Dual (Super) Conformal Symmetry and Integrability in $AdS_5 \times S^5$

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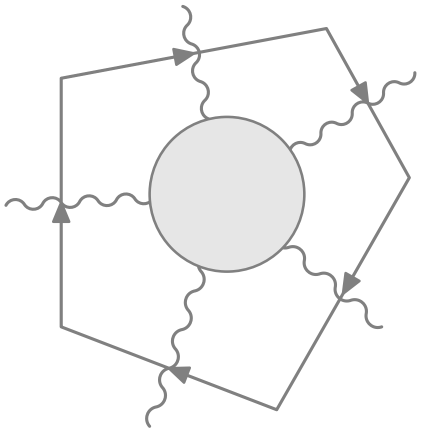
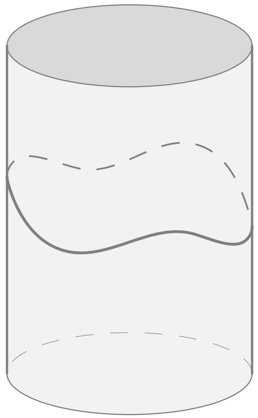




Fundamental Aspects of String Theory
KITP Santa Barbara, February 17, 2009

Work in collaboration with R. Ricci, A. Tseytlin, M. Wolf.
References: 0807.3228; also 0705.0303, 0807.1095, 0807.3196.

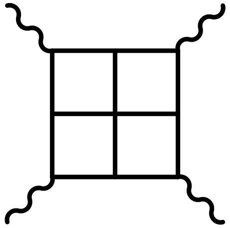
Gluon Amplitudes, Wilson Loops, Integrability and AdS/CFT

Planar Gluon Scattering Amplitudes

Intriguing result in $\mathcal{N}=4$ SYM in the planar limit $N_\mathrm{c}\to\infty$:

Four-gluon scattering amplitude obeys BDS relation [Anastasiou, Bern, Dixon, Kosower] [Bern, Dixon, Smirnov]

$$A(p,\lambda)\simeq A^{(0)}(p)\exp\Big(2D_\mathrm{cusp}(\lambda)M^{(1)}(p)\Big)\,.$$



Only required data: • tree level, • one loop, • cusp dimension.

- Captures IR singularities correctly.
- No finite remainder function $F(p,\lambda)$.

Gluon scattering amplitudes constructible by unitarity and suitable ansatz.

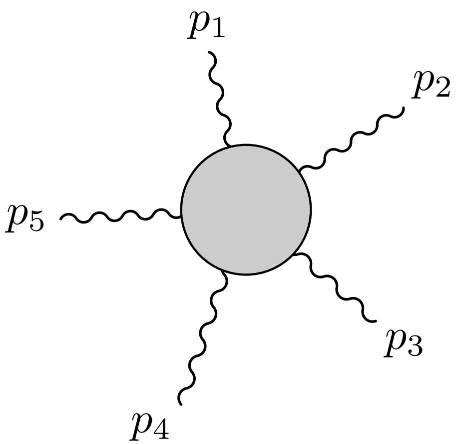
Verified BDS relation at $\mathcal{O}(\lambda^4)$ with [Bern, Dixon, Smirnov] [Bern, Czakon, Dixon, Kosower, Smirnov]

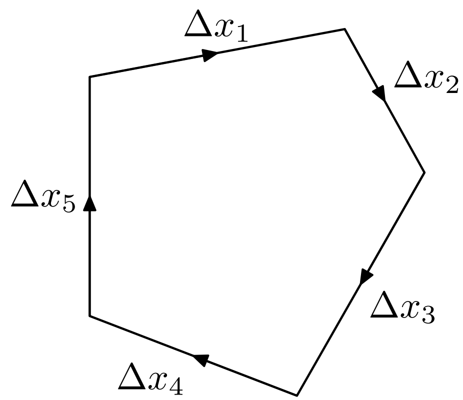
$$D_\mathrm{cusp}(\lambda)=\frac{1}{2}\frac{\lambda}{\pi^2}-\frac{1}{96}\frac{\lambda^2}{\pi^2}+\frac{11}{23040}\frac{\lambda^3}{\pi^2}-\left(\frac{73}{2580480}+\frac{\zeta(3)^2}{1024\pi^6}\right)\frac{\lambda^4}{\pi^2}\pm\ldots\,.$$

Light-Like Wilson Loops

What does scattering correspond to in the dual string theory on $AdS_5 \times S^5$?

After a T-duality it relates to a light-like Wilson loop! [Alday Maldacena]





- light-like momenta $p_k^2 = 0$
- momentum conservation $\sum_k p_k = 0$
- polarisations

- light-like separations $\Delta x_k^2 = 0$
- closure $\sum_k \Delta x_k = 0$
- ? (Only MHV? Only prefactor?)

Set $p_k = \Delta x_k$ and match Wilson loop expectation value with amplitude.

