Black hole entropy and anomalies

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Introduction

- Recent work has provided a detailed match between micro/macro computations of black hole entropy. Subleading corrections to Bekenstein-Hawking formula:

$$S(Q) = \frac{A(Q)}{4}\left(1 + \frac{c_1}{Q} + \frac{c_2}{Q^2} + \dots\right) + c_0 \ln Q$$

Cardoso, de Wit, Kappeli, Mohaupt; Wald; Ooguri, Strominger, Vafa; Dabholkar, Denef, Moore, Pioline; Sen; . . .

- Also have results when “corrections" are leading effect, e.g. fundamental heterotic string.
- Connection to topological string via OSV conjecture.
- Agreement at first seems mysterious, since higher derivatives are included in non-systematic way (only F-terms).
- I’ll argue that agreement has origin in exact statements about anomalies. Also use underlying AdS_3 geometry.
- Generalizes to non-BPS black holes. Heterotic strings with both left and right moving excitations.

Overview

- General comments on micro/macro comparisons
- Entropy in general higher derivative theories
- Use of gravitational anomalies
- Examples
- Holography in presence of gravitational anomalies

Microscopic degeneracy

- Consider bound state degeneracy of BPS system of branes. Counting done in worldvolume CFT

- Two kinds of charges

  Heavy charges P: define CFT (e.g. D1-D5)

  Light charges Q : excitations in CFT (e.g. momentum)

- Natural object is “mixed" partition function

$$Z_{\rm CFT}(P,\phi) = \sum_Q \Omega_{\rm CFT}(P,Q) e^{-\phi\cdot Q}$$

  Degeneracies $\Omega_{\rm CFT}(P,Q)$ recovered by inverse Laplace transform.

- For CFT dual to CY black holes considered here, have

$$\ln Z_{\rm CFT}(P,\phi) = \frac{\pi}{6\phi^0}\left(C_{IJK}P^I P^J P^K + \frac{1}{2}c_{2I}P^I\right) - \frac{\pi}{2}\frac{C_{IJ}\phi^I\phi^J}{\phi^0} + \dots$$

  Accounts for power law corrections to degeneracies. Omitted terms suppressed for small ϕ_0.

Macroscopic degeneracy

- Near horizon geometry: $\mathrm{AdS}_3 \times S^p \times X$. AdS_3 black hole reduces to AdS_2 upon KK reduction.
- Have standard AdS partition function $Z_{\mathrm{AdS}}(P, \phi)$ where

  Heavy charges P: set size of AdS

  potentials ϕ : boundary conditions for sugra fields
- Full partition function is

$$Z_{\mathrm{AdS}}(P, \phi) = \sum_{\mathrm{geometries}} \int \prod_i \mathcal{D}\Phi_i \, e^{-S}$$

- Sum over all geometries with specified boundary conditions. Integrate over fluctuations of massive and massless string modes, including string/brane instantons.
- Too hard. But to reproduce $\frac{1}{\phi^0}$ terms just need local contributions from sugra action, including all higher derivative terms from integrating out massive modes.

- Topological string captures contributions of form

$$\mathcal{L} \sim \sum_g c_g R^2 W^{2g-2} + \dots$$

Including just these terms reproduces CFT partition function.

- Doesn't capture other terms, e.g. R^4. Such terms do contribute individually at this order, and would seem to spoil agreement.
- What's going on? Answer: local part of partition function governed by symmetries and anomalies. Top. string gets these right. All other terms will cancel — nonrenormalization.
- Argument uses susy of underlying theory, but results can be applied to compute entropy of non-susy black holes in these theories.

