

L0

# On a possibility of refining the topological vertex

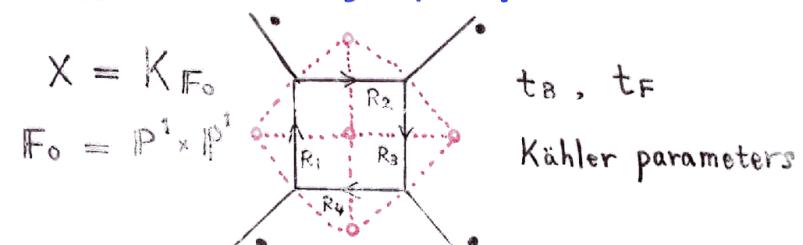
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based on a work w/ H. AWATA

hep-th / 0502061.

- Reports on some "EXPERIMENTS"  
( still incomplete  
for a final answer )

The topological vertex  $C_{R_1 R_2 R_3}(q)$ <sup>11</sup>  
gives all genus (A-type) topological  
string amplitude  $Z_A^X(q; t_i)$  on  
local toric CY<sub>3</sub> X.



AKMV hep-th / 0305132

$$Z_A^{K_{F_0}}(q; t_B, t_F) = \sum_{R_1 \dots R_4} W_{R_1 R_2} W_{R_2 R_3} W_{R_3 R_4} \\ \times W_{R_4 R_1} e^{-t_B(l_{R_2} + l_{R_4}) - t_F(l_{R_1} + l_{R_3})}$$

$$W_{RS}(q) = e^{\frac{ks}{2}} C_{\cdot RS} \\ = \lim_{N \rightarrow \infty} q^{\frac{-N}{2}(l_R + l_S)} \frac{S_{RS}(q, N)}{S_{..}(q, N)}$$

$$q := \exp(i g_s) = \exp\left(\frac{2\pi i}{N+k}\right)$$

$Z_A^X(q; t_i)$  is connected with  
(local)  
the enumerative issues on  $CY_3$ .

On the other hand we have the partition  
function of instanton counting  
in 4D (or 5D) Yang-Mills theory.

N. Nekrasov hep-th/0206161

$$\begin{aligned} Z_{\text{inst}}^{\text{SU}(N)}(q_1, q_2; \vec{a}, \Lambda) &= \sum_{\vec{Y}} \frac{\Lambda^{|\gamma_1| + |\gamma_2| + \dots + |\gamma_N|}}{\prod_{\alpha, \beta=1}^N N_{\alpha, \beta}(\varepsilon_1, \varepsilon_2; \vec{a})} \end{aligned}$$

a sum over (isolated) fixed points of toric  
action

$|\gamma_1| + |\gamma_2| + \dots + |\gamma_N| = k$  : instanton number

$$(e^{\varepsilon_1}, e^{\varepsilon_2}) : T^2 \hookrightarrow \mathbb{C}^2 \simeq \mathbb{R}^4$$

$$e^{\vec{a}} : T^{N-1} \hookrightarrow \text{SU}(N), \sum_{i=1}^N a_i = 0$$

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When  $\varepsilon_1 + \varepsilon_2 = 0$ ,

$Z_{\text{inst}}^{\text{SU}(N)}$  (5D version) is nothing but

$$Z_A^X \text{ for } X = \text{ALE fibration of type } A_{N-1} \text{ over } \mathbb{P}^{1, -1}$$

Iqbal - Kashini - Poor hep-th/0212279, 0306032

Later the proof was completed and generalized in

Hollowood, Iqbal, Vafa hep-th/0310272

T. Eguchi and H. K. hep-th/0310235  
Jian Zhou math.AG/0311237.

Identification of the parameters  
(pure  $SU(2)$  for simplicity)

$$q_F := e^{i\beta_S} = e^{-2\pi i \beta} (\beta = \varepsilon_1)$$

$$Q_F := e^{-t_F} = e^{-4\pi i \beta}$$

$$Q_B := e^{-t_B} = (\beta \Lambda)^4 \times Q_F$$

$\beta$ : the radius of  $S^1$

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A natural question is 14  
if we have a similar story  
when  $\varepsilon_1 + \varepsilon_2 \neq 0$

At first sight this does not look promising, since  $\varepsilon_1 + \varepsilon_2 \neq 0$  breaks  $N=2$  SUSY.