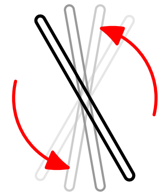
- Functional form agrees with BDS relation at strong coupling! [Alday Maldacena]
- Amplitudes dual to Wilson loops at also weak coupling! [Drummond Korchemsky Sokatchev]

More Legs and Loops

Further results on duality between amplitudes and light-like Wilson loops:

- 4 legs, strong coupling: Agreement with spinning string energy [Gubser Klebanov Polyakov]

$$D_{\text{cusp}}(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \mathcal{O}(1/\sqrt{\lambda}^{0}).$$



- n legs, 1 loop: General agreement. [Drummond Korchemsky Sokatchev]
- 4 legs, 2 loops: Agreement (adjust renormalisation). [Drummond, Henn Korchemsky Sokatchev]
- 6 legs, 2 loops: Agreement, but $F(p,\lambda)$ needed! [Bern, Dixon, Kosower Roiban, Spradlin Vergu, Volovich] [Drummond, Henn Korchemsky Sokatchev]
- ∞ legs, strong coupling: $F(p,\lambda)$ required! [Alday Maldacena]
- Further indications for $F(p,\lambda)$. [Drummond, Henn Korchemsky Sokatchev] [Bartels Lipatov Sabio Vera]

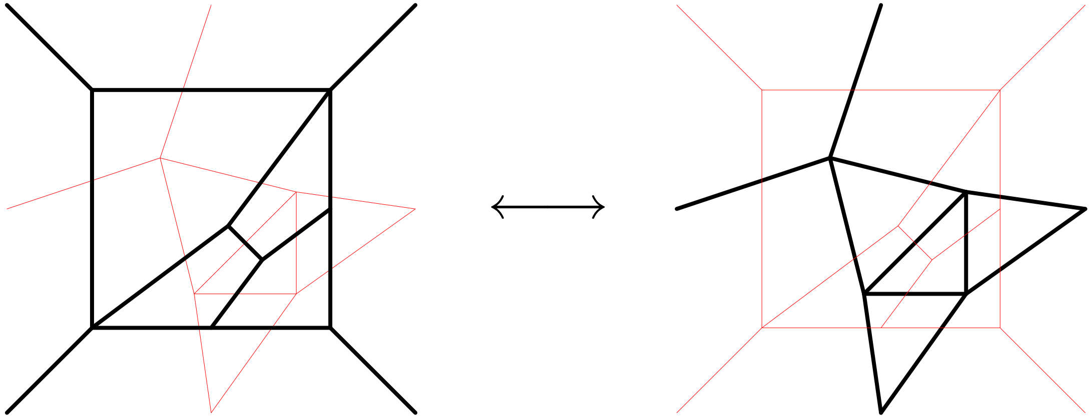
MHV amplitudes expected to obey more general relation

$$A(p,\lambda) \simeq A^{(0)}(p) \exp\left(2D_{\text{cusp}}(\lambda) M^{(1)}(p) + F(p,\lambda)\right).$$

Simplicity and Dual Conformal Symmetry

Planar amplitudes and integrals simpler than expected:

- No bubbles, no triangles, typically scalar boxes.
- Similarity of momentum and position space propagators in $D = 4$.

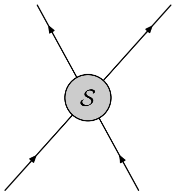


- Dual amplitudes and integrals are conformal. [Drummond Korchemsky Sokatchev]...
- Combine MHV amplitudes into superspace amplitudes. One multiplet!
  Dual superconformal symmetry of superspace amplitude. [Drummond, Henn Korchemsky Sokatchev]
- Self-duality of superstrings requires also fermionic T-duality. [Berkovits Maldacena]
- Dual superconformal symmetry $\widehat{=}$ symmetry of T-dual model.
- Dual superconformal symmetry allows $F(p, \lambda)$ only for $n \geq 6$ legs.

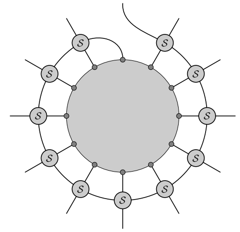
Cusp Dimension from Bethe Equations

Cusp dimension known from AdS/CFT planar integrable system!

Compute cusp dimension using Bethe equations. Integral eq.: [Eden Staudacher]



$$\psi(x) = K(x,0) - \int_0^\infty K(x,y)\,\frac{dy\,y}{e^{y/2g}-1}\,\psi(y).$$



Kernel $K = K_0 + K_1 + K_\mathrm{d}$ with [NB, Eden Staudacher]

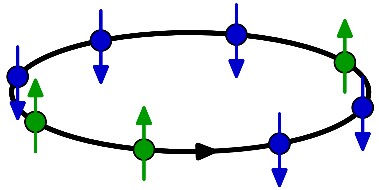
$$K_0(x,y) = \frac{x\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - y\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2},$$

$$K_1(x,y) = \frac{y\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - x\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2},$$

$$K_\mathrm{d}(x,y) = 2\int_0^\infty K_1(x,z)\,\frac{dz\,z}{e^{z/2g}-1}\,K_0(z,y).$$

Cusp anomalous dimension: $D_\mathrm{cusp} = 16g^2\psi(0)$.

