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STABILITY CONDITIONS

$$\begin{array}{ccc}
 \mathcal{D} & \rightsquigarrow & \text{Stab}(\mathcal{D}) \\
 \text{triangulated} & & \text{complex} \\
 \text{category} & & \text{manifold}
 \end{array}$$

Think of  $\mathcal{D}$  as the category of branes in a topological twisting of some  $N=2$  SUSY field theory.

$$\text{eg } \mathcal{D} = \mathcal{D}^b \text{Coh}(X), \quad \mathcal{D} = \mathcal{D}^b \text{Fuk}(Y)$$

The SUSY field theory has deformations which don't change the topological twist  $\mathcal{D}$ .

Main Idea (Douglas): These extra parameters determine a subcategory of stable branes

$$\mathcal{P} = \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset \mathcal{D}$$

Aim: Axiomatise properties of  $\mathcal{P} \subset \mathcal{D}$  and obtain a space  $\text{Stab}(\mathcal{D})$  as the set of all such subcategories.

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Example (heuristic!)

Take  $\mathcal{D} = \mathcal{D}^b \text{Fuk}(Y)$  for some symplectic manifold  $Y$ .

We expect

$$\mathcal{M}_{\mathbb{C}}(Y) \longleftrightarrow \text{Stab}(\mathcal{D})$$

moduli of complex structures on  $Y$

Given a complex structure can define special Lagrangians

$$\Omega|_L = e^{i\phi} \text{vol}_L$$

This defines stable branes  $\bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset \mathcal{D}$

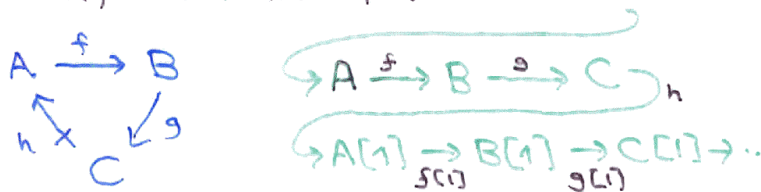
Note that each  $L \in \mathcal{D}$  has a "central charge"

$$Z(L) = \int_L \omega \in \mathbb{C}$$

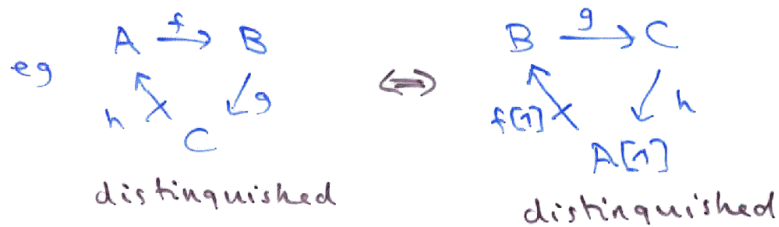
and  $L \in \mathcal{P}(\phi) \Rightarrow Z(L) \in \mathbb{R}_{>0} e^{i\pi\phi}$

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Recall: a triangulated category has a shift functor  $[1] : \mathcal{D} \xrightarrow{\cong} \mathcal{D}$  and distinguished triangles



satisfying some axioms



The Grothendieck group  $K(\mathcal{D})$  is the free abelian group on IM classes of objects of  $\mathcal{D}$  modulo relations

$[B] = [A] + [C]$  if 
$$\begin{array}{ccc}
 A & \rightarrow & B \\
 \uparrow & & \downarrow \\
 & C &
 \end{array}$$
 distinguished

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**Definition 1** A stability condition on  $\mathcal{D}$  consists of a full additive subcategory  $\mathcal{P}(\phi) \subset \mathcal{D}$  for each  $\phi \in \mathbb{R}$ , and a group homomorphism  $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$ , such that

(a) if  $E \in \mathcal{P}(\phi)$  then  $Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi)$ ,

(b)  $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$  for all  $\phi \in \mathbb{R}$ ,

(c) if  $\phi_1 > \phi_2$  and  $A_j \in \mathcal{P}(\phi_j)$  then

$\text{Hom}_{\mathcal{D}}(A_1, A_2) = 0,$

(d) for each  $0 \neq E \in \mathcal{D}$  there is a finite sequence of real numbers

$\phi_1 > \phi_2 > \dots > \phi_n$

and a collection of triangles



with  $A_j \in \mathcal{P}(\phi_j)$  for all  $j$ .

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Given a stability condition  $\sigma = (Z, \mathcal{P})$  on  $\mathcal{D}$  and an object  $0 \neq E \in \mathcal{D}$ , the filtrations of axiom (d) are unique up to isomorphism. Thus we can define

$$\phi_{\sigma}^{+}(E) = \phi_1, \quad \phi_{\sigma}^{-}(E) = \phi_n,$$

$$m_{\sigma}(E) = \sum_{i=1}^n |Z(A_i)| \in \mathbb{R}_{>0}.$$

The expression

$$\sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi_{\sigma}^{\pm}(E) - \phi_{\tau}^{\pm}(E)|, \left| \log \frac{m_{\sigma}(E)}{m_{\tau}(E)} \right| \right\}$$

defines a metric  $d(\sigma, \tau) \in [0, \infty]$  on the set of all stability conditions on  $\mathcal{D}$ .

Write  $\text{Stab}(\mathcal{D})$  for the set of “locally-finite” stability conditions on  $\mathcal{D}$  with the topology induced by this metric. There is a continuous map

$$\mathcal{Z}: \text{Stab}(\mathcal{D}) \longrightarrow \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

sending a stability condition  $\sigma = (Z, \mathcal{P})$  to its central charge  $Z$ .

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**Theorem 1** *For each connected component  $\Sigma \subset \text{Stab}(\mathcal{D})$  there is a linear subspace*

$$V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

*with a well-defined linear topology such that the map  $\mathcal{Z}$  induces a local homeomorphism  $\mathcal{Z}: \Sigma \rightarrow V(\Sigma)$  onto an open subset.*

It follows that  $\text{Stab}(\mathcal{D})$  is a (possibly infinite-dimensional) complex manifold.

If  $X$  is a smooth projective complex variety, set

$$\mathcal{D} = \mathcal{D}^b \text{Coh}(X).$$

Let  $\text{Stab}(X)$  be the subset of  $\text{Stab}(\mathcal{D})$  for which  $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$  factors via the Chern character

$$\text{ch}: K(\mathcal{D}) \longrightarrow H^*(X, \mathbb{Q}).$$

Then  $\text{Stab}(X)$  is a finite-dimensional complex manifold.