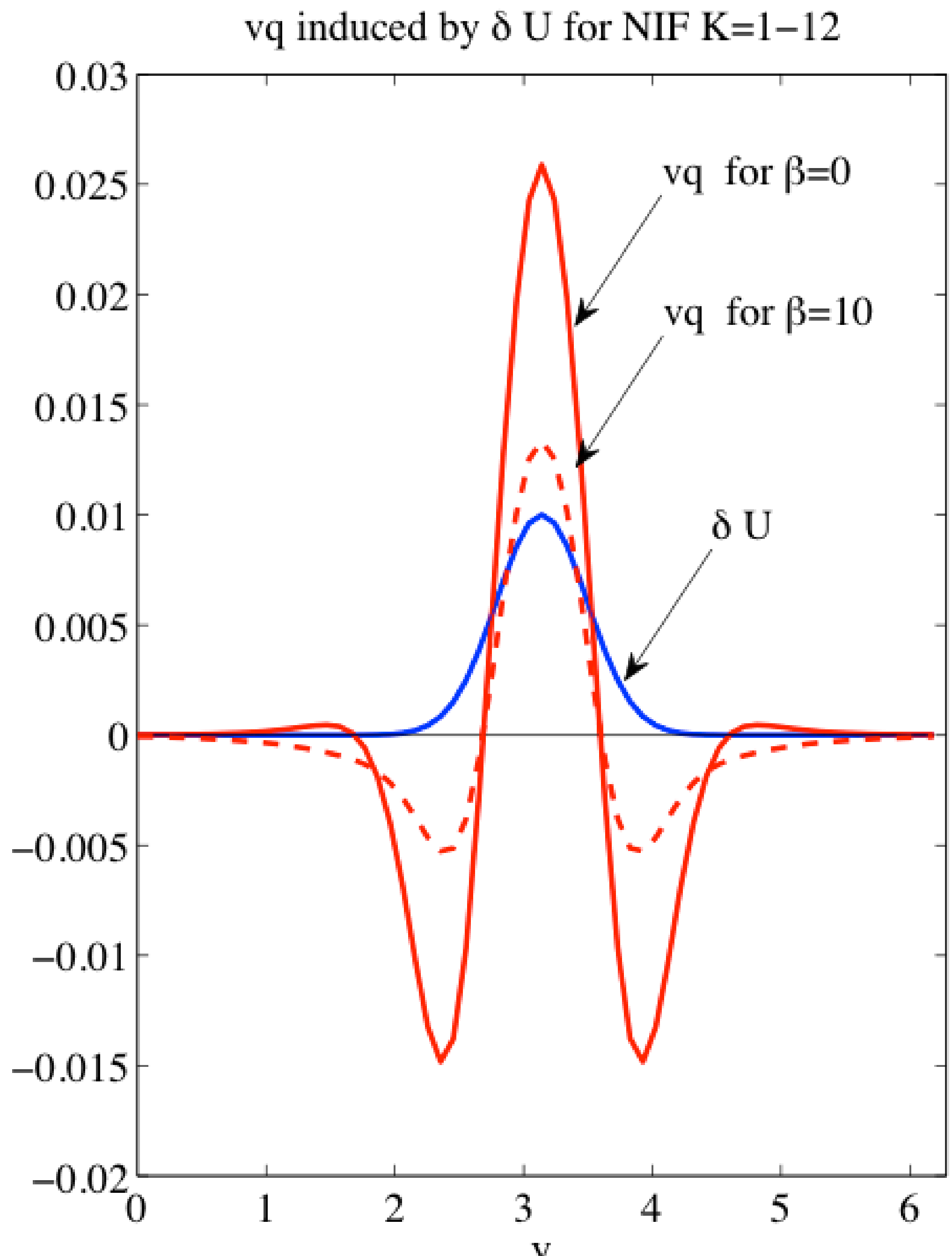
$$\cancel{\frac{dC}{dt}} = A(\delta U)C + CA(\delta U) + Q$$

Spatial covariance of the stochastic excitation maintaining a background of isotropic turbulence

Impose a test function δU in A and solve for the steady Reynolds stress using perturbation covariance C

Structural Instability of a Turbulent State

Consider β-plane turbulence with isotropic stochastic excitation maintaining the turbulence. There are no ***coherent*** Reynolds stresses. However, an imposed test function jet breaks the symmetry of this turbulence resulting in **upgradient** momentum fluxes:



Note:

- Fluxes are upgradient (opposite to eddy diffusion).
- Jet acceleration is maximum for β=0 (opposite to wave radiation and vorticity mixing theories for jet formation).

