Having the World on a String

(toying with de Sitter space and inflation)

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based on

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- Liouville gravity & 2d de Sitter space
- Closed string tachyons
- 2d models of inflation
- de Sitter thermodynamics
- Quantum de Sitter
- Localized tachyons
- Delusions of grandeur

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Introduction

The theoretical foundations of quantum cosmology are poorly understood. The appropriate ingredients are a subject of much debate and speculation. In trying to understand basic phenomena in quantum field theory, it often helps to find a class of solvable examples in order to build up intuition, for instance much of the structure of the duality revolution in string theory (D-branes, matrix models, holography, *etc*) was foreshadowed in the arena of solvable models of $2d$ gravity. One might hope that solvable models of de Sitter quantum gravity and inflation have a similar useful role to play.

The existing state of the art in quantum cosmology involves $0+1d$ minisuperspace QM. We need a spatial dimension in order to model the effects of spatial scales (*e.g.* structure formation, inhomogeneous matter, *etc.*).

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Liouville gravity

Liouville theory is the appropriate $2d$ model of pure (anti) de Sitter gravity. In the classical limit, its equation of motion is

$$R = \mu \, .$$

For $\mu < 0$ one has AdS geometry; $\mu > 0$ gives dS.

All $2d$ metrics are $g = e^{\gamma\phi}\hat{g}$ up to diffeomorphism. The equation $R = \mu$ is then $-\hat{\nabla}^2\phi = \frac{\mu}{\gamma}e^{\gamma\phi}$; the general classical solution can be expressed locally as

$$e^{\gamma\phi} = -\frac{8}{\mu}\frac{\partial A(x^+)\partial B(x^-)}{[A(x^+) - B(x^-)]^2} \, ,$$

where $x^\pm = \tau \pm \sigma$.

Simple spatially homogeneous (σ-indep) solutions correspond to simple homogeneous cosmologies. For example, let us consider $A = e^{i\epsilon x^+}$, $A = x^+$, or $A = e^{\varepsilon x^+}$ (and similarly for $B(x^-)$):

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For the dS case $\mu > 0$, these yield three spatially homogeneous solutions:

$$e^{\gamma\phi} = \begin{cases} \dfrac{8}{\mu}\dfrac{\epsilon^2}{\sin^2(\epsilon\tau)} & , \quad E_L = -\dfrac{\epsilon^2}{2\gamma^2} < 0 \\[3mm] \dfrac{8}{\mu}\dfrac{1}{\tau^2} & , \quad E_L = 0 \\[3mm] \dfrac{8}{\mu}\dfrac{\varepsilon^2}{\sinh^2(\varepsilon\tau)} & , \quad E_L = \dfrac{\varepsilon^2}{2\gamma^2} > 0 \end{cases}$$

corresponding to the classes of motion in the exponential potential provided by the cosmological term $\frac{\mu}{\gamma^2}e^{\gamma\phi}$,
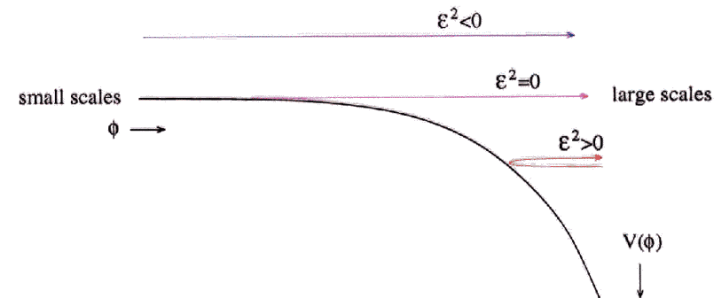
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The constant curvature geometry is unique. Each of the above solutions covers a portion of the global conformal diagram of de Sitter space:
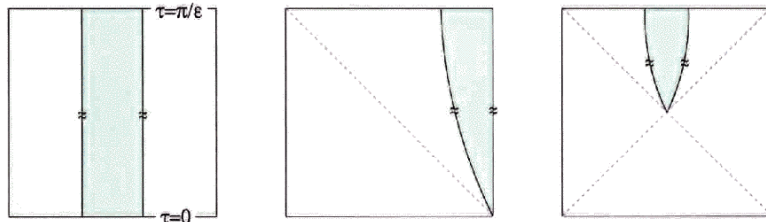
The identifications indicated arise if we impose a closed universe topology $\sigma \sim \sigma + 2\pi$. This also leads to a Casimir shift

