Bootstrapping string theory

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2310.12322 with L. F. Alday, D. Dorigoni, M. Green and C. Wen

What is non-perturbative string theory?

- String theory is perturbatively defined via the worldsheet.
- Use to compute flat space string S-matrix in small string coupling g_s expansion at finite string length $\ell_s = \sqrt{\alpha'}$.
- Q1: How do we study string theory at **finite** g_s and ℓ_s?
- Q2: How do we study string theory at **finite** curvature and Ramond-Ramond flux (i.e. most realistic compactifications)?
  - Some progress from Green-Schwarz [Green, Schwartz '81] and pure spinor [Berkovitz '00] approaches, but still no systematic answer.

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Answer: Holography (?)

- Non-perturbative string theory in AdS via holography, e.g.:
  - IIB string theory on $AdS_5 \times S^5$ with string length ℓ_s and complex string coupling $\tau_s = \chi + i/g_s$, is dual to [Maldacena '97] :
  - 4d $\mathcal{N} = 4$ $SU(N)$ Super-Yang-Mills (SYM) with $\tau \equiv \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}$.
- From the dictionary $L^4/\ell_s^4 = g_{\text{YM}}^2 N\,, \quad \tau = \tau_s$, we see that $AdS_5 \times S^5$ supergravity describes large N strongly coupled SYM.
  - Use supergravity to compute double trace scaling dimension [D'Hoker, Mathur, Matusis, Rastelli '99] : $\Delta = 4 - 16/N^2 + O(N^{-7/2})$.
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Weak coupling: finite N, small $g_{\rm YM}$

- When $\lambda \equiv g_{\rm YM}^2 N$ is small, can study SYM with Feynman diagrams for any N like any weakly coupled gauge theory.

- E.g. lowest unprotected singlet (the Konishi) has [Velizhanin '09] :

$$\Delta = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} + \frac{\lambda^4\left(-1440\left(\frac{12}{N^2}+1\right)\zeta(5) + 576\zeta(3) - 2496\right)}{65536\pi^8} + O(\lambda^5)$$

  - First non-planar correction only at 4-loops!

- But bulk dual is very stringy in this regime, no gravity approximation, no black holes.

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Planar integrability: finite λ, large N

- Can compute all scaling dimensions for $N \to \infty$ and finite λ from quantum spectral curve [Gromov, Kazakov, Leurent, Volin '14] .
  - Implemented numerically for entire spectrum just recently [Gromov, Hegedus, Julius, Sokolova '23] .
- At small λ matches weak coupling, at large λ single trace operators like Konishi match stringy prediction:

$$\Delta_{\text{Kon}} = 2\lambda^{1/4} - 2 + 2/\lambda^{1/4} + \dots ,$$

- But planar limit only classical string theory (leading g_s), e.g.:
  - Higher traces just trivial products of single traces, e.g. lowest double trace has $\Delta = 2 + 2$, missing $1/N^2$ correction.

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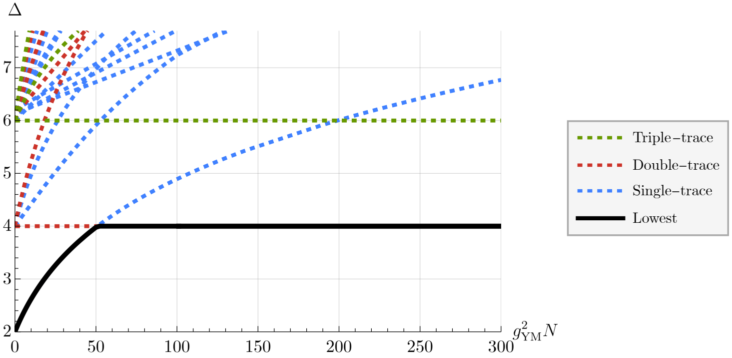
Planar spectrum: limitations



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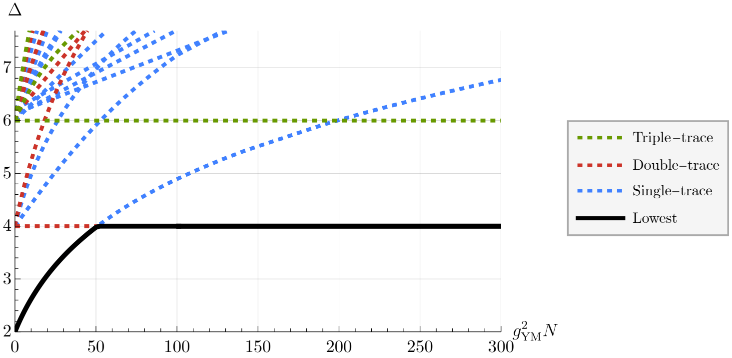

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Combine non-perturbative methods, bootstrap and supersymmetric localization, to study stress tensor correlator for all N and τ.

Outline:

- Basics of stress tensor correlator.
- Non-perturbative constraints at large or finite N.
- Numerical bootstrap bounds
  - Compare to weak and strong coupling perturbative results.
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$\mathcal{N} = 4$ SYM basics

- All $\mathcal{N} = 4$ CFTs have $SU(4)$ R-symmetry, and are conformal manifolds with one complex parameter τ.
  - Defined by values of central charge $c = \dim(G)/4$ and complex τ.
- $\mathcal{N} = 4$ SYM is gauge theory where matter transform in adjoint of gauge group G, which must be compact classical lie group.
  - For this talk, we take $G = SU(N)$, with $c = \frac{N^2-1}{4}$.
- Duality group of $\mathcal{N} = 4$ $SU(N)$ SYM is $SL(2,\mathbb{Z})$.
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Stress tensor correlator

- 4-point function of stress-tensor superprimary S^a with $\mathbf{20}'$ index a:

$$\langle S^a(x_1)S^b(x_2)S^c(x_3)S^d(x_4)\rangle = \frac{G^{abcd}(U,V)}{x_{12}^4 x_{34}^4}\,, \quad U\equiv \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,, V\equiv \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\,.$$

- $\langle SSSS\rangle$ Ward identity has formal solution [Dolan, Osborn '02] :

$$G^{abcd}(U,V) = G^{abcd}(U,V)_{\text{short}} + \Theta^{abcd}(U,V)\mathcal{T}(U,V)\,.$$

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- $\langle SSSS\rangle$ Ward identity has formal solution [Dolan, Osborn '02] :

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Block expansion

- Expand $\mathcal{T}(U, V)$ in even spin ℓ 4d conformal blocks $g_{\Delta,\ell}(U, V)$:

$$\mathcal{T} = U^{-2} \sum_{\ell,\Delta\geq\ell+2} \lambda^2_{\Delta,\ell} g_{\Delta+4,\ell}(U, V) + F^{(0)}_{\text{short}}(U, V) + \frac{1}{c} F^{(1)}_{\text{short}}(U, V)\,.$$

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- Impose that $\langle S^a(x_1)S^b(x_2)S^c(x_3)S^d(x_4)\rangle$ is permutation invariant.
- Fixes large $c \sim N^2$ correlator in terms of finite # of coeffs b_i at each $1/c$ [Heemskerk, Penedones, Polchinski, Sully '09; Alday, Bissi, Lukowski '14] :

$$\mathcal{T} = \frac{\mathcal{T}_R}{c} + b_1 \frac{\mathcal{T}_{R^4}}{c^{7/4}} + \frac{\mathcal{T}_{R|R} + b_2 \mathcal{T}_{R^4}}{c^2} + \frac{b_3 \mathcal{T}^1_{D^4R^4} + b_4 \mathcal{T}^2_{D^4R^4}}{c^{9/4}} + \dots .$$

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- Derivatives of free energy $F(m)$ deformed by hyper mass relate to S^4 integrals of correlator [Binder, SMC, Pufu, Wang '19; SMC, Pufu '20] :

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Mass deformed sphere partition function

- Computed using localization in terms of rank(G) dimensional matrix model integral for gauge group G [Pestun '08] .
- For $SU(N)$ we have explicitly (with $a_{ij} \equiv a_i - a_j$):

$$Z = \int \frac{d^{N-1}a}{N!} \frac{\prod_{i<j} a_{ij}^2 H^2(a_{ij})}{H(m)^{N-1} \prod_{i\neq j} H(a_{ij}+m)} e^{-\frac{8\pi^2}{g_{\text{YM}}^2}\sum_i a_i^2} |Z_{\text{inst}}(m,\tau,a_{ij})|^2 \,.$$

  - $H(z)$ is product of Barnes G-functions.
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- Expand $Z_{\text{inst}}(m, \tau, a_{ij})$ in instanton number k as

$$Z_{\text{inst}}(m, \tau, a_{ij}) = \sum_{k=0}^{\infty} e^{2\pi i k \tau} Z_{\text{inst}}^{(k)}(m, a_{ij}) \,.$$

- Using Z_{inst}, computed $\mathcal{F}_2(\tau)$ and $\mathcal{F}_4(\tau)$ to any order in $1/N$ and finite τ [SMC, Green, Pufu, Wang, Wen '19; Alday; Dorigoni; SMC, Green, Wen '23] :

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Localization inputs for any N

- Eisenstein series diverges for weak coupling due to $1/g_{\text{YM}}$ terms:

$$E(\tfrac{3}{2};\tau) = \frac{16\pi^{3/2}\zeta(3)}{g_{\text{YM}}^3} + \frac{1}{3}\pi^{3/2}g_{\text{YM}} + \sum_{k=1}^{\infty} \frac{32\cos(\theta)\pi^{3/2}k\sigma_{-2}(k)K_1\left(\frac{8k\pi^2}{g_{\text{YM}}^2}\right)}{g_{\text{YM}}}$$

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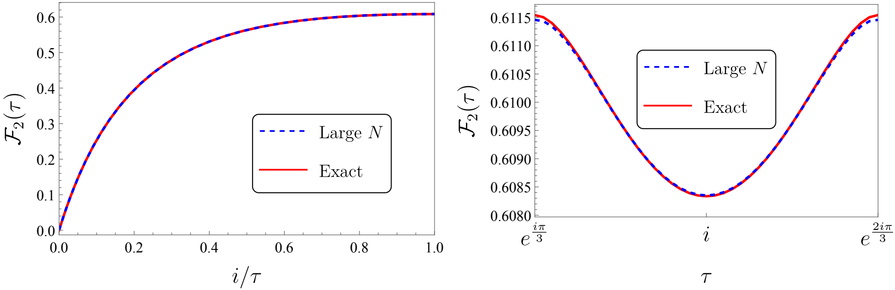

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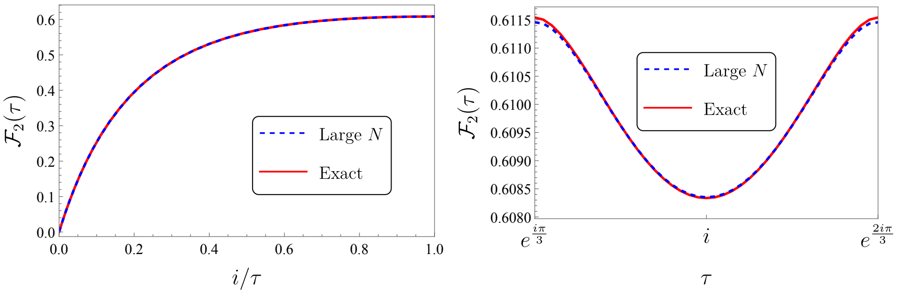

Numerical bootstrap+localization

- Combine all non-perturbative constraints (unitarity, crossing, localization) to bootstrap CFT data [SMC, Dempsey, Pufu '21] .
  - Input N via c in short contributions.
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  - Impose crossing and localization inputs as linear constraints, bounds improve monotonically with truncation size Λ of infinite crossing constraints.
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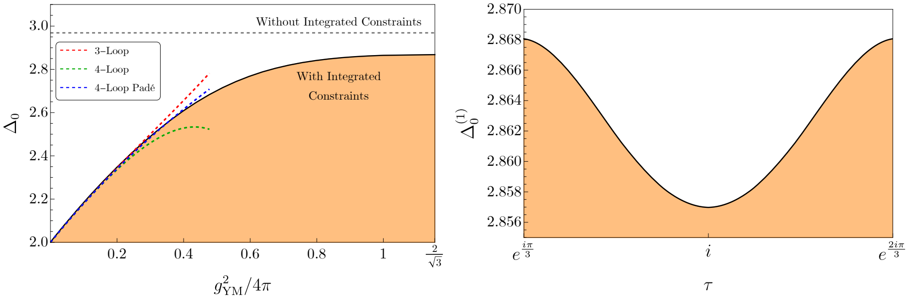

- All these bounds with truncation $\Lambda = 39$, converged for $SU(2)$.
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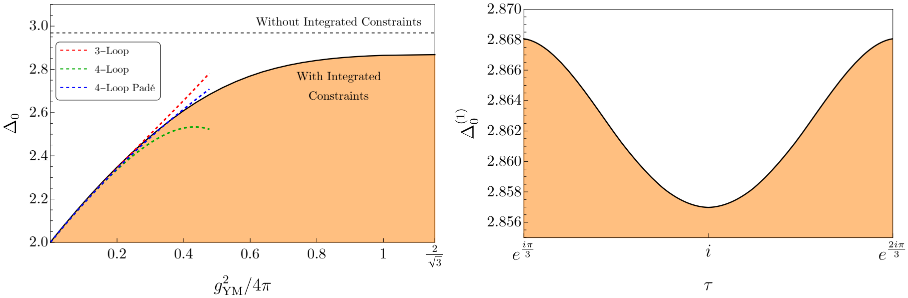

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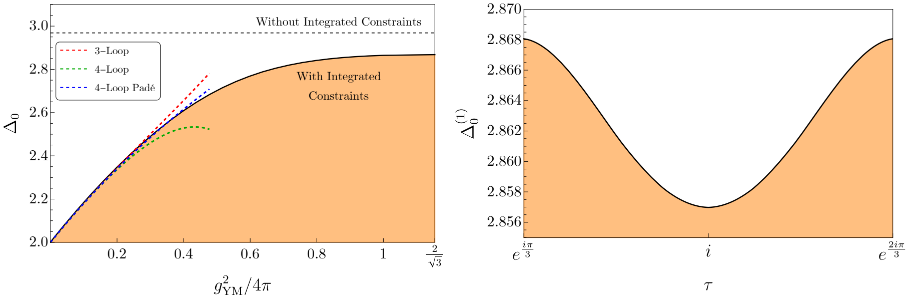

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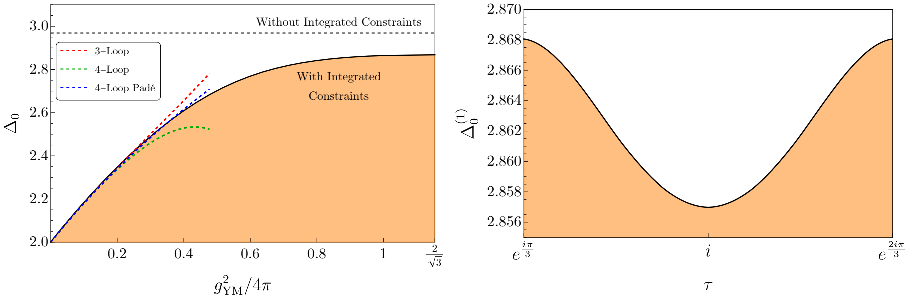

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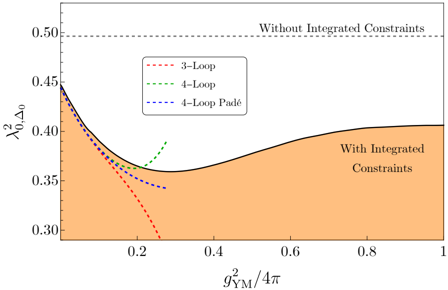

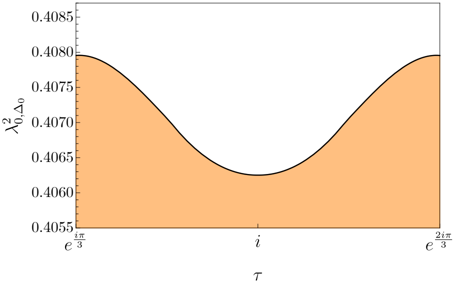

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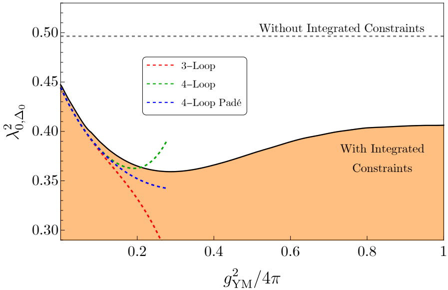

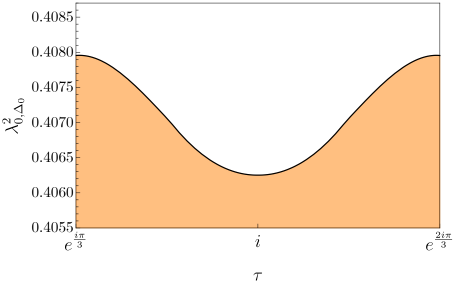

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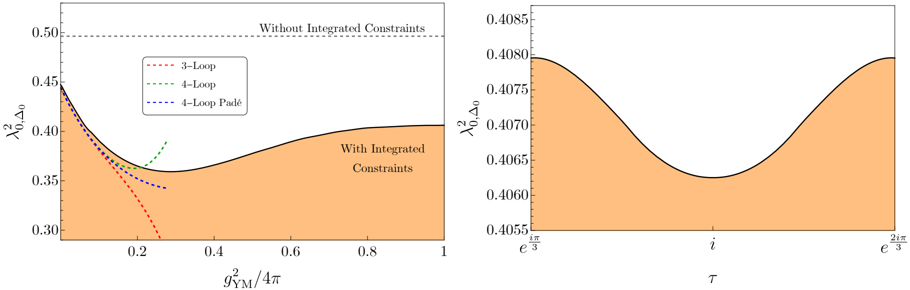

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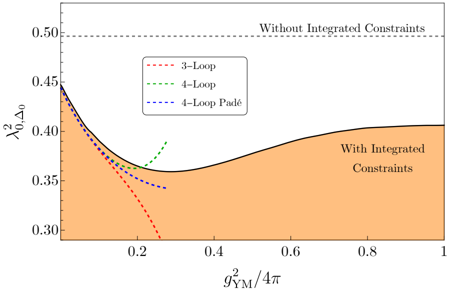

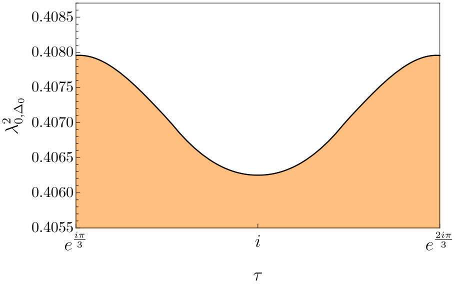

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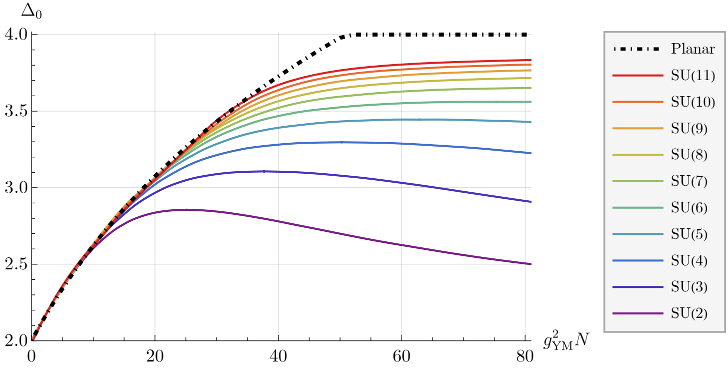

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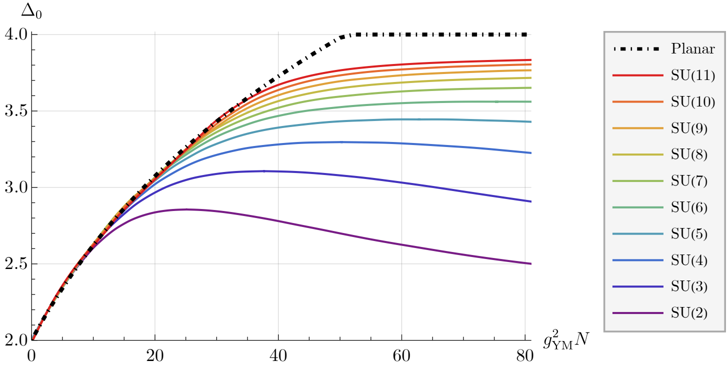

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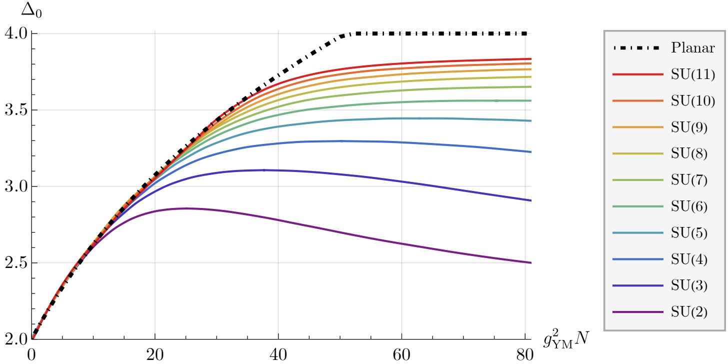

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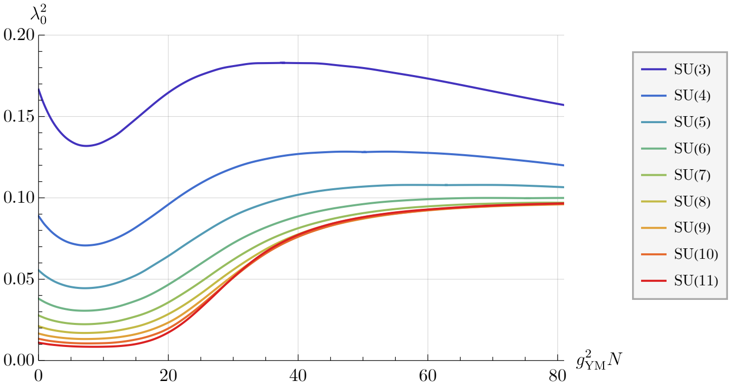

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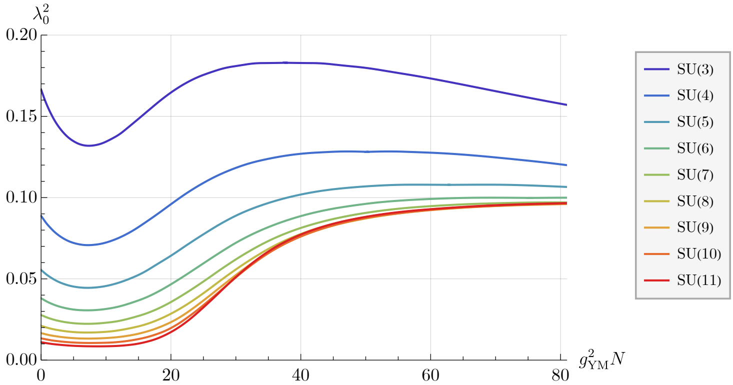
- No planar integrability results to compare to now.

Bounds on lowest λ^2 for various N



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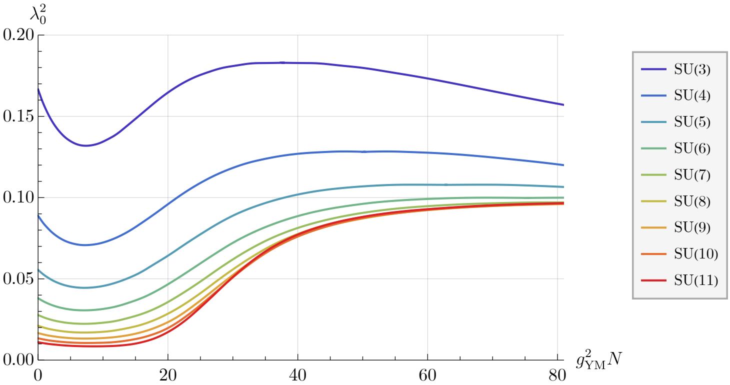

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Analytic bootstrap+localization

- Recall that analytic bootstrap (i.e. crossing, pole structure of Witten diagrams, and flat space limit) fixes correlator to:

$$\mathcal{T} = \frac{\mathcal{T}_R}{c} + b_1 \frac{\mathcal{T}_{R^4}}{c^{7/4}} + \frac{\mathcal{T}_{R|R} + b_2 \mathcal{T}_{R^4}}{c^2} + \frac{b_3 \mathcal{T}^1_{D^4 R^4} + b_4 \mathcal{T}^2_{D^4 R^4}}{c^{9/4}} + \dots .$$

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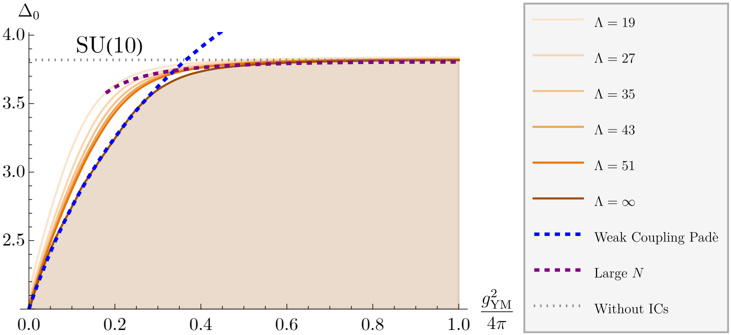

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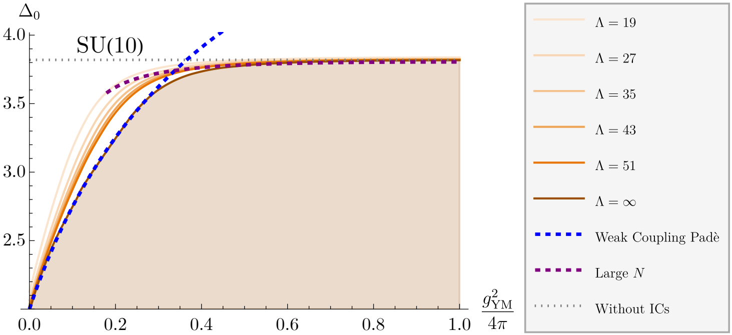

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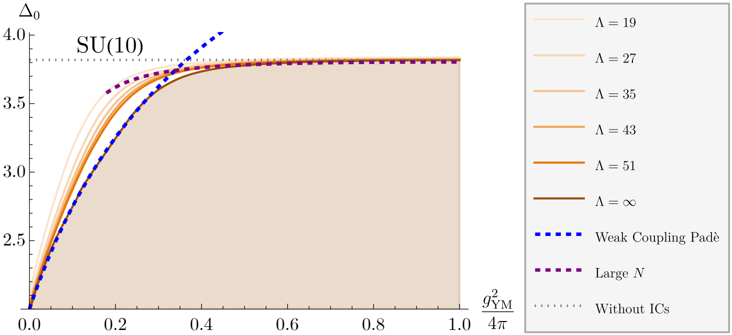

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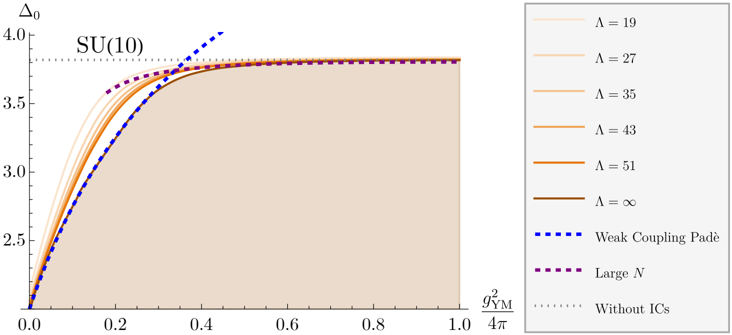

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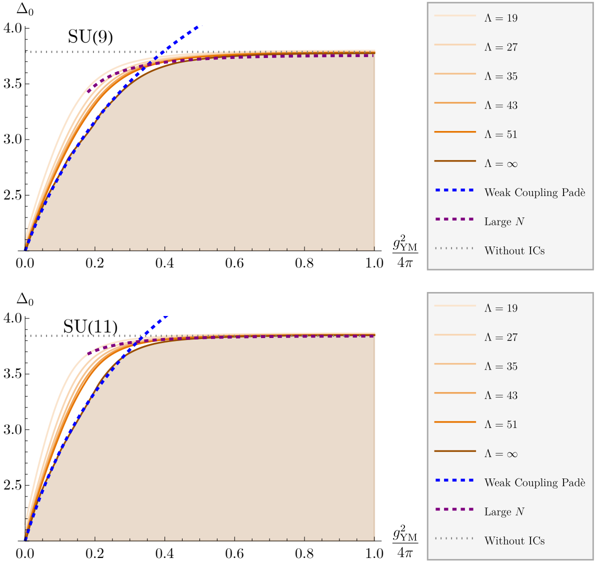

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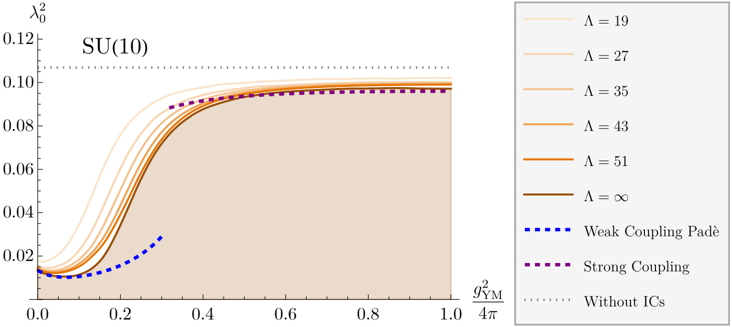

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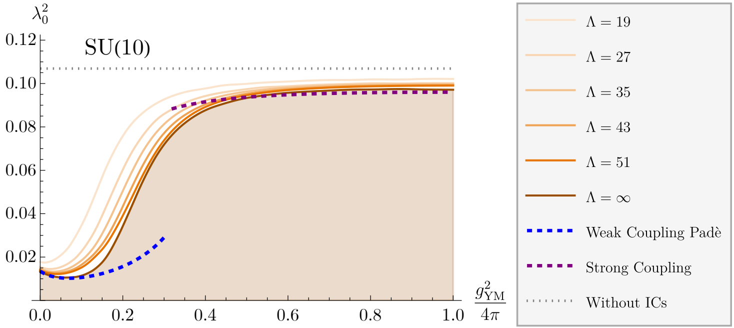

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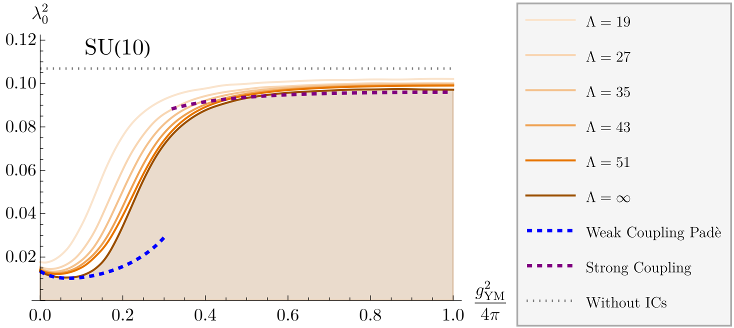

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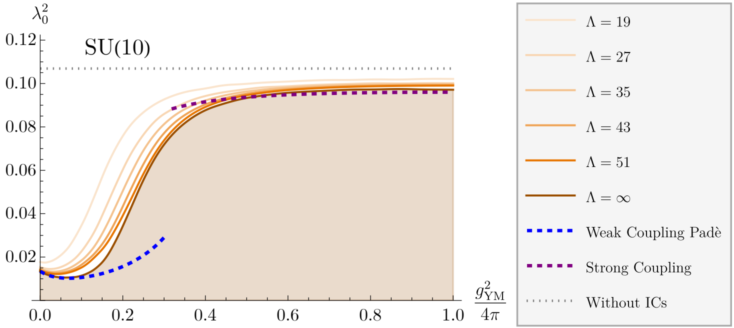

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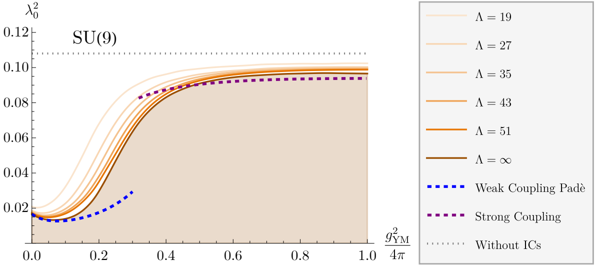

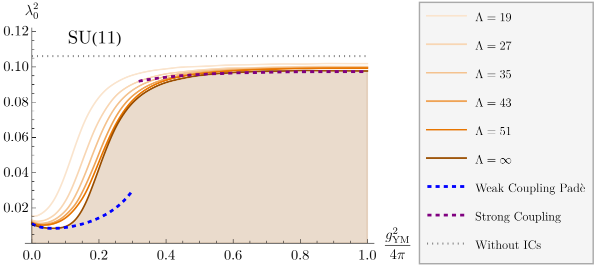

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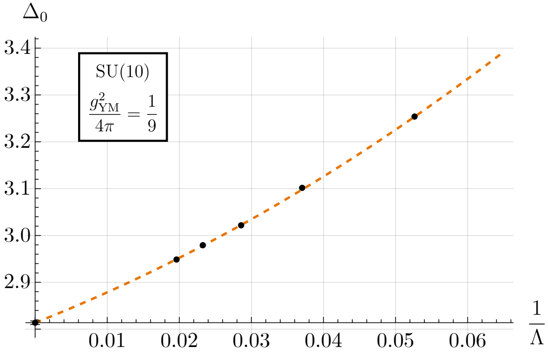

- We use simple polynomial ansatz for extrapolation: $\Delta = \Delta_0 + \Delta_1/\Lambda + \Delta_2/\Lambda^2$.
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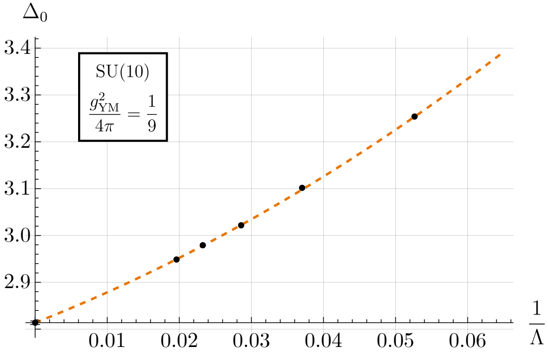