Weak/Strong Expansion

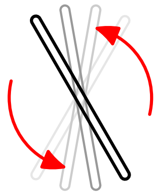
Weak coupling expansion of integral equation  [NB, Eden, Staudacher]

$$D_{\text{cusp}}(\lambda) = \frac{1}{2}\frac{\lambda}{\pi^2} - \frac{1}{96}\frac{\lambda^2}{\pi^2} + \frac{11}{23040}\frac{\lambda^3}{\pi^2} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\frac{\lambda^4}{\pi^2} \pm \ldots .$$

Agreement with gluon scattering amplitudes. [Bern, Dixon, Smirnov][Bern, Czakon, Dixon, Kosower, Smirnov]

Strong coupling asymptotic expansion of integral equation [Casteill, Kristjansen][Basso, Korchemsky, Kotański]

$$E_{\text{cusp}}(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi} - \frac{\beta(2)}{\pi\sqrt{\lambda}} + \ldots .$$



Agreement with semiclassical energy of spinning string. [Gubser, Klebanov, Polyakov][Frolov, Tseytlin][Roiban, Tirziu, Tseytlin]

Questions

- Same cusp dimension from amplitudes & integrable system:
  How to apply integrability to scattering amplitudes?
- Can one compute remainder function $F(p, \lambda)$ (like $D_{\mathrm{cusp}}(\lambda)$)?
- Relation between (dual) superconformal symmetry and integrability?
- What about non-planar corrections?
- What about non-MHV amplitudes?
- How to relate scattering amplitudes to Wilson loops in gauge theory?
- How does the T-self-duality work for $AdS_5 \times S^5$? Fermionic T-duality?!

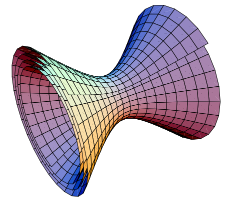
Outline of this Talk

- AdS_{n+1} sigma model & T-self-duality. [Alday Maldacena]
- Sketch of superstrings on $AdS_5 \times S^5$ & fermionic T-duality. [Berkovits Maldacena]
- T-duality for symmetries & integrable structure. [NB, Ricci Tseytlin, Wolf] [Berkovits Maldacena]

AdS_{n+1} Sigma Model

AdS_{n+1} Coset Space Sigma Model

AdS_{n+1} is the coset space



$$AdS_{n+1} = \widetilde{\mathrm{SO}}(n,2)/\mathrm{SO}(n,1) = \mathrm{G}/\mathrm{H}.$$

Setup: Group-valued field and equivalence classes of coset space G/H

$$g(\sigma,\tau) \in \widetilde{\mathrm{SO}}(n,2), \qquad g \simeq gh, \quad h(\sigma,\tau) \in \mathrm{SO}(n,1).$$

Algebra-valued Maurer–Cartan form, split into $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$,

$$J = g^{-1}dg = J_0 + J_1, \qquad J_0 \in \mathfrak{h}, \quad J_1 \in \mathfrak{g}/\mathfrak{h}.$$

$\mathbb{Z}_2$ automorphism $\Omega(J_k) := (-1)^k J_k$ and action

$$S = \int \tfrac{1}{2}\,\mathrm{Tr}\, J_1 \wedge *J_1 = \int \tfrac{1}{4}\,\mathrm{Tr}\, J \wedge *\big(J - \Omega(J)\big).$$

Lax Connection

Maurer–Cartan equations and equations of motion

$$0 = dJ_0 + J_0 \wedge J_0 + J_1 \wedge J_1,$$

$$0 = dJ_1 + J_0 \wedge J_1 + J_1 \wedge J_0,$$

$$0 = d{*J_1} + J_0 \wedge {*J_1} + {*J_1} \wedge J_0.$$

Introduce Lax connection

$$A(z) = J_0 + \tfrac{1}{2}(z + z^{-1})J_1 + \tfrac{1}{2}(-z + z^{-1}){*J_1}.$$

MCE and EOM are equivalent to flatness of Lax connection

$$dA(z) + A(z) \wedge A(z) = 0.$$

Integrability! Lax monodromy $\mathrm{P}\oint \exp A(z)$ leads to integrable structure.

Poincaré Coordinates

Present $\mathfrak{so}(n,2)$ as conformal algebra on $\mathbb{R}^{n-1,1}$

$$[\mathfrak{D},\mathfrak{P}_\mu]=+\mathfrak{P}_\mu,\quad [\mathfrak{D},\mathfrak{K}_\mu]=-\mathfrak{K}_\mu,\quad [\mathfrak{P}_\mu,\mathfrak{K}_\nu]=2\mathfrak{L}_{\mu\nu}+2\eta_{\mu\nu}\mathfrak{D}.$$

Then $\mathfrak{L},\mathfrak{P}+\mathfrak{K}\in\mathfrak{h}$ and $\mathfrak{D},\mathfrak{P}-\mathfrak{K}\in\mathfrak{g}/\mathfrak{h}$. $\mathbb{Z}_2$ automorphism Ω

$$\Omega(\mathfrak{L}_{\mu\nu})=\mathfrak{L}_{\mu\nu},\quad \Omega(\mathfrak{P}_\mu)=\mathfrak{K}_\mu,\quad \Omega(\mathfrak{K}_\mu)=\mathfrak{P}_\mu,\quad \Omega(\mathfrak{D})=-\mathfrak{D}.$$