$$E_L \to E_L - \frac{Q^2}{8} - 1 = E_L - \frac{c-1}{24} \ .$$

The elliptic solution describes eternal de Sitter space in 'global' coordinates; the parabolic solution describes eternal de Sitter space in 'flat' coordinates, and the hyperbolic one describes a universe which has a past Milne-type singularity at $\tau = -\infty$ and expands into an asymptotically de Sitter future as $\tau \to 0$.

Being two-dimensional gravity, there is a dual interpretation of the Liouville + matter action

$$\mathcal{S}_L = \frac{1}{4\pi} \int \sqrt{-\hat{g}} \left(-\tfrac{1}{2}(\hat{\nabla}\phi)^2 - \tfrac{Q}{2}\hat{R}\phi + \tfrac{1}{2}(\hat{\nabla}\vec{X})^2 - \tfrac{\mu}{\gamma^2} e^{\gamma\phi} \right)$$

as the worldsheet description of a test string propagating in a background

$$\mathcal{G}_{\mu\nu} = \eta_{\mu\nu}$$
$$\mathcal{D} = -\tfrac{1}{2}Q\phi \ , \qquad (Q=\sqrt{\tfrac{1}{3}(c-25)})$$
$$\mathcal{T} = \tfrac{\mu}{\gamma^2} e^{\gamma\phi}$$

in string theory in $c \geq 25$ target space dimensions.

The (log) scale factor ϕ is the target time coordinate X^0; the spatial target coordinates X^i play the role of (conformal) matter fields coupled to this dynamical metric. The negative metric for the conformal mode is a precise analogue of the situation in higher dimensional gravity.
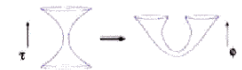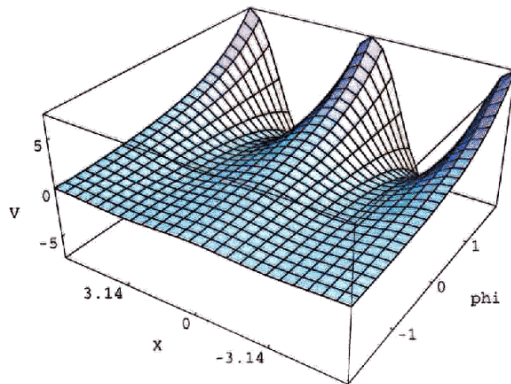
Inflation

Tachyon condensates that are inhomogeneous in target space describe gravitationally dressed matter potentials; suitable choices of matter potential give rise to interesting two dimensional models of inflation. Consider, for instance, a gravitationally dressed cosine matter potential – the Liouville-Sine-Gordon model

$$S_{\mathrm{LSG}} = \frac{1}{4\pi} \int d\tau d\sigma \sqrt{\hat{g}} \left(-\frac{1}{2}(\hat{\nabla}\phi)^2 - \frac{1}{2}Q\hat{R}\phi \right.$$

$$\left. +\frac{1}{2}(\hat{\nabla}X)^2 - \frac{\lambda}{\alpha^2} e^{\alpha\phi} \cos kX \right)$$

To compare the dynamics of this model with that of inflation, define the 'proper time' $dt = e^{\alpha\phi/2}d\tau \equiv \mathrm{a}d\tau$, the 'Hubble parameter' $H = \dot{\mathrm{a}}/\mathrm{a}$, and the matter potential $V = \frac{\lambda}{\alpha^2}\cos(kX)$.