Black hole entropy and higher derivatives (see also Sen, hep-th/0506177)

- Focus on cases with IR geometry

$$\mathrm{AdS}_3 \times S^P \times X$$

Canonical example: M-theory on CY_3 with wrapped M5-branes and M2-branes. Related via dualities to examples including

heterotic string on T^5

D1-D5 on $T^4 \times S^1$, K3 $\times S^1$

- Start with general diff. invariant theory admitting $\mathrm{AdS}_3 \times S^P \times X$. Reduce to $D = p + 3$ action

$$I = \frac{1}{16\pi G_{p+3}} \int d^{p+3}x \sqrt{g}\, \mathcal{L}_{p+3} + S_{\mathrm{bndy}} + S_{\mathrm{CS}}$$

- Assume solution:

$$ds^2 = \ell^2_{\mathrm{AdS}} d\hat{s}^2_{\mathrm{AdS}} + \ell^2_{\mathrm{S^p}} d\hat{s}^2_{S^p}$$

$$\text{diff/gauge invariant quantities constant}$$

- Local variation of radii:

$$\delta I = \frac{1}{16\pi G_{p+3}} \int d^{p+3}x \left\{ \frac{\partial(\sqrt{g}\mathcal{L}_{p+3})}{\partial \ell}\delta\ell \quad + \quad \underbrace{\nabla_\mu(\cdots)\delta\ell}_{=\,0\,\mathrm{by\,cov.\,constancy}} \right\}$$

- So radii found by extremizing central charge function

$$\boxed{c(\ell_{\mathrm{AdS}}, \ell_{\mathrm{S^p}}) = \frac{3\Omega_2\Omega_p}{32\pi G_{p+3}} \ell_{\mathrm{AdS}}^3 \ell_{\mathrm{S^p}}^p \mathcal{L}_{p+3}}$$

- To understand prefactor consider

$$ds^2_{\mathrm{AdS}} = \ell^2_{\mathrm{AdS}}(d\eta^2 + \sinh^2\eta d\Omega_2^2)$$

- Regulate bulk action with finite η cutoff:

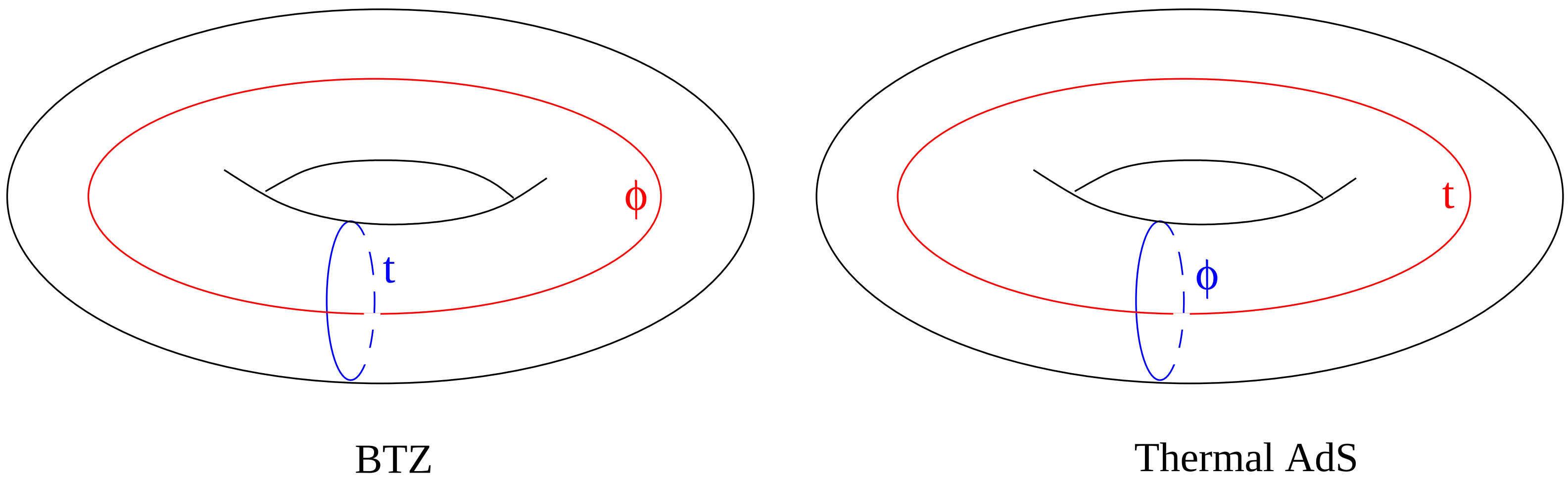
$$\delta I = -\frac{\Omega_2\Omega_p \ell_{\mathrm{AdS}}^3 \ell_{\mathrm{S^p}}^p \mathcal{L}_{p+3}}{32\pi G_{p+3}}\delta\eta_{\mathrm{max}}$$