Jet formation instability is inherent in isotropic turbulence

First form the perturbation SSD equations around an equilibrium turbulent state, (U_e, C_e).

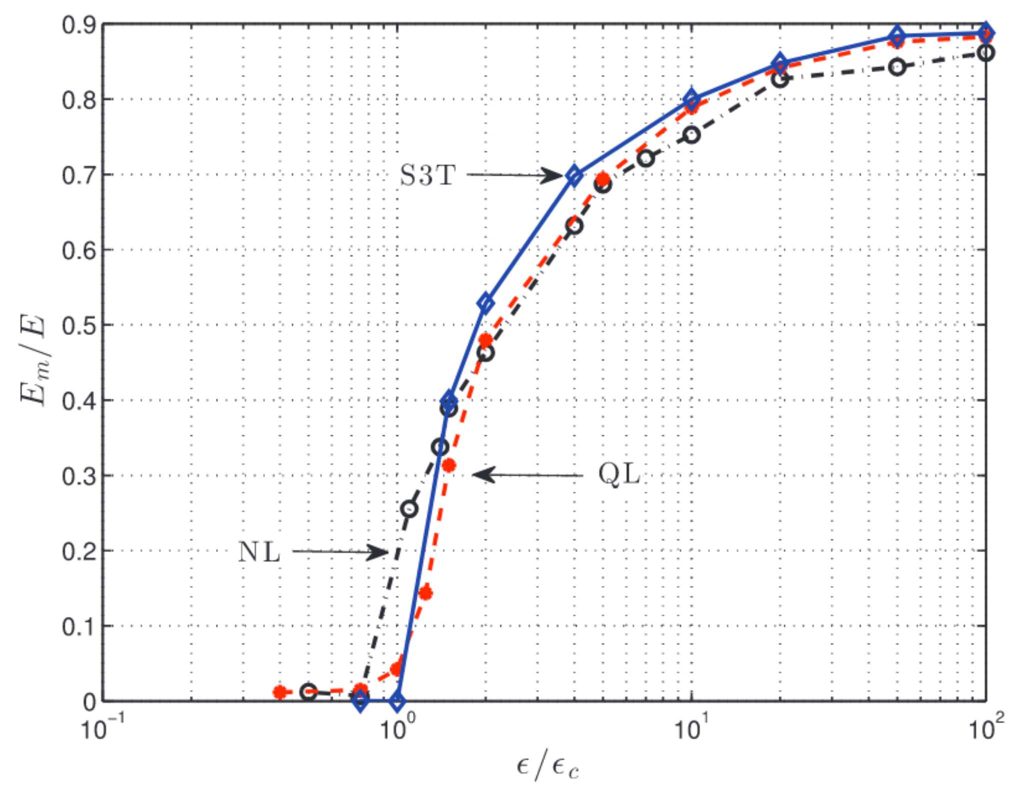
Linear function of δC

$$\frac{d\delta U}{dt} = -r\delta U + (v'q')_{\delta C}$$

$$\frac{d\delta C}{dt} = (\delta A)C_e + C_e(\delta A)^\dagger + A(U_e)\delta C + \delta C A^\dagger(U_e)$$

Then obtain the eigenmodes and growth rate of the jet forming instability $(\delta U_i, \delta C_i)e^{\sigma_i t}$