Then the equations of motion and the Hamiltonian constraint become (for spatially homogeneous fields, and setting $\hat{R} = 0$)

$$H^2 = \frac{\alpha^2}{2(d-1)} \left(\frac{1}{2}\dot{X}^2 + V(X) - \mathrm{a}^{-2}(\frac{c-1}{24}) + \rho_\perp \right)$$

$$\dot{H} = -\frac{\alpha^2}{2} \left(\frac{1}{2}\dot{X}^2 - \mathrm{a}^{-2}(\frac{c-1}{24}) + \rho_\perp + P_\perp \right)$$

$$0 = \ddot{X} + (d-1)H\dot{X} + V'(X)$$

where for us $d = 2$. For d dimensions, these are precisely the Friedmann equations of a homogeneous cosmology if we substitute $\frac{1}{2}\alpha^2 \rightarrow 8\pi G_d$, i.e. the Liouville coupling plays the role of Newton's constant.[1]

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[1]Note that the zero-point energy $-\frac{1}{8}Q^2 - 1$ plays the role of spatial curvature. The sign is appropriate to a closed spatial universe in higher dimensions.

The conditions for slow-roll inflation are then that the dimensionless parameters

$$\epsilon \equiv \frac{1}{2}\frac{2}{\alpha^2}\left(\frac{V'}{V}\right)^2 = (k/\alpha)^2\tan^2 kX$$

$$\eta \equiv \frac{2}{\alpha^2}\left(\frac{V''}{V}\right) = 2(k/\alpha)^2$$

have magnitude much smaller than one. Both are well satisfied for $k << \alpha$ and an initial matter distribution that starts near the top of the matter potential, $X \sim 0$.

For $k = \alpha$, the theory is exactly solvable — the action separates into a Liouville theory for $\Phi \equiv \phi + iX$ plus that of its complex conjugate:

$$\mathcal{S} = \frac{-1}{4\pi}\int d\tau d\sigma \sqrt{-\hat{g}}\left(\frac{1}{4}(\hat{\nabla}\Phi)^2 + \frac{1}{4}Q\hat{R}\Phi + \frac{\lambda}{2\alpha^2}e^{\alpha\Phi}\right) + \text{c.c.}$$

The classical equations of motion thus have the general solution $e^{\alpha\Phi} = -\frac{8}{\lambda}\frac{A'B'}{(A-B)^2}$, where now we allow the functions $A(x^+)$ and $B(x^-)$ to be generically complex.
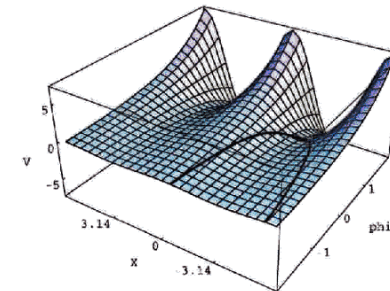
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For example, the zero energy solution

$$e^{\alpha\Phi} = \frac{2}{\lambda}\frac{1}{(\tau+ib)^2}$$

$$e^{\alpha\phi/2} = a(\tau) = \sqrt{\frac{2}{\lambda(\tau^2+b^2)}}$$

$$\tan(\alpha X/2) = -b/\tau$$

describes a scalar field X that starts off at the top of its cosine potential at $\tau = -\infty$, rolls down to the potential minimum at $\tau = 0$, and then climbs back up to the top of the potential as $\tau \to \infty$.

As this occurs, the geometry is nearly an expanding de Sitter space for large negative τ; it reaches a maximum scale factor $e^{\alpha\phi} = \frac{2}{\lambda}b^{-2}$ at $\tau = 0$, at which point it enters a contracting phase and returns to zero scale factor as $\tau \to +\infty$.

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Topological inflation

Another scenario for inflation uses topological defects as seeds for 'eternal' inflation. Since a scalar field at the core of such a defect is pinned at the maximum of its potential, if the characteristic size of the defect exceeds the Hubble scale, the interior of the defect will inflate.