- Think of CFT on S^2. Conformal anomaly: $T^i_i = -\frac{c}{24\pi}R^{(2)}$. Gives

$$\delta I = \frac{1}{2}\int d^2x\sqrt{g}\,T^{ij}\delta g_{ij} = -\frac{c}{3}\delta\eta_{\mathrm{max}}$$

Entropy from central charge

- Knowledge of central charge leads to black hole free energy.
- Recall relation between BTZ and thermal AdS (Maldacena, Strominger):



Related by modular transformation

$$\tau_{\mathrm{BTZ}} = -\frac{1}{\tau_{\mathrm{thermal}}}$$

- More generally have "$SL(2,Z)$" family of black holes.

- Warmup: consider case where only charges are $M,\ J$.
- Think of geometries as contributions to partition function

$$Z = e^{-I} = \mathrm{Tr}\ e^{2\pi i\tau(L_0 - \frac{c}{24}) - 2\pi i\bar{\tau}(\tilde{L}_0 - \frac{\tilde{c}}{24})}$$

- Ignoring nonlocal contributions have

$$L_0^{\mathrm{thermal}} = \tilde{L}_0^{\mathrm{thermal}} = 0 \qquad \Rightarrow \qquad I_{\mathrm{thermal}} = \frac{i\pi}{12}(c\tau - \tilde{c}\bar{\tau})$$
$$\Rightarrow \qquad I_{BTZ} = -\frac{i\pi}{12}(\frac{c}{\tau} - \frac{\tilde{c}}{\bar{\tau}})$$

- Entropy follows from

$$\Omega = e^S = \int d\beta d\mu\, e^{-I_{\mathrm{BTZ}} + \beta H + \mu J}$$
$$\tau = i(\beta - \mu)/2\pi$$

- Saddle point evaluation gives Cardy formula:

$$S = 2\pi\left(\sqrt{\frac{c}{6}h_L} + \sqrt{\frac{\tilde{c}}{6}h_R}\right)$$

$$h_L = \frac{1}{2}(M - J), \quad h_R = \frac{1}{2}(M + J)$$

Comments

- Gives entropy for BTZ black hole in arbitrary diff. invariant theory. $S \neq \frac{A}{4}$ in general.
- Effect of higher derivatives absorbed into c, as determined by c-extremization.
- Saddle point result equivalent to Wald's formula

$$S = -\frac{1}{8G_D}\int_{\mathrm{hor}} d^{D-2}x\sqrt{h}\,\frac{\delta\mathcal{L}_D}{\delta R_{\mu\nu\alpha\beta}}\epsilon^{\mu\nu}\epsilon^{\alpha\beta}$$

Show by using c-extremization and dim. analysis to write generalization of Brown-Henneaux result:

$$c = \frac{\ell_{\mathrm{AdS}}}{2G_3} g_{\mu\nu}\frac{\partial\mathcal{L}_3}{\partial R_{\mu\nu}}$$

Find: Wald formula = Cardy formula (Saida, Soda).

Including other potentials

- Include gauge fields A^I_μ in AdS$_3$:

$$S_A \sim \int_{\rm AdS} (F^I \wedge^\star F^I + C_{IJ} A^I \wedge F^J) + \int_{\partial {\rm AdS}} C_{IJ} A^I \wedge^\star A^J$$

Boundary term needed for good variational principle. Boundary current becomes chiral $J_I(z)$, as in CFT.