Fix a gauge s.t. $J_\mathfrak{L}=J_\mathfrak{K}=0$: Poincaré coordinates x^μ,φ for AdS_{n+1}

$$g=\exp(ix^\mu\mathfrak{P}_\mu)\exp(\varphi\mathfrak{D}),\qquad J=ie^{-\varphi}dx^\mu\,\mathfrak{P}_\mu+d\varphi\,\mathfrak{D}=J_\mathfrak{P}+J_\mathfrak{D}.$$

Resulting action is quadratic in dx^μ: $\operatorname{Tr}\mathfrak{D}\mathfrak{D}=1$, $\operatorname{Tr}\mathfrak{P}_\mu\mathfrak{K}_\nu=2\eta_{\mu\nu}$

$$S=\int\left(\tfrac{1}{2}d\varphi\wedge *d\varphi+\tfrac{1}{2}e^{-2\varphi}dx^\mu\wedge *dx_\mu\right).$$

T-Self-Duality Transformation

Perform formal T-duality along coordinates x^μ, drop boundary terms:

$$
\begin{aligned}
S \quad &= \quad \int \left(\tfrac{1}{2} d\varphi \wedge *d\varphi + \tfrac{1}{2} e^{-2\varphi} dx^\mu \wedge *dx_\mu\right) \\
&\overset{dx^\mu \to \Lambda^\mu}{\simeq} \int \left(\tfrac{1}{2} d\varphi \wedge *d\varphi + \tfrac{1}{2} e^{-2\varphi} \Lambda^\mu \wedge *\Lambda_\mu - \tilde{x}_\mu \wedge d\Lambda^\mu\right) \\
&\simeq \quad \int \left(\tfrac{1}{2} d\varphi \wedge *d\varphi + \tfrac{1}{2} e^{-2\varphi} \Lambda^\mu \wedge *\Lambda_\mu - \Lambda^\mu \wedge d\tilde{x}_\mu\right) \\
&\overset{\Lambda^\mu \to e^{2\varphi} * d\tilde{x}^\mu}{\simeq} \int \left(\tfrac{1}{2} d\varphi \wedge *d\varphi - \tfrac{1}{2} e^{+2\varphi} * d\tilde{x}^\mu \wedge d\tilde{x}_\mu\right) \\
&\overset{\varphi \to -\tilde{\varphi}}{\simeq} \int \left(\tfrac{1}{2} d\tilde{\varphi} \wedge *d\tilde{\varphi} + \tfrac{1}{2} e^{-2\tilde{\varphi}} d\tilde{x}^\mu \wedge *d\tilde{x}_\mu\right) = \tilde{S}.
\end{aligned}
$$

Action T-self-dual! Relations: $dx_\mu = e^{-2\tilde{\varphi}} * d\tilde{x}_\mu$ and $\varphi = -\tilde{\varphi}$.

Implications of T-Self-Duality

Compare model expressed through original x^μ, φ and dual variables $\tilde{x}^\mu, \tilde{\varphi}$

original variables		dual variables
equation of motion	$\Leftrightarrow$	integrability condition
integrability condition	$\Leftrightarrow$	equation of motion
local quantities	$\Rightarrow$	non-local quantities
non-local quantities	$\Leftarrow$	local quantities
Noether charge	$\Rightarrow$	non-local charge
non-local charge	$\Leftarrow$	Noether charge
Lax connection	$\Rightarrow$	dual Lax connection
dual Lax connection	$\Leftarrow$	Lax connection

However:

- What are the dual Noether symmetries?
- Two distinct integrable structures from Lax connections?!

T-Self-Duality on Phase Space

Relation between Maurer–Cartan forms: $J_{\mathfrak{P}} = *\tilde{J}_{\mathfrak{P}}$ and $J_{\mathfrak{D}} = -\tilde{J}_{\mathfrak{D}}$.

System of Maurer–Cartan equations and equations of motion

$$\begin{aligned}
0 &= dJ_{\mathfrak{D}}, \\
0 &= d{*J_{\mathfrak{D}}} - \tfrac{1}{2} J_{\mathfrak{P}} \wedge *\Omega(J_{\mathfrak{P}}) - \tfrac{1}{2} {*\Omega(J_{\mathfrak{P}})} \wedge J_{\mathfrak{P}}, \\
0 &= dJ_{\mathfrak{P}} + J_{\mathfrak{D}} \wedge J_{\mathfrak{P}} + J_{\mathfrak{P}} \wedge J_{\mathfrak{D}}, \\
0 &= d{*J_{\mathfrak{P}}} - J_{\mathfrak{D}} \wedge *J_{\mathfrak{P}} - *J_{\mathfrak{P}} \wedge J_{\mathfrak{D}},
\end{aligned}$$