The Liouville-Sine-Gordon theory provides a simple model of this sort as well — we simply compactify the scalar X on a circle of radius $2/k$ (in string units) and consider winding strings. A prototypical solution of this sort in the $\alpha = k$ model has

$$A(x^+) = e^{(\varepsilon - iw)x^+} , \quad B(x^-) = e^{(-\varepsilon - iw)x^-} ,$$
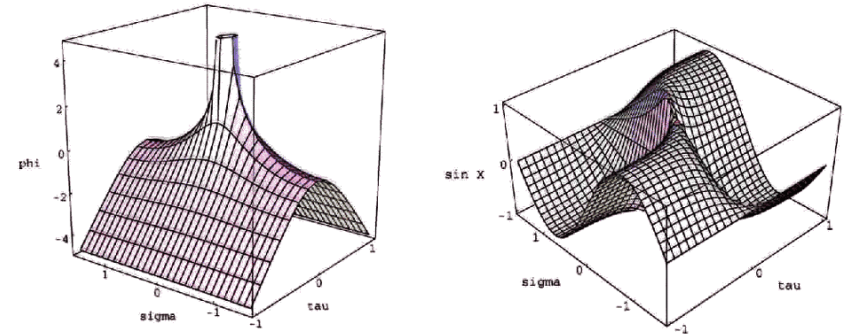
leading to

$$\exp[\alpha\Phi] = \frac{2}{\lambda} \frac{\varepsilon^2 + w^2}{\sinh^2(\varepsilon\tau - iw\sigma)} .$$

At large τ, one sees that the imaginary part $X = \operatorname{Im}\Phi$ winds w times.

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The plots of ϕ and X (for $w = 1$)

show that this solution describes a pair of oppositely wound strings which annihilate, leaving an unwound string perched at the top of the cosine potential. It is interesting to note that, even though we naively start off with a coordinate domain $\mathbb{R} \times S^1$ that we wish to interpret as a single closed universe cosmology, the actual dynamics grows a second 'child universe' that absorbs the domain wall core. The exterior of the domain wall recollapses, as will the child universe after fluctuations drive X off its potential maximum.

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Entropy

Bousso has argued that the gravitational entropy of de Sitter space should bound the 'observable' entropy of matter in a space that is asymptotically de Sitter in both the past and future. The basic idea is that extracting too much entropy from the de Sitter horizon of a static observer requires putting too much energy into the spacetime, and hence a cosmological singularity develops either in the past or future. Let us see how this argument works in two dimensions.

Global de Sitter space is the vacuum $\epsilon = 1$ of the negative energy Liouville solution, where the matter fields are also unexcited. As we populate the levels of the matter system, ϵ decreases due to the Hamiltonian constraint
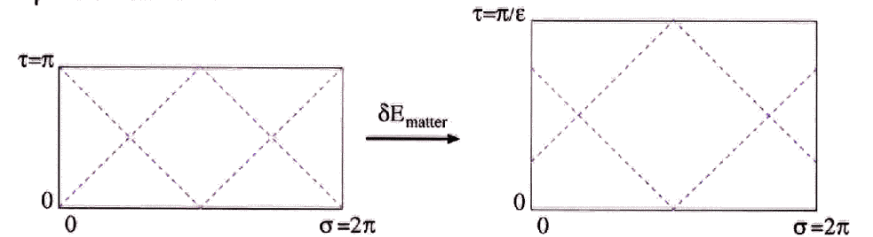
$$\frac{\epsilon^2}{2\gamma^2} + E_{\mathrm{matter}} - \frac{c-1}{24} = 0$$
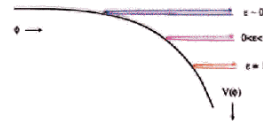
until it reaches zero at $E_{\mathrm{matter}} = \frac{c-1}{24}$. At this point the geometry is on the cusp of having a singularity, and indeed there will be a big crunch (Milne singularity) if we add more matter energy, since the solutions will cross over into the positive energy class.