- Satisfy boundary conditions with flat connection A^I_μ. Induces charge due to CS term:

$$Q_I \sim \int C_{IJ} A^J$$

- Metric dependence of boundary term shifts stress tensor

$$\tilde{L}_0 \to \tilde{L}_0 + C_{IJ} A^I A^J$$

- Repeating previous analysis now gives

$$I_{\rm BTZ} = -\frac{i\pi}{12}(\frac{c}{\tau} - \frac{\tilde{c}}{\overline{\tau}}) + \frac{1}{\overline{\tau}} C_{IJ} A^I A^J$$

- After relabelling,

$$\overline{\tau} \to \frac{i}{2}\phi^0, \quad A^I \to \phi^I$$

and specializing to extremal black hole ($\frac{1}{\tau} = 0$), this agrees with CFT result

$$\ln Z_{\rm CFT}(P,\phi) = \frac{\pi}{6\phi^0}\left(C_{IJK}P^I P^J P^K + \frac{1}{2}c_{2I}P^I\right) - \frac{\pi}{2}\frac{C_{IJ}\phi^I\phi^J}{\phi^0}$$

provided

$$\tilde{c} = C_{IJK}P^I P^J P^K + \frac{1}{2}c_{2I}P^I$$

- Summary: Computation of degeneracy, including subleading corrections boils down to computation of central charge $\tilde{c}$.

Central charge: two-derivative example

- Basic example: M-theory on CY_3. 2-derivative bosonic Lagrangian for gravity/vector multiplets:

$$-R + \frac{1}{2}G_{IJ}\partial_\mu X^I \partial^\mu X^J + \frac{1}{4}G_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{6}C_{IJK}F^I \wedge F^J \wedge A^K$$

- Solution takes form (Gutowski,Reall; Gauntlett,Gutowski)

$$\begin{aligned} ds^2 &= f^2(dt+\omega)^2 + f^{-1}ds_B^2 \\ F^I &= d[fX^I(dt+\omega)] + \Theta^I \end{aligned}$$

with

$$\begin{aligned} \Theta^I &= {}^\star\Theta^I \\ \nabla^2(f^{-1}X_I) &= \frac{1}{4}C_{IJK}\Theta^J \cdot \Theta^K \\ d\omega + {}^\star d\omega &= -f^{-1}X_I\Theta^I \end{aligned}$$

- Take base to be Taub-NUT, and solve in terms of 4 sets of harmonic functions

$$\begin{aligned} H^I &\sim M5\text{−branes} \\ H_I &\sim M2\text{−branes} \\ H^0 &\sim KK\text{−monopole} \\ H_0 &\sim \text{momentum} \end{aligned}$$

- IIA interpretation in terms of D0-D2-D4-D6-P black hole.
- Simplest to set number of KK-monopoles (D6-branes) to zero.
- Get near horizon geometry (locally)

$$\mathrm{AdS}_3 \times S^2 \times CY_3$$

- “Heavy" charges are M5-branes wrapped on 4-cycles

$$P^I = -\frac{1}{2\pi} \int_{S^2} F^I$$

- c-extremization gives

$$c = \tilde{c} = C_{IJK} P^I P^J P^K$$

- Theory has $(4,0)$ superconformal symmetry.
- To compute corrections using c-extremization we need full Lagrangian.

Corrections and nonrenormalization

- At 2-derivative order theory has solution AdS$_3 \times S^2 \times X$ with $(4,0)$ superconformal symmetry.
- Assume this persists in presence of higher derivatives.
- Observation (Harvey, Minasian, Moore): symmetries relate central charges to M5-brane grav. anomalies.
- c related by symmetry to $SU(2)_R$ anomaly. $c - \tilde{c}$ given by gravitational anomaly. Corrections to both given by single R^2 term in $D = 5$. No further corrections.
- Conclusion: R^2 terms which have been studied preserve symmetry and account for anomaly, so give exact result.
- Individual R^4 terms can correct central charge, but complete set consistent with $(4,0)$ symmetry will give zero.

