The Minimal Length

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Based on

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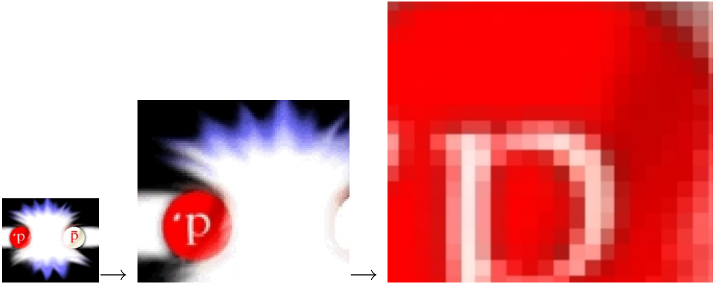
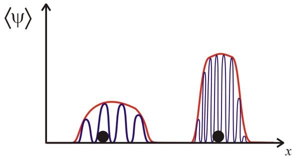
The Minimal Length Scale

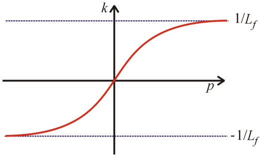


- Very general expectation for quantum gravity: fluctuations of spacetime itself disable resolution of small distances
- Can be found e.g. in String Theory, Loop Gravity, NCG, etc.
- Minimal length scale acts as UV cutoff
- Large extra dimensions lower the Planck scale $\longrightarrow$ LHC

Finite Resolution of Structures



- For large momenta, p, Compton-wavelength $\lambda = 1/k$ can not get arbitrarily small $\lambda > L_{\rm min} = 1/M_{\rm f}$



Quantum Mechanics with a Minimal Length

- Modify wave-vector k and commutation relations
  $k = k(p) = \hbar p + a_1 p^3 + a_2 p^5 \ldots \Rightarrow [p_i, x_j] = i \partial p_i / \partial k_j$
- Results in a generalized uncertainty principle

$$\Delta x \Delta p \geq \frac{1}{2} \hbar \left(1 + b_1 L_{\rm min}^2 \langle p^2 \rangle \right)$$

- And a squeezed phase space at high energies

$$\langle p | p' \rangle = \frac{\partial p}{\partial k} \delta(p - p') \Rightarrow dk \to \hbