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STABILITY CONDITIONS

$$\mathcal{D} \rightsquigarrow \mathrm{Stab}(\mathcal{D})$$

triangulated category — complex manifold

Think of $\mathcal{D}$ as the category of branes in a topological twisting of some $N=2$ SUSY field theory.

eg $\mathcal{D} = D^b \mathrm{Coh}(X)$, $\mathcal{D} = D^b \mathrm{Fuk}(Y)$

The SUSY field theory has deformations which don't change the topological twist $\mathcal{D}$.

Main Idea (Douglas): These extra parameters determine a subcategory of stable branes

$$\mathcal{P} = \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset \mathcal{D}$$

Aim: Axiomatise properties of $\mathcal{P} \subset \mathcal{D}$ and obtain a space $\mathrm{Stab}(\mathcal{D})$ as the set of all such subcategories.

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Example (heuristic!)

Take $\mathcal{D} = D^b \mathrm{Fuk}(Y)$ for some symplectic manifold Y.

We expect

$$\mathcal{M}_{\mathbb{C}}(Y) \hookrightarrow \mathrm{Stab}(\mathcal{D})$$

moduli of complex structures on Y

Given a complex structure can define special Lagrangians

$$\Omega|_L = e^{i\phi}\, \mathrm{vol}_L$$

This defines stable branes $\bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset \mathcal{D}$

Note that each $L \in \mathcal{D}$ has a "central charge"

$$Z(L) = \int_L \Omega \in \mathbb{C}$$

and $L \in \mathcal{P}(\phi) \Rightarrow Z(L) \in \mathbb{R}_{>0}\, e^{i\pi\phi}$

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Recall : a triangulated category has a shift functor $[1] : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ and distinguished triangles

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & A[1] \arrow[r, "{f[1]}"] & B[1] \arrow[r, "{g[1]}"] & C[1] \arrow[r] & \cdots \end{tikzcd}$$

satisfying some axioms

eg

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd} \iff \begin{tikzcd} B \arrow[rr, "g"] & & C \arrow[dl, "h"] \\ & {A[1]} \arrow[ul, "{f[1]}"] & \end{tikzcd}$$

distinguished distinguished

The Grothendieck group $K(\mathcal{D})$ is the free abelian group on iso classes of objects of $\mathcal{D}$ modulo relations

$$[B] = [A] + [C] \quad \text{if} \quad \begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul] & \end{tikzcd}$$

distinguished

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Definition 1 *A stability condition on $\mathcal{D}$ consists of a full additive subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, and a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$, such that*

(a) if $E \in \mathcal{P}(\phi)$ then $Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi)$,

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all $\phi \in \mathbb{R}$,

(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then

$$\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0,$$

(d) for each $0 \neq E \in \mathcal{D}$ there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{tikzcd} 0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\ & A_1 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] & \end{tikzcd}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j.

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Given a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ and an object $0 \neq E \in \mathcal{D}$, the filtrations of axiom (d) are unique up to isomorphism. Thus we can define

$$\phi_\sigma^+(E) = \phi_1, \qquad \phi_\sigma^-(E) = \phi_n,$$

$$m_\sigma(E) = \sum_{i=1}^{n} |Z(A_i)| \in \mathbb{R}_{>0}.$$

The expression

$$\sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi_\sigma^\pm(E) - \phi_\tau^\pm(E)|, |\log \frac{m_\sigma(E)}{m_\tau(E)}| \right\}$$

defines a metric $d(\sigma, \tau) \in [0, \infty]$ on the set of all stability conditions on $\mathcal{D}$.

Write $\mathrm{Stab}(\mathcal{D})$ for the set of "locally-finite" stability conditions on $\mathcal{D}$ with the topology induced by this metric. There is a continuous map

$$\mathcal{Z} \colon \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

sending a stability condition $\sigma = (Z, \mathcal{P})$ to its central charge Z.

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Theorem 1 *For each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ there is a linear subspace*

$$V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

with a well-defined linear topology such that the map $\mathcal{Z}$ induces a local homeomorphism $\mathcal{Z} \colon \Sigma \to V(\Sigma)$ onto an open subset.

It follows that $\mathrm{Stab}(\mathcal{D})$ is a (possibly infinite-dimensional) complex manifold.

If X is a smooth projective complex variety, set

$$\mathcal{D} = \mathcal{D}^b \,\mathrm{Coh}(X).$$

Let $\mathrm{Stab}(X)$ be the subset of $\mathrm{Stab}(\mathcal{D})$ for which $Z \colon K(\mathcal{D}) \to \mathbb{C}$ factors via the Chern character

$$\mathrm{ch} \colon K(\mathcal{D}) \longrightarrow H^*(X, \mathbb{Q}).$$

Then $\mathrm{Stab}(X)$ is a finite-dimensional complex manifold.