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Note that the addition of matter makes the de Sitter space 'taller':

Higher-dimensional gravity also exhibits this property (Gao and Wald).

Due to the exponential growth in their density of levels, the total number of matter states in the class of geometries which is asymptotically de Sitter both in the past and in the future is bounded by the density of matter levels at $E_{\mathrm{matter}} = \frac{c-1}{24}$; in this way, one reaches an analogue of Bousso's 'N-bound'

$$S_{\mathrm{matter}} = 4\pi\sqrt{\frac{c}{6}E_{\mathrm{matter}}} \leq \frac{\pi}{3}c$$

If we take $A_{\mathrm{hor}} = 2$ the 'area' of an S^{d-2} as $d \to 2$, and[2] $G = \gamma^2/16\pi$ the effective Newton's constant, then one finds $S_{\mathrm{grav}} = A_{\mathrm{hor}}/4G = \frac{2\pi}{3}c$. The factor of two is not understood, but indeed $S_{\mathrm{matter}} < S_{\mathrm{grav}}$.

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[2]For instance, as suggested by the Friedman equations.

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Quantum de Sitter space

We wish to construct de Sitter Liouville partition/correlation functions corresponding to the classical geometries
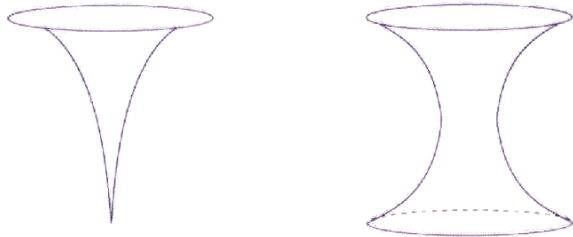
The first is a punctured disk, with the wavefunction at the big bang/crunch corresponding to a vertex operator in the combined Liouville + matter system. The other, asymptotically dS region is a conformal boundary, and so should correspond to a boundary state in Liouville. Similarly, the second geometry is an annulus diagram with appropriate asymptotically dS conformal boundary states in Liouville.

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The appropriate boundary state was constructed by the brothers Zamolodchikov, who built a family of states associated to Virasoro highest weight representations of conformal dimension

$$h_{mn} = -\tfrac{1}{8}Q^2 + \tfrac{1}{8}(2m/\gamma - n\gamma)^2$$

having null vector at level mn. The choice $m = n = 1$ (*i.e.* $h_{mn} = 0$) has the null vector $L_{-1}|\Psi_{1,1}\rangle = 0$. The associated boundary state is asymptotically SL(2) invariant; all vertex operators approach the identity operator when taken to the boundary. Note that in two dimensions, the AdS isometry $SO(d-1,2)$ and the dS isometry $SO(d,1)$ are both $SO(2,1) = SL(2,\mathbb{R})$. The Zamolodchikovs thought of the $m = n = 1$ state as implementing the conformal boundary of AdS; here we use it for dS.

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The annulus partition function appropriate to asymptotic dS boundary conditions in both past and future is then

$$Z = \int_0^\infty d\tau \; Z_{\text{matter}} \cdot Z_{\text{ghost}} \cdot Z_{\text{Liouville}}$$

$$= \int d\tau \; \chi_{\text{matter}}(q) \cdot \eta^2(q) \cdot \int d\nu \; \frac{q^{\nu^2/2}}{\eta(q)} \, 2\sin(2\pi\nu/\gamma)\sin(\pi\nu\gamma)$$

(where as usual $q = e^{-2\pi\tau}$). An appropriate matter partition function is the character $\chi_{\text{matt}} = \frac{q^{E_m}}{[\eta(q)]^c}$ of the Ishibashi state $|B_\Psi\rangle$ associated to the matter highest weight state $|\Psi_{\text{matt}}\rangle$ of interest. The integral over the annulus modulus τ and the Liouville momentum $\nu = \epsilon/\gamma$ then has a double saddle at

$$\hat\tau \simeq \frac{i}{\epsilon} \;, \qquad \hat\nu^2 = \frac{\epsilon^2}{\gamma^2} = \frac{c_m - 1}{12} - 2E_m \;.$$

The saddle in τ reproduces the classical 'height' of the conformal diagram, while the saddle in ν is the classical Hamiltonian constraint.