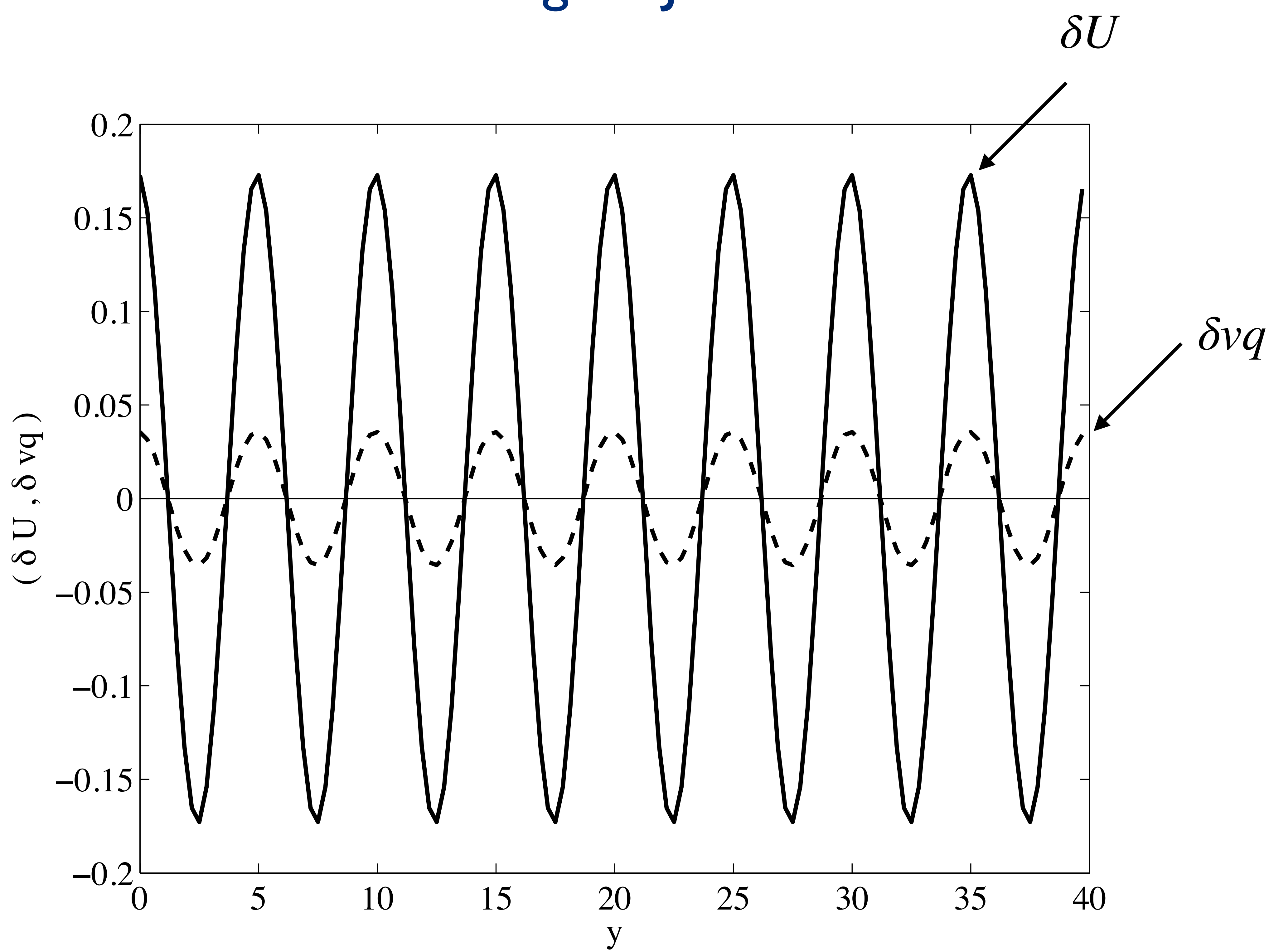
Jet formation instability is inherent to isotropic turbulence



- When the background turbulence is sufficiently strong the coherent jet forming Reynolds stresses exceed jet damping resulting in a linear jet formation instability and eventually emergence of a nonlinearly equilibrated finite amplitude coherent jet in the nonlinear SSD.

- This jet formation instability is distinct from hydrodynamic instability. Jets arise from interaction between the mean flow and the perturbation covariance which produce a structural bifurcation of the statistical state with no counterpart in the 2D N-S equations expressed using state variables.