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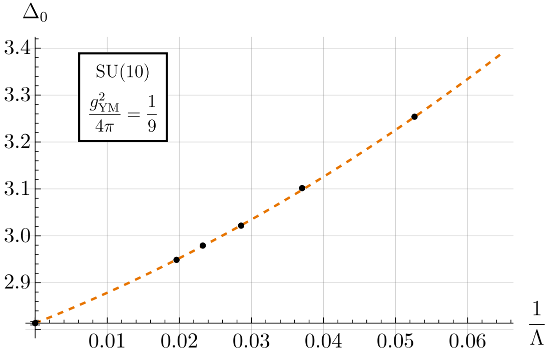

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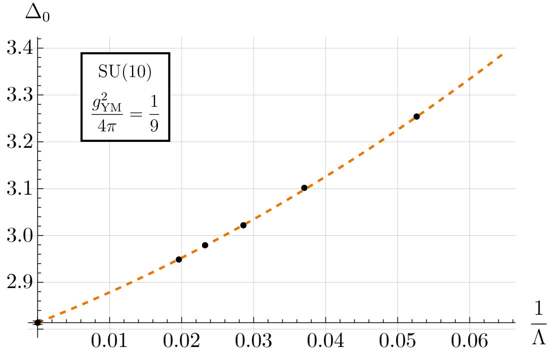

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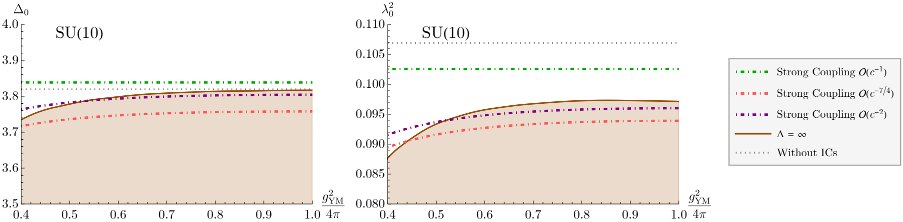

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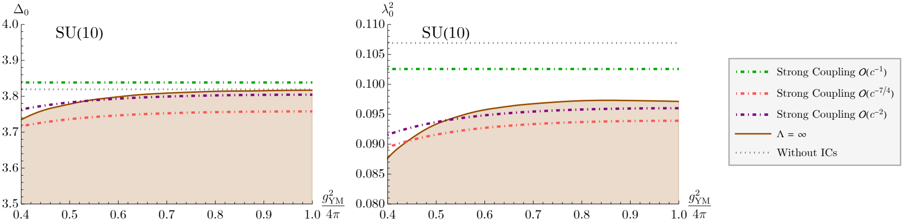

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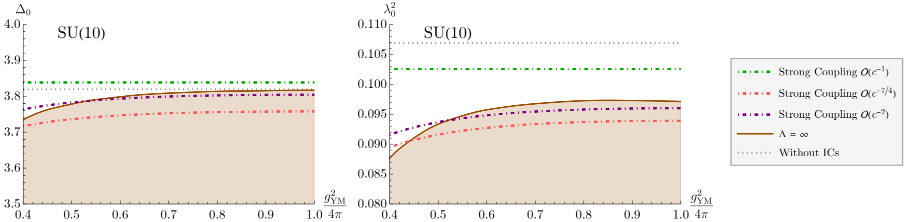

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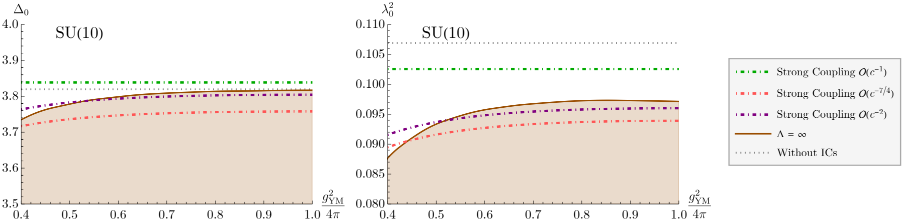

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Near future directions

- More accurate bounds, sensitive to higher twist or spin operators.
  - In bootstrap without localization, higher spins usually easier to access than higher twists, but with localization both equally hard.
- Get greater accuracy from imposing more localization constraints (e.g. from the squashed sphere), or from mixing with lowest dimension long operator (which is also relevant).
- If we are sensitive to second lowest twist, then impose ≤ 2 relevant operators to get islands for each τ, rigorously solve SYM!
- Combine localization + bootstrap to numerically solve ANY 3d $\mathcal{N} = 2$, 4d $\mathcal{N} = 2$, or 5d $\mathcal{N} = 1$ Lagrangian CFT, e.g.:
  - 4d $\mathcal{N} = 2$ dual to open strings [SMC '22; Behan, SMC, Ferrero '23] .
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See you in Kyoto!

- Bootstrap, Localization, and Holography, May 20-24
- Some funding for students, poster session!