Corrections to central charge

((Maldacena, Strominger, Witten; Harvey, Minasian, Moore)

- Geometry and central charge are corrected by higher derivative terms. Correction to central charge follows from $D = 11$ term

$$\epsilon_{11} C_3 \left[\mathrm{Tr}\, R^4 - \frac{1}{4} (\mathrm{Tr}\, R^2)^2 \right]$$

Coefficient known from M5-brane anomaly cancellation
(Vafa,Witten; Duff, Liu, Minasian)

- Reduce to $D = 5$ in presence of M5-brane wrapping 4-cycle $P_0 = P_0^I \sigma_I$

$$S_{\mathrm{anom}} = \frac{\overbrace{c_2}^{\text{2nd Chern}} \cdot P_0}{48} \int A \wedge \underbrace{p_1}_{1^{\text{st}}\text{Pontryagin}} = \frac{1}{2} \left(\frac{1}{2\pi} \right)^2 \frac{c_2 \cdot P_0}{4\pi} \int F \wedge \underbrace{\omega_3}_{\text{Lorentz CS}}$$

- Bulk action not invariant under local Lorentz transformations:

$$\delta\omega = d\Theta + [\omega, \Theta]$$

- Pick up contribution at AdS_3 boundary

- $D = 1 + 1$ gravitational anomaly:

$$\delta S_{\rm anom} = -\frac{1}{2}\frac{c_2 \cdot P}{48}\frac{1}{2\pi}\int_{\partial \rm AdS_3} \rm Tr\,(\Theta d\omega)$$

- Compare to variation of $(4, 0)$ CFT partition function

$$-\delta \ln Z_{CFT} = \frac{c - \tilde{c}}{48}\frac{1}{2\pi}\int_{\partial \rm AdS_3} \rm Tr\,(\Theta d\omega)$$

$$\Rightarrow \quad c - \tilde{c} = -\frac{1}{2}c_2 \cdot P$$

- R-symmetry anomaly (diffeomorphisms on sphere) gives shift in c

$$\Delta c = \frac{1}{2}c_2 \cdot P$$

- Exact central charges are then (Maldacena, Strominger, Witten; Harvey, Minasian, Moore)

$$\begin{aligned} c &= C_{IJK}P^I P^J P^K + \frac{1}{2}c_2 \cdot P \\ \tilde{c} &= C_{IJK}P^I P^J P^K + c_2 \cdot P \end{aligned}$$

Application: heterotic string spectrum

$$\text{M}-\text{theory on } K3\times T^2\times S^1 \quad \cong \quad \text{Het. on } T^6$$
$$M5 \text{ wrapped } w \text{ times on } K3\times S^1 \quad \cong \quad \text{Het. string wrapped } w \text{ times on } S^1$$

$$c_2(K3)=24 \quad \Rightarrow \quad c=12w, \quad \tilde{c}=24w$$

Corresponds to transverse left and right movers on heterotic worldsheet.

- Saddle point entropy of small BPS black hole yields entropy of DH states (Dabholkar)

$$S=2\pi\sqrt{\frac{\tilde{c}}{6}h_R}=4\pi\sqrt{nw}$$

  Power law corrections reproduced upon fixing gauge potentials and performing $d\phi$ integral (Dabholkar, Denef, Moore, Pioline)

- Can also consider non-extremal small black hole vs. non-BPS heterotic states:

$$S=2\pi\sqrt{2}\sqrt{wh_L}+4\pi\sqrt{wh_R}$$

Holography with gravitational anomalies

- Gravity with negative Λ:

$$S = \frac{1}{16\pi G}\int d^D x\sqrt{g}\,(R - 2\Lambda) + S_{\rm bndy}$$

- Metric admits Fefferman-Graham expansion

$$ds^2 = d\eta^2 + e^{2\eta/\ell} g^{(0)}_{ij} dx^i dx^j + g^{(2)}_{ij} dx^i dx^j + \ldots$$