Unstable Emergent Jet Structure



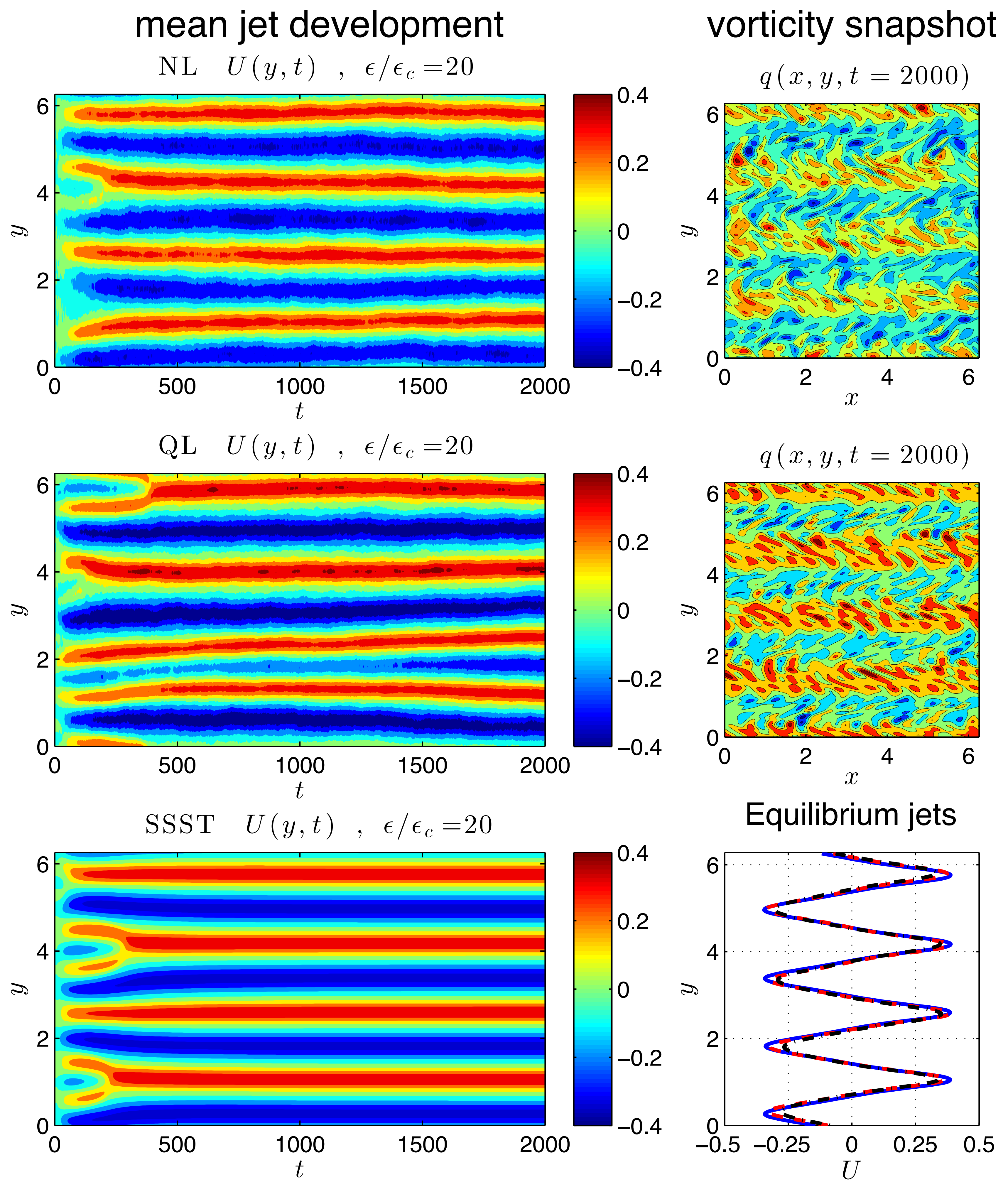
The most unstable eigenfunction at eddy excitation $\Theta = 1.5\ mW/kg$. Shown is the mean velocity component, δU, of the eigenfunction (solid) and the momentum flux divergence from the perturbation covariance, $\delta \boldsymbol{C}$, (dashed).