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- The $m = n = 1$ Liouville boundary state is SL(2) invariant – for Euclidean AdS it describes the quantum boundary of the Poincare disk; for Lorentzian dS it describes de Sitter asymptotia. So in contrast to some recent claims in the literature, dS quantum gravity – at least in $d = 2$ – is *not* a quantum mechanics of a finite number of degrees of freedom (it's a rather conventional conformal QFT), and it *does* respect the classical dS isometries at the quantum level.

- Approaching the threshold at $\epsilon = 0$ of the cosmological singularity, $\hat\tau \to \infty$ and naively the annulus pinches off to make a crunch/bang geometry

$\epsilon^2 > 0$ $\epsilon^2 < 0$

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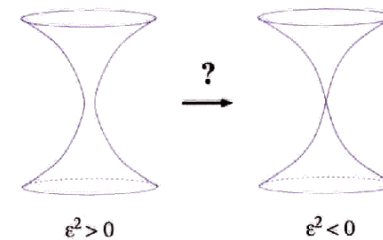
However, the width of the saddle blows up in the limit $\epsilon \to 0$, and a more careful analysis of the integrals involved indicates that one cannot push the geometry through such a crunch.

- As another example of the utility of explicit constructions, one can examine the effect of replacing the standard matter vacuum by some exotic state, such as the 'α-vacua' which have been considered as possible alternative dS invariant states, which might exhibit exotic 'trans-Planckian' effects on inflationary fluctuations. These states are defined by a Bogolubov rotation of the standard vacuum

$$|\alpha\rangle = \exp\left[\frac{1}{2}\tanh\alpha \sum_{n>0}(a^2_{-n} + \tilde{a}^2_{-n})\right]|0\rangle \ ,$$

so e.g. $(\text{ch}\alpha\, a_n - \text{sh}\alpha\, a_{-n})|\alpha\rangle = 0$. Unfortunately, these states do not lie in any sensible Virasoro representation, so it seems difficult to impose BRST (gauge) invariance. It is doubtful that they can be consistently coupled to Liouville gravity.

- One of our motivations for constructing solvable models of inflation was to test various proposals such as the eternal/chaotic inflationary universe, where quantum fluctuations of the inflaton drive the volume-weighted average of the field to climb its potential. The domains of large proper volume so obtained occupy a vanishingly small (conformal) coordinate domain, which is compensated by a huge conformal factor in that region, leading to the typical fractal conformal diagram of the late-time structure of eternal/chaotic inflation
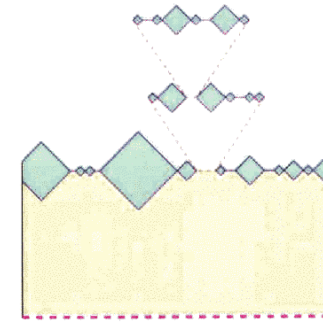
But how are we to distinguish these violent fluctuations of the conformal factor from the standard fluctuations of a quantum field on all scales, that we are accustomed to regularizing and renormalizing away?

The essential driving force behind this violent oscillation of the scale factor is the negative metric on the kinetic energy of the scale factor. In two-dimensional models, most if not all of the fluctuations of the scale factor (Liouville mode) are gauge artifacts, which are eliminated by the BRST (Hamiltonian) constraints on the Wheeler-DeWitt wavefunction.

The physical state space has positive metric, and the scale factor is determined by the matter state. Violent fluctuations of the scale factor would have to be built in by the choice of highly excited matter states, which is hardly the appropriate starting point for inflation. Put differently, finite entropy of the matter Hilbert space at fixed energy is incompatible with the arbitrarily large entropy production of eternal inflation.