maps to equivalent system

$$\begin{aligned}
0 &= -d\tilde{J}_{\mathfrak{D}}, \\
0 &= -d{*\tilde{J}_{\mathfrak{D}}} - \tfrac{1}{2} {*\tilde{J}_{\mathfrak{P}}} \wedge \Omega(\tilde{J}_{\mathfrak{P}}) - \tfrac{1}{2} \Omega(\tilde{J}_{\mathfrak{P}}) \wedge *\tilde{J}_{\mathfrak{P}}, \\
0 &= d{*\tilde{J}_{\mathfrak{P}}} - \tilde{J}_{\mathfrak{D}} \wedge *\tilde{J}_{\mathfrak{P}} - *\tilde{J}_{\mathfrak{P}} \wedge \tilde{J}_{\mathfrak{D}}, \\
0 &= d\tilde{J}_{\mathfrak{P}} + \tilde{J}_{\mathfrak{D}} \wedge \tilde{J}_{\mathfrak{P}} + \tilde{J}_{\mathfrak{P}} \wedge \tilde{J}_{\mathfrak{D}}.
\end{aligned}$$

Sign of equations for $J_{\mathfrak{D}}$ flipped; equations for $J_{\mathfrak{P}}$ exchanged.

T-Duality for Noether Charges

Conserved Noether current k and charge Q

$$k = g\big(2J_{\mathfrak{D}} + J_{\mathfrak{P}} - \Omega(J_{\mathfrak{P}})\big)g^{-1}, \qquad d{*}k = 0, \qquad Q = \oint {*}k.$$

Noether current components in Poincaré gauge:

$$\begin{aligned}
k_{\mathfrak{K}} &= -ie^{-2\varphi}dx^\mu \mathfrak{K}_\mu,\\
k_{\mathfrak{D}} &= 2(d\varphi + e^{-2\varphi}x_\mu dx^\mu)\mathfrak{D},\\
k_{\mathfrak{L}} &= 2e^{-2\varphi}x^\mu dx^\nu \mathfrak{L}_{\mu\nu},\\
k_{\mathfrak{P}} &= i(dx^\mu - 2x^\mu d\varphi + e^{-2\varphi}x^2 dx^\mu - 2e^{-2\varphi}x^\mu x_\nu dx^\nu)\mathfrak{P}_\mu.
\end{aligned}$$

Relation between charges and dual charges on periodic solutions

$$\tilde{Q}_{\mathfrak{K}} = 0, \quad \tilde{Q}_{\mathfrak{L}} - \tilde{Q}_{\mathfrak{D}} = Q_{\mathfrak{L}} + Q_{\mathfrak{D}} + [Q_{\mathfrak{K}}, ix^\mu \mathfrak{P}_\mu], \quad Q_{\mathfrak{P}}, \tilde{Q}_{\mathfrak{P}} \text{ unrelated.}$$

Some dual charges are trivial, some are related, some unrelated!

T-Self-Duality for Lax Connection

Lax connection in Poincaré gauge

$$
\begin{aligned}
A(z) = &+\tfrac{1}{2}z^{-1}(z^2+1)J_{\mathfrak{D}} - \tfrac{1}{2}z^{-1}(z^2-1){*}J_{\mathfrak{D}} \\
&+ \tfrac{1}{4}z^{-1}(z+1)\big((z+1)J_{\mathfrak{P}} - (z-1){*}J_{\mathfrak{P}}\big) \\
&+ \tfrac{1}{4}z^{-1}(z-1){*}\Omega\big((z+1)J_{\mathfrak{P}} - (z-1){*}J_{\mathfrak{P}}\big).
\end{aligned}
$$

Dual Lax connection with substitution of $\tilde{J}_{\mathfrak{P}} = {*}J_{\mathfrak{P}}$ and $\tilde{J}_{\mathfrak{D}} = -J_{\mathfrak{D}}$

$$
\begin{aligned}
\tilde{A}(z) = &-\tfrac{1}{2}z^{-1}(z^2+1)J_{\mathfrak{D}} + \tfrac{1}{2}z^{-1}(z^2-1){*}J_{\mathfrak{D}} \\
&+ \tfrac{1}{4}z^{-1}(z+1){*}\big((z+1)J_{\mathfrak{P}} - (z-1){*}J_{\mathfrak{P}}\big) \\
&+ \tfrac{1}{4}z^{-1}(z-1)\Omega\big((z+1)J_{\mathfrak{P}} - (z-1){*}J_{\mathfrak{P}}\big).
\end{aligned}
$$

Related by z-dependent automorphism:

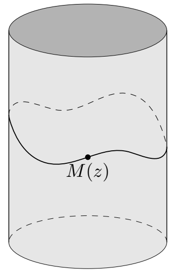
$$
\tilde{A}(z) = \left(\frac{z+1}{z-1}\right)^{\mathfrak{D}} \Omega\big(A(z)\big) \left(\frac{z-1}{z+1}\right)^{\mathfrak{D}}.
$$

Integrable structures equivalent! [NB, Ricci Tseytlin, Wolf]

Multi-Local Charges

Lax monodromy $M(z)$ is conserved modulo similarity transformation

$$M(z) = g(0)\left[\overrightarrow{\mathrm{P}\exp}\oint A(z)\right] g(2\pi)^{-1}.$$