The same jet formation instability is obtained in NL, (DNS), QL (quasilinear) and SSD dynamics

NL

QL

SSD



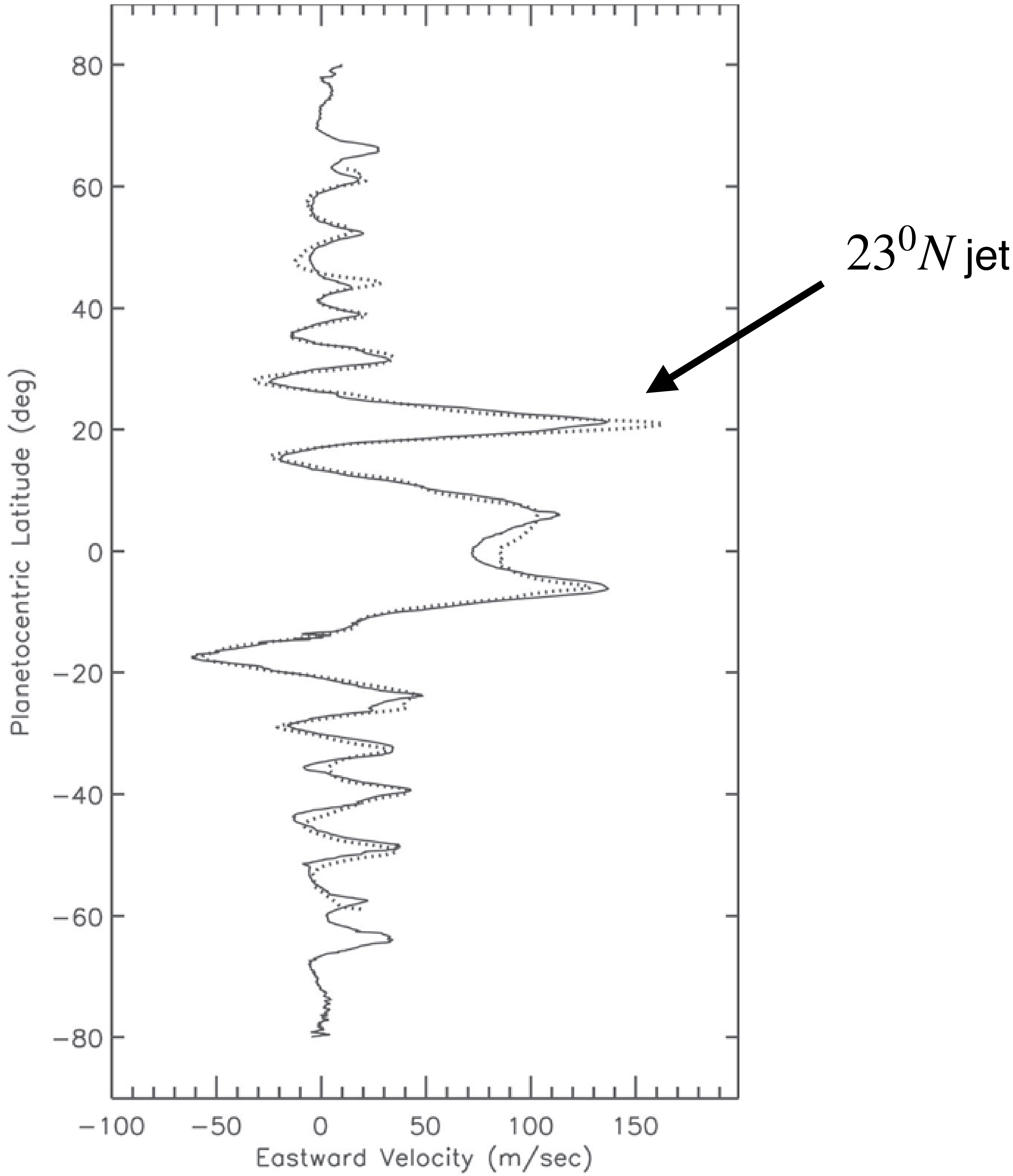
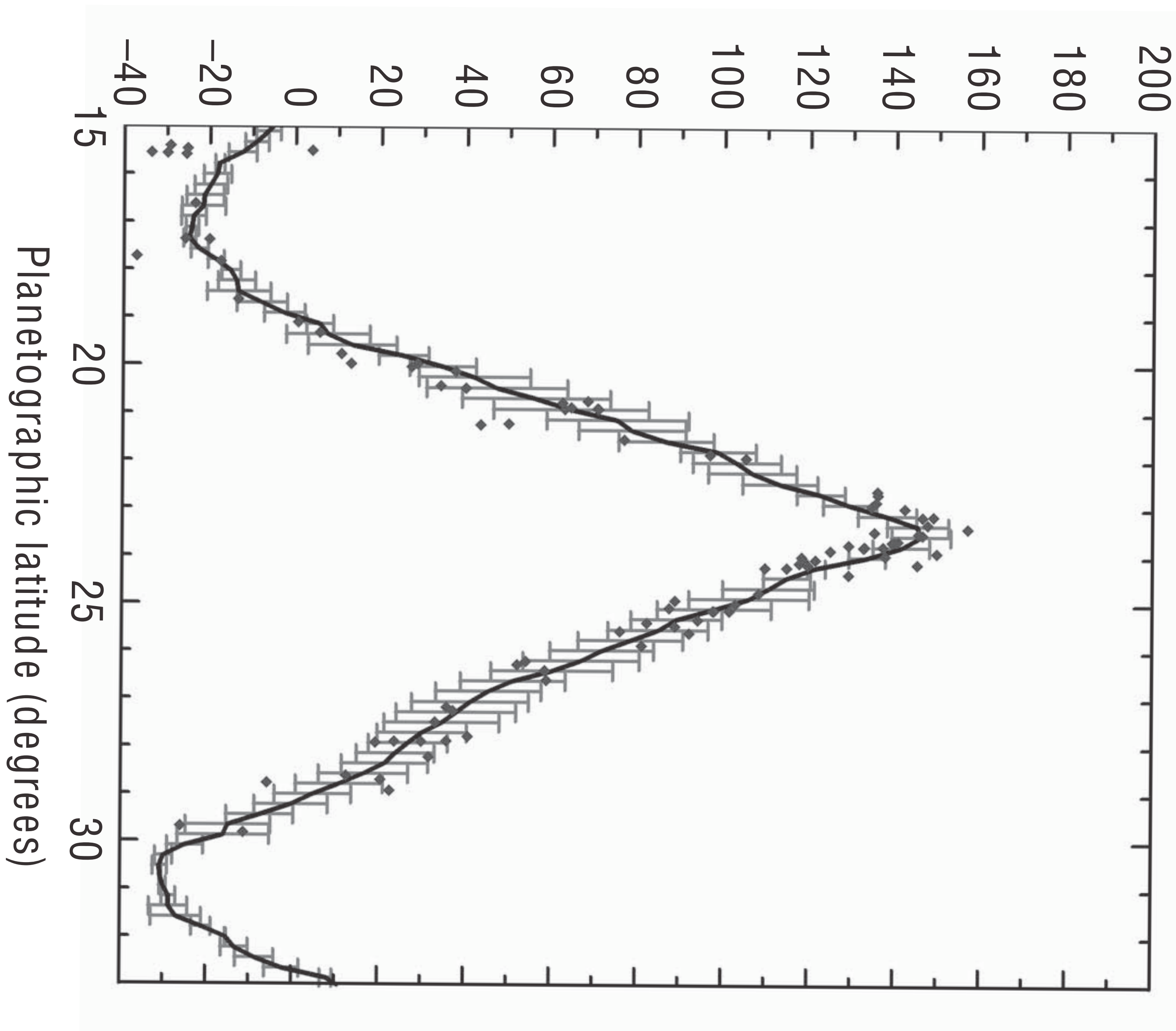