- Stress tensor defined as variation of action (Balasubramanian, PK)

$$\delta S = \frac{1}{2}\int_{\partial AdS} d^{D-1}x\sqrt{g}\;T^{ij}\delta g^{(0)}_{ij}$$

$$D = 3: \quad T_{ij} = \frac{1}{8\pi G\ell}(g^{(2)}_{ij} - g^{(2)}{}^k{}_k g^{(0)}_{ij})$$

- Stress tensor obeys: $\nabla_i T^{ij} = 0,\ T^i_i = -\frac{c}{12}R$ (Henningson,Skenderis).
- Brown-Henneaux formula: $c = \tilde{c} = \frac{3\ell}{2G}$.

- Now add non diff. invariant term (Deser,Jackiw,Templeton)

$$S_{CS}(\Gamma) = \beta \int \mathrm{Tr}(\Gamma d\Gamma + \frac{2}{3}\Gamma^3) \qquad ({\Gamma^i}_j = \Gamma^i_{jk} dx^k)$$

- Under $x^i \to x^i - \xi^i(x)$ action varies as

$$\delta S = \beta \int_{\partial \mathrm{AdS}} \mathrm{Tr}(v d\Gamma), \quad ({v^i}_j = \partial_j \xi^i) \qquad \text{(Gen. coord. anomaly)}$$

Implies nonconserved stress tensor

$$\nabla_i T^{ij} = g^{ij} \epsilon^{kl} \partial_k \partial_m \Gamma^m_{il}$$

- CFT has anomaly proportional to $c - \tilde{c}$. Find

$$c = c_0 + 48\pi\beta, \quad \tilde{c} = c_0 - 48\pi\beta$$

- Full stress tensor is

$$T^{ij} = T^{ij}_0 + \frac{2\beta}{\ell^2}(g^{ik}_{(2)} \epsilon^{lj} + g^{jk}_{(2)} \epsilon^{li}) g^{(0)}_{kl} + X^{ij}(g^{(0)})$$

- Apply to BTZ labelled by (m, j). Geometry uncorrected.

$$M = m - \frac{32\pi\beta G_3}{\ell^2} j, \quad J = j - 32\pi\beta G_3 m$$

- Global AdS$_3$ has $m = -1/8G_3,\ j = 0$:

$$M = -\frac{1}{8G_3}, \quad J = 4\pi\beta = \frac{c - \tilde{c}}{24}$$

- BH entropy given by Cardy formula

$$S = 2\pi \left(\sqrt{\frac{c}{6} h_L} + \sqrt{\frac{\tilde{c}}{6} h_R} \right)$$

$$h_L = \frac{1}{2}(M\ell - J), \quad h_R = \frac{1}{2}(M\ell + J)$$

- Amusing example: $\beta = c_0/48\pi$

$$c = 96\pi\beta, \quad \tilde{c} = 0$$

Extremal black hole $M\ell - J = 0$ has nonzero horizon area but $S = 0$.

Conclusions / Questions

- Symmetries and anomalies are very powerful in determining black hole entropy, including corrections.
- Need better understanding of non-perturbative effects related to exponentially small terms.
- Entropy of Type II strings / D1-D5 on T^4 remains to be understood. Gravitational anomaly vanishes.

  Exact worldsheet analysis of spacetime central charge

  (Kutasov, Seiberg; Kutasov, Larsen, Leigh)

$$\begin{aligned} \text{Heterotic}: \quad c &= c_0 + 12, \quad \tilde{c} = c_0 + 24 \\ \text{Type II}: \quad c &= c_0, \quad \tilde{c} = c_0 \end{aligned}$$

$$c_0 = 6 N_{NS5} N_{KK}$$