However, if we abandon the single-universe interpretation of 2d inflationary cosmology in favor of a multiverse interpretation (which might be forced on us by the quantum theory), a mechanism similar to eternal/chaotic inflation appears − there is rampant string production (universe creation) from the exponential time dependence of the background.

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Localized tachyons

The multiverse interpretation of 2d cosmology can be thought of in string terms. The 2d cosmological constant is a closed string tachyon condensate, whose unlimited exponential growth is unrealistic; the tachyon condensate rapidly grows so large that one cannot ignore its back-reaction on the target geometry. Ultimately, one will need to know the late-time state that the condensate evolves toward. The fate of generic closed string tachyon condensates in string theory is an open question.

However, certain *localized* closed string tachyon condensates are understood in the superstring, in the critical dimension $d = 10$. The test string which is our two-dimensional cosmology can be situated in the target space in a region of localized instability, where it experiences inflation during the finite period of time when the tachyon condenses; away from this region of field space, string theory is stable.

Note that the potential landscape experienced by a 2d universe depends on the epoch of scale factor (time) under consideration.

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The localized closed string tachyon condensation leads to a stable remnant, or flat spacetime, with an outgoing pulse of radiation.

radiation

unstable defect

produced strings
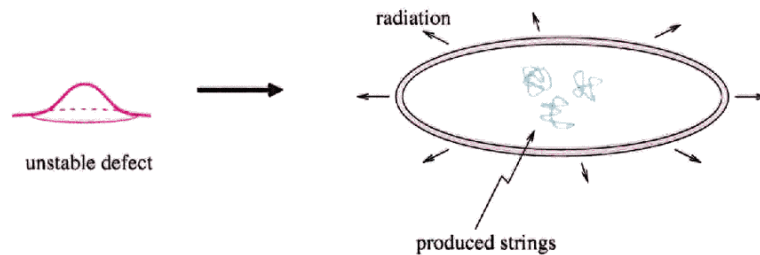
As viewed by a test string near the origin, this process would appear as an exponential potential for the timelike Liouville field coupled to a localized matter perturbation, which then relaxes at finite ϕ (*i.e.* time) to a smooth final configuration. In other words, the $2d$ cosmological constant dynamically relaxes to zero at large scale factor ϕ! We thus find an explicit and self-consistent realization of a 'baby universe' mechanism (*à la* Coleman) for relaxing the cosmological constant.

There is also string (universe) pair production, which is large due to the exponential growth in the density of string states. One might expect most non-empty universes to arise from such quantum processes.

Entertaining fantasies

It would certainly be interesting if such a scenario could be exhibited in more realistic models. There are features of string theory that parallel what we have found here. In known string theory constructions, de Sitter space is metastable. The qualitative structure of the configuration space of string theory is the same as in our two-dimensional model – the 'tachyon' (the cosmological constant) is 'localized' in field space; there always seem to be one or more 'flat directions to infinity', with the nontrivial potential only in some finite region of the low-energy field space.[3]

V_{eff}

'Localized Tachyon'

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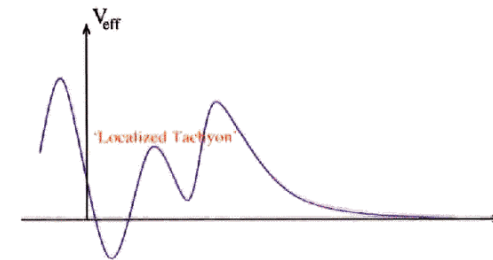
[3]Thus the 'Dine-Seiberg problem' of string theory is a feature, not a bug!

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In other words, the instability to exponential growth of the scale factor only exists in a localized region of field space, much as in the $2d$ model of localized tachyons.   Topology change is expected to occur in string theory; could topology change result in 'universal pair production'?   Does this idea explain why the typical universe at large scale emerged from an inflationary phase?   And why the cosmological constant has relaxed to a small value today? And ... ?

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