Expansion at $z=1$ yields multi-local charges $Y^{(k)} \in \mathfrak{so}(n,2)$

$$M\left(\frac{1-\epsilon}{1+\epsilon}\right) = \exp\left(\sum_{r=1}^{\infty} \epsilon^r Y^{(r)}\right).$$

Local charge $Y^{(1)}$ is Noether charge Q, next higher charge $Y^{(2)}$ is bi-local:

$$Y^{(1)} = \oint *k = Q, \qquad Y^{(2)} = \tfrac{1}{2}\iint_{\sigma_1<\sigma_2} [*k_1, *k_2] + \oint k, \qquad \ldots.$$

How are the multi-local charges $Y^{(k)}$ and dual charges $\tilde{Y}^{(k)}$ related?

Mapping of Local and Multi-Local Charges

Self-duality of dual Lax connections lifts to relation of monodromies

$$e^{-ix(0)\cdot\mathfrak{P}}\,M(z(\epsilon))\,e^{+ix(0)\cdot\mathfrak{P}}\,\exp(-\epsilon Q_{\mathfrak{K}})$$
$$=(-\epsilon)^{\mathfrak{D}}\,\Omega\big(e^{-i\tilde{x}(0)\cdot\mathfrak{P}}\,\tilde{M}(z(\epsilon))\,e^{+i\tilde{x}(0)\cdot\mathfrak{P}}\big)(-\epsilon)^{-\mathfrak{D}}$$

Leads to following mapping of charges (up to commutators)

$$\begin{array}{lll} & & 0 = \tilde{Q}_{\mathfrak{K}} \\ Q_{\mathfrak{K}} & Q_{\mathfrak{L},\mathfrak{D}} \sim \pm\tilde{Q}_{\mathfrak{L},\mathfrak{D}} & -Q_{\mathfrak{P}} \sim \tilde{Y}^{(2)}_{\mathfrak{K}} \\ Y^{(2)}_{\mathfrak{K}} \sim -\tilde{Q}_{\mathfrak{P}} & Y^{(2)}_{\mathfrak{L},\mathfrak{D}} \sim \pm\tilde{Y}^{(2)}_{\mathfrak{L},\mathfrak{D}} & -Y^{(2)}_{\mathfrak{P}} \sim \tilde{Y}^{(3)}_{\mathfrak{K}} \\ Y^{(3)}_{\mathfrak{K}} \sim -\tilde{Q}^{(2)}_{\mathfrak{P}} & Y^{(3)}_{\mathfrak{L},\mathfrak{D}} \sim \pm\tilde{Y}^{(3)}_{\mathfrak{L},\mathfrak{D}} & -Y^{(3)}_{\mathfrak{P}} \sim \tilde{Y}^{(4)}_{\mathfrak{K}} \end{array}$$

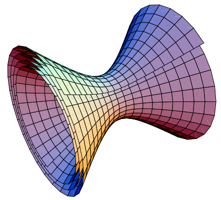
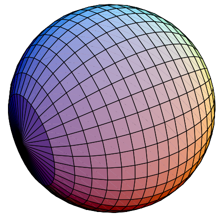
- Charges $Y^{(r)}_{\mathfrak{L},\mathfrak{D}}$ mapped to same level $\tilde{Y}^{(r)}_{\mathfrak{L},\mathfrak{D}}$,
- charges $Y^{(r)}_{\mathfrak{K}}$ mapped to $Y^{(r-1)}_{\mathfrak{P}}$ at lower level. $\tilde{Q}_{\mathfrak{P}}$ mapped to $Y^{(2)}_{\mathfrak{K}}$.

Superstrings on $AdS_5 \times S^5$

Super Coset Model

Superstrings on $AdS_5 \times S^5$ based on the coset

$$\frac{\mathrm{G}}{\mathrm{H}} = \frac{\widetilde{\mathrm{PSU}}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)} = AdS_5 \times S^5 \times \mathbb{R}^{0|32}.$$

 $\times$  $\times$ fermi.

Coset corresponds to $\mathbb{Z}_4$ automorphism

$$J = g^{-1} dg = J_0 + J_1 + J_2 + J_3, \qquad \Omega(J_k) = i^k J_k.$$

Gauge connection J_0, bosonic momenta J_2, fermions $J_{1,3}$. Action

$$S = \int \tfrac{1}{2} \mathrm{STr}\big(J_2 \wedge *J_2 + J_1 \wedge J_3\big).$$

Action is invariant under diffeomorphisms, kappa symmetry and local H.