However, there are some supporting evidences.

1.  $SU(2)_L \times SU(2)_R$  structure of  $Z_{SU(2)}^{\text{inst}}(\varepsilon_1, \varepsilon_2; a, \Lambda)$   
(to be shown in a moment)

2. <sup>3</sup>Categorifications of link invariants  
(Khovanov-Rozansky homology)

[Ref] Gukov-Schwarz-Vafa  
hep-th/0412243.

If we can work out the problem, 15  
it would give us a refinement  
of the topological vertex  $C_{R_1 R_2 R_3}(q, t)$ .

$$C_{R_1 R_2 R_3}(q) = q^{-\frac{1}{2} k_{R_3}} S_{R_2}(q^{-p}) \\ \times \sum_q S_{R_1/q}(q^{-\mu_{R_2}^t - p}) S_{R_3/q}(q^{-\mu_{R_3}^t - p})$$

Okounkov-Reshetikhin-Vafa hep-th/0309208.

Our attempt is just to promote the (skew) Schur functions  $S_{R/q}(x)$  to the (skew) Macdonald functions  $P_{R/q}(x; q, t)$  and choose a specialization appropriately. (A motivation for it will be given later by considering the case of  $U(1)$  theory.)

This rather naive attempt does work  
for obtaining a generating function  
of equivariant  $X_g$  or Elliptic genera  
of  $\text{Hilb}^n \mathbb{C}^2$ ; the partition  
function of instanton counting for  
 $U(1)$  theory on  $\mathbb{R}^4 \times S^2$  or  $\mathbb{R}^4 \times T^2$ .

But ----,

1.  $C_{R_1 R_2 R_3}(g, t)$  loses  
the cyclic symmetry among  $R_i$ 's
2. We are not able to reproduce  
 $Z_{\text{inst}}^{\text{SU}(2)}$  completely at the moment

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There is another approach

to the problem. from Donaldson

- Thomas theory or 6D gauge theory.

$$C_{R_1 R_2 R_3}(g) \rightsquigarrow C_{R_1 R_2 R_3}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \text{ for CY}$$

Nekrasov-Okounkov : Equivariant Vertex  
(to appear soon ?)

H. Nakajima (private communication)

A.  $SU(2)_L \times SU(2)_R$  spin  
decomposition of  $F = \log Z_{\text{inst}}^{\text{SU}(2)}$

B. Generating function of  
 $X_g(\text{Hilb}^n(\mathbb{C}^2))$  and  $\text{Ell}(\text{Hilb}^n(\mathbb{C}^2))$   
from  $C_{R_1 R_2 R_3}(g, t)$

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L8

$$A. \quad SU(2)_L \times SU(2)_R$$

We expect the following structure  
of the free energy  $F$   
from the argument of Gopakumar-Vafa type

$$F = \sum_{\beta \in H_2(x, z)} \sum_{n=1}^{\infty} \sum_{(j_L, j_R)} Q_\beta^n \cdot N_\beta^{(j_L, j_R)} \\ \times \frac{((gt)^{-n-j_L} + \dots + (gt)^{n-j_L}) \left( \left(\frac{t}{z}\right)^{-n-j_R} + \dots + \left(\frac{t}{z}\right)^{n-j_R} \right)}{n \cdot (g^{j_L} - g^{-j_L}) \cdot (t^{n/2} \pm t^{-n/2})}$$

$$Q_\beta := e^{-S_\beta w}$$

$N_\beta^{(j_L, j_R)}$  : the multiplicity of "BPS" states  
with spins  $(j_L, j_R)$  and charge  $\beta$

$$g := e^{\varepsilon_1} \quad t := e^{-\varepsilon_2}$$

$$\begin{aligned} S_1 &= F_{12} & \varepsilon_2 &= F_{34} \\ &= [D_1, D_2] & &= [D_3, D_4] \end{aligned}$$

L1

For  $SU(2)$  Yang-Mills theory.

$$X = K F_0 \Rightarrow \beta = k B + n F$$

↑  
instanton number

Compute  $F_{k=1}, F_{k=2}, \dots$

$$\text{from } Z_{\text{inst}}^{\text{SU}(2)} = \sum_{k=0}^{\infty} \Lambda^k Z_k.$$