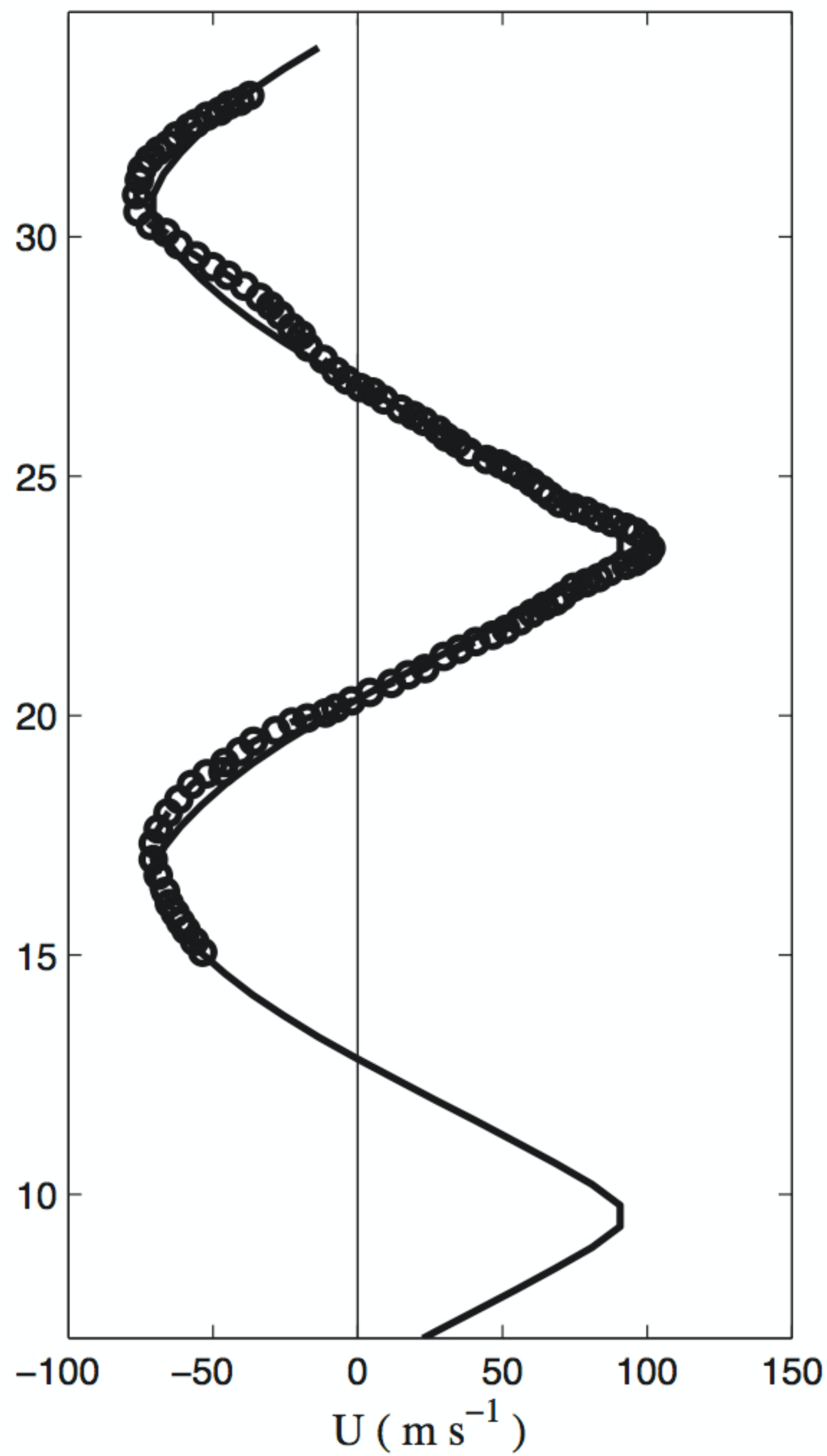
Figure 6.2. Zonal winds vs. latitude in 1979 and 2000. The dashed line is from ***Voyager*** (Limaye 1986) and the solid line is from ***Cassini*** (Porco ***et al.*** 2003).

Jupiter's 23^0 N Jet



Winds from HST images (error bars represent the standard deviation of the measurements)
(Sanchez-Levega etal. (2008))

Jupiter's 23^0 N Jet Compared with SSD Equilibrium



No free parameter

Conclusions

- The Fundamental dynamics of turbulence is revealed only when the dynamics is written using statistical variables (in SSD form).
- Excepting only IHT, turbulence is essentially linear (i.e. quasilinear) as is revealed when the dynamics is written using statistical variables (in SSD form).
- Coherent structures (jets, layers) arise as linear instabilities only when the dynamics is written using statistical variables (in SSD form).
- Nonlinear equilibria proceeding from unstable modes of the SSD comprise analytic solution of their associated turbulence problem.