Super-Poincaré Coordinates

Structure of superconformal algebra $\mathfrak{psu}(2,2|4)$:

$$\mathfrak{P}$$

$$\mathfrak{Q} \qquad \bar{\mathfrak{Q}}$$

$$\mathfrak{R}, \mathfrak{D}, \mathfrak{L}$$

$$\bar{\mathfrak{S}} \qquad \mathfrak{S}$$

$$\mathfrak{K}$$

$\mathbb{Z}_4$ automorphism: (sketch)

$$\begin{aligned} \Omega(\mathfrak{P}) &= \mathfrak{K}, & \Omega(\mathfrak{K}) &= \mathfrak{P}, \\ \Omega(\mathfrak{Q}) &= i\mathfrak{S}, & \Omega(\mathfrak{S}) &= i\mathfrak{Q}, \\ \Omega(\bar{\mathfrak{Q}}) &= i\bar{\mathfrak{S}}, & \Omega(\bar{\mathfrak{S}}) &= i\bar{\mathfrak{Q}}, \\ \Omega(\mathfrak{D}) &= -\mathfrak{D}, & \Omega(\mathfrak{R}) &= \pm\mathfrak{R}. \end{aligned}$$

- Use local H symmetry to gauge away $J_{\mathfrak{K}}$.
- Use kappa symmetry to gauge away half of fermions, e.g. $J_{\mathfrak{S}}, J_{\bar{\mathfrak{S}}}$.

Group element and expansion of Maurer–Cartan form (qualitatively)

$$g = \exp(x\mathfrak{P})\exp(\theta\bar{\mathfrak{Q}})\exp(\bar{\theta}\mathfrak{Q})\exp(\varphi\mathfrak{D})\exp(y\mathfrak{R}),$$

$$J = e^{-\varphi}(dx + \bar{\theta}d\theta)\mathfrak{P} + e^{-\varphi/2}yd\bar{\theta}\mathfrak{Q} + e^{-\varphi/2}yd\theta\bar{\mathfrak{Q}} + d\varphi\mathfrak{D} + dy\mathfrak{R}.$$

Maurer–Cartan form quadratic in θ: action quartic! [Kallosh Rahmfeld][Pesando hep-th/9808020]

Bosonic T-Duality

Action in super-Poincaré gauge (sketch)

$$S \sim \int \left(e^{-2\varphi}(dx + \bar{\theta} d\theta)^2 + e^{-\varphi} y d\theta^2 + e^{-\varphi} y d\bar{\theta}^2 + d\varphi^2 + dy^2\right).$$

Action quadratic in dx, perform T-duality on dx: [Alday Maldacena]

$$S \sim \int \left(e^{2\varphi} d\tilde{x}^2 + \tilde{x} d\bar{\theta} d\theta + e^{-\varphi} y d\theta^2 + e^{-\varphi} y d\bar{\theta}^2 + d\varphi^2 + dy^2\right).$$

- Fermionic terms are not similar to original action.
- T-duality maps D3-brane to smeared D-instanton.
- Action now quadratic in θ.
- Dilaton shifts by $+4\varphi$.

Fermionic T-Duality

Action after T-duality on dx (sketch)

$$S \sim \int \left(e^{2\varphi} d\tilde{x}^2 + \tilde{x}\, d\bar{\theta}\, d\theta + e^{-\varphi} y\, d\theta^2 + e^{-\varphi} y\, d\bar{\theta}^2 + d\varphi^2 + dy^2\right).$$

Action quadratic in $d\theta$, perform T-duality on $d\theta$: [Berkovits Maldacena]

$$S \sim \int \Big(e^{2\varphi} d\tilde{x}^2 + e^{\varphi} y (d\tilde{\theta} + \tilde{x}\, d\bar{\theta})^2 + e^{-\varphi} y\, d\bar{\theta}^2 + d\varphi^2 + dy^2\Big).$$

- T-duality could map smeared D-instanton back to D3-brane.
- Dilaton shifts by -4φ. Back to original value.

Quadratic Gauge

Finally invert $\varphi \to -\tilde{\varphi}$

$$S \sim \int \left(e^{-2\tilde{\varphi}} d\tilde{x}^2 + e^{-\tilde{\varphi}} y (d\tilde{\theta} + \tilde{x} d\bar{\theta})^2 + e^{\tilde{\varphi}} y d\bar{\theta}^2 + d\tilde{\varphi}^2 + dy^2 \right).$$

This action agrees with a different gauge of kappa symmetry: [Roiban Siegel]

$$g = \exp(\bar{\theta}\bar{\mathfrak{S}}) \exp(\tilde{x}\mathfrak{P}) \exp(\tilde{\theta}\mathfrak{Q}) \exp(\tilde{\varphi}\mathfrak{D}) \exp(y\mathfrak{R}),$$
$$J = e^{-\tilde{\varphi}} d\tilde{x}\mathfrak{P} + e^{-\tilde{\varphi}/2} y (d\tilde{\theta} + \tilde{x} d\bar{\theta})\mathfrak{Q} + e^{\tilde{\varphi}/2} y d\bar{\theta}\bar{\mathfrak{S}} + d\tilde{\varphi}\mathfrak{D} + dy\mathfrak{R}.$$

Compare with structure of superconformal algebra $\mathfrak{psu}(2,2|4)$:

$\mathfrak{P}$

$\mathfrak{Q}$ $\bar{\mathfrak{Q}}$

$\mathfrak{R}, \mathfrak{D}, \mathfrak{L}$

$\bar{\mathfrak{S}}$ $\mathfrak{S}$

$\mathfrak{K}$

Summary of Super-T-Self-Duality

Back to superstrings on $AdS_5 \times S^5$ in different gauge.