We find the expected structure  
up to  $k=3$ .

modulo the overall factor of  $(\frac{g}{t})^k$   
(cf: Def of 5D partition fn by Nakajima)

$$k=1 \quad N_{B+nF}^{(j_L, j_R)} = \delta_{j_L, 0} \delta_{j_R, n+\frac{1}{2}}$$

$$\begin{aligned} k=2 \quad &\oplus N_{2B+nF}^{(j_L, j_R)} (j_L, j_R) \\ &= \bigoplus_{l=1}^n \bigoplus_{m=1}^{n-l+1} \left[ \frac{m+1}{2} \right] \left( \frac{l-1}{2}, \frac{3l+2m}{2} \right) \\ &= \left( \frac{n-1}{2}, \frac{3n+2}{2} \right) \oplus (\text{lower } j_L) \end{aligned}$$

1/0

This "BPS" state counting is  
to be compared with the moduli  
space  $\tilde{M}_\beta$  of D2-branes over  $\beta$

$\pi : \tilde{M}_\beta \rightarrow M_\beta$  : the fiber is  
the moduli sp. of flat U(1) bdls  
over  $m \in M_\beta$

$SU(2)_L \times SU(2)_R$  = the Lefshetz action  
fiber base on  $\tilde{M}_\beta$

Hosono - Saito - Takahashi hep-th/990115)

Klemm - Katz - Vafa hep-th/9910181

$\beta = kB + nF \subset \mathbb{F}_0$  has "generically"

$$d = k+n \quad g = (k-1)(n-1)$$

$$\text{and} \quad M_\beta \simeq \mathbb{P}^{(k+1)(n+1)-1}$$

$k=1$  base =  $\mathbb{P}^{2n+1}$  fiber : pt.

$k=2$  base =  $\mathbb{P}^{3n+2}$  fiber =  $T^{2n-2}$

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### B. U(1) gauge theory

(Equivariant genera of  $\text{Hilb}^n(\mathbb{C}^2)$ )

$$\begin{aligned} Z^{U(1)} &= \sum_{\mu} Q^{|\mu|} \prod_{s \in \mu} \frac{1}{1 - t^{-l(s)} q^{-a(s)-1}} \\ &\quad \times \frac{1}{1 - t^{l(s)+1} q^{a(s)}} \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{Q^n}{n(1-t^n)(1-q^{-n})} \right) \\ &= \sum_{n=0}^{\infty} Q^n \chi_0(\text{Hilb}^n(\mathbb{C}^2))(t, q) \end{aligned}$$

$$t = \frac{q}{g}$$

$$\begin{aligned} Z^{U(1)} &= \exp \left( \sum_{n=1}^{\infty} \frac{-Q^n}{n(g^{n/2} - q^{-n/2})^2} \right) \\ &= \prod_{m=1}^{\infty} (1 - g^m Q)^{-m} \end{aligned}$$

topological string amplitude of conifold.

Macdonald function  $P_R(x; q, t)$   
<sup>L12</sup>

basis of  $\Lambda \otimes \mathbb{Q}(q, t)$

orthogonal w.r.t a certain inner product

$\Lambda$ : the ring of symmetric functions  
 in  $x_1, x_2, x_3, \dots$

The power sum functions  $p_n(x)$

$$p_n(x) = \sum_{i=1}^{\infty} x_i^n$$

are generators of  $\Lambda$ .

$$P_{D_1}(x; q, t) = p_1(x)$$

$$P_{D_2}(x; q, t) = \frac{(1+t)(1-q)}{1-qt} \frac{p_2(x)}{2} + \frac{(1+q)(1-t)}{(1-qt)} \frac{p_1(x)^2}{2}$$

$$P_B(x; q, t) = -\frac{p_2(x)}{2} + \frac{p_1(x)^2}{2}$$

$P_R(x; q, t)$  is a "master" fn. <sup>L13</sup>  
 for a family of symmetric functions.

$$(1) \quad P_R(x; q, t) = S_R(x) \text{ the Schur fn.}$$

$$(2) \quad P_R(x; 0, t) = H_R(x; t) \text{ the Hall}$$

- Littlewood fn.

$$(3) \quad \lim_{q \rightarrow 1} P_R(x; q, q^\beta) = J_R^{\beta^{-1}}(x)$$

the Jack fn (polynomial)

A specialization is a homomorphism

$$\varepsilon : \Lambda \otimes \mathbb{Q}(q, t) \longrightarrow \mathbb{Q}(q, t)$$

$\varepsilon$  is defined by giving  $\varepsilon(p_n(x))$

$$\text{e.g. } \varepsilon_{a,b,q}(p_n) = \frac{a^n - b^n}{1 - q^n}$$

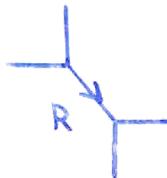
Why are these relevant?