Super-T-Duality maps $AdS_5 \times S^5$ superstring to itself:

$$\begin{array}{ccc}
AdS_5 \times S^5 & \xrightarrow{\mathfrak{S}\bar{\mathfrak{S}}\mathfrak{K}\ \text{gauge}} & \text{super-Poincaré gauge} \\
\bar{\mathfrak{Q}}\mathfrak{S}\mathfrak{K} \downarrow \text{gauge} & & \updownarrow \mathrm{T}(dx) \\
\text{quadratic gauge} & \xleftrightarrow{\mathrm{T}(d\theta)} & \text{D-instanton}
\end{array}$$

Super-T-self-duality maps Maurer–Cartan forms as follows

$$\tilde{J}_{\mathfrak{P}} = *J_{\mathfrak{P}}, \quad \tilde{J}_{\mathfrak{D}} = -J_{\mathfrak{D}}, \quad \tilde{J}_{\mathfrak{Q}} = iJ_{\mathfrak{Q}}, \quad \tilde{J}_{\bar{\mathfrak{S}}} = \Omega(J_{\bar{\mathfrak{Q}}}), \quad \tilde{J}_{\mathfrak{R}} = \Omega(J_{\mathfrak{R}}).$$

Maurer–Cartan equations and equations of motion are dual.

What about Lax connection and integrable structure?

Lax Connection

Supercoset model has Lax connection [Bena, Polchinski, Roiban][NB, Kazakov, Sakai, Zarembo]

$$A(z) = J_0 + \tfrac{1}{2}(z^2 + z^{-2})J_2 + \tfrac{1}{2}(-z^2 + z^{-2}){*J_2} + zJ_1 + z^{-1}J_3.$$

Bosonic part dual as before. Fermionic part of Lax connection and dual:

$$A_{\mathrm{F}}(z) = \tfrac{1}{2}(z + z^{-1})\big(J_{\mathfrak{Q}} + J_{\bar{\mathfrak{Q}}}\big) + \tfrac{1}{2}(z - z^{-1})\big(-i\Omega(J_{\mathfrak{Q}}) - i\Omega(J_{\bar{\mathfrak{Q}}})\big),$$
$$\tilde{A}_{\mathrm{F}}(z) = \tfrac{1}{2}(z + z^{-1})\big(+iJ_{\mathfrak{Q}} + \Omega(J_{\bar{\mathfrak{Q}}})\big) + \tfrac{1}{2}(z - z^{-1})\big(\Omega(J_{\mathfrak{Q}}) + iJ_{\bar{\mathfrak{Q}}}\big).$$

Requires the following transformation [NB, Ricci, Tseytlin, Wolf]

$$\tilde{A}(z) = \left(\frac{z + z^{-1}}{z - z^{-1}}\right)^{\mathfrak{D}+\mathfrak{B}} \Omega\big(A(z)\big) \left(\frac{z - z^{-1}}{z + z^{-1}}\right)^{\mathfrak{D}+\mathfrak{B}}.$$

Mapping of Local and Non-Local Charges

Expected mapping of local and non-local charges

$$Q^{(r)}_{\mathfrak{P}} \sim \tilde{Q}^{(r-1)}_{\mathfrak{K}}$$

$$Q^{(r)}_{\mathfrak{Q}} \sim \tilde{Q}^{(r-1)}_{\mathfrak{S}} \qquad Q^{(r)}_{\bar{\mathfrak{Q}}} \sim \tilde{Q}^{(r\pm 0)}_{\bar{\mathfrak{S}}}$$

$$Q^{(r)}_{\mathfrak{R},\mathfrak{D},\mathfrak{L}} \sim \tilde{Q}^{(r\pm 0)}_{\mathfrak{R},\mathfrak{D},\mathfrak{L}}$$

$$Q^{(r)}_{\bar{\mathfrak{S}}} \sim \tilde{Q}^{(r\pm 0)}_{\bar{\mathfrak{Q}}} \qquad Q^{(r)}_{\mathfrak{S}} \sim \tilde{Q}^{(r+1)}_{\mathfrak{Q}}$$

$$Q^{(r)}_{\mathfrak{K}} \sim \tilde{Q}^{(r+1)}_{\mathfrak{P}}$$

Confirmed for local charges. [Berkovits Maldacena]

Conclusion: Superconformal and dual superconformal symmetry are two partially overlapping superconformal algebras in the integrable structure.

Non-planar corrections violate integrability:
Dual superconformal symmetry is probably violated as well.

Conclusions

Conclusions

⋆ **Super-T-Self-Duality**

- Superstrings on $AdS_5 \times S^5$ self-dual under super-T-duality.
- Symmetries of dual model map into novel dual symmetries (non-local).
- Integrable structure maps to itself (modulo automorphism).
- Conformal and dual conformal symmetry part of integrable structure.

⋆ **Open Problems**

- Apply integrability to the construction of scattering amplitudes:
  Can we compute finite remainder function $F(p, \lambda)$ efficiently?
- Action of bosonic T-duality well-understood: D-branes.
  How does fermionic T-duality act on D-branes in general?