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$$a = \lambda^{-\frac{1}{2}} q^{\frac{1}{2}} \quad b = \lambda^{\frac{1}{2}} q^{\frac{1}{2}}$$

$$\mathcal{E}_{(a,b),q}(S_R) = W_R(q, \lambda) \quad \lambda = q^N$$

$$\xrightarrow{\text{large } N} C_{R..}(q)$$



$$\sum_R C_{R..} C_{R..}^t (-1)^{l_R} e^{-t \cdot l_R} \\ = \exp \left( \sum_{n=1}^{\infty} \frac{e^{-t \cdot n}}{n (q^{\frac{n}{2}} - q^{-\frac{n}{2}})^2} \right)$$

Specialization of  $P_R(x; q, t)$

$$\mathcal{E}_{(a,b),q}(P_R) = \prod_{s \in \mu} \frac{at^{l(s)} - b q^{a(s)}}{1 - q^{a(s)} t^{l(s)+1}}$$

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$$C_{R_1 R_2 R_3}(q, t) := t^{-n(R_3^t)} q^{n(R_3)} \\ \times \tilde{P}_{R_2}(t^{-p}; q, t) \sum_R \iota \tilde{P}_{R/R}(q^{\mu^{R_2} p}; q, t) \\ \times \tilde{P}_{R_3^t/R}(q^{-\mu^{R_2}} t^{-p}; q, t)$$

$$q^{-\mu} t^{-p}: \quad x_i = q^{-\mu_i} t^{i-\frac{1}{2}}$$

$$\iota: \Lambda \rightarrow \Lambda \quad \text{involution} \quad \iota(p_n) = -p_n$$

$$C^{R_1 R_2 R_3}(q, t) := C_{R_1 R_2 R_3}(t, q)$$

$$C_{R..}(q, t) = C_{..R}(q, t) = C_{..R}(q, t)$$

$C_{R_1 R_2 R_3}(q, q)$  = the topological vertex.

Prop 1

$$\sum_R (-Q)^{|\mu^R|} C_{..R} C^{..R^t} \\ = \sum_{n=0}^{\infty} \tilde{Q}^n \chi_0(\mathrm{Hilb}^n(\mathbb{C}^2))$$

$$\tilde{Q} := Q \left( \frac{t}{q} \right)^{\frac{1}{2}}$$

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$$\sum_{R_1, R_2} (-Q_1)^{|\mu^{R_1}|} (-Q_2)^{|\mu^{R_2}|} C_{R_1 R_2} C^{t t}_{R_1 R_2}$$

$$= \sum_{n=0}^{\infty} \tilde{Q}^n \chi_y(\text{Hilb}^n(\mathbb{C}^2))(q, t)$$

(modulo "perturbative" factor)

$$\tilde{Q} := Q_1 \left(\frac{t}{q}\right)^{\frac{1}{2}} \quad y := Q_2 \left(\frac{q}{t}\right)^{\frac{1}{2}}$$

Prop. 3

$$\sum_{R_1, R_2, R_3} (-Q_1)^{|\mu^{R_1}|} (-Q_2)^{|\mu^{R_2}|} (-Q_3)^{|\mu^{R_3}|} C_{R_1 R_2 R_3} C^{t t t}_{R_1 R_2 R_3}$$

$$= \sum_{n=0}^{\infty} \tilde{Q}^n E_{(y, p)}(\text{Hilb}^n(\mathbb{C}^2))(q, t)$$

$$\tilde{Q} := Q_2 \left(\frac{t}{q}\right)^{\frac{1}{2}} \quad y := Q_3 \left(\frac{q}{t}\right)^{\frac{1}{2}} \quad p := Q_1 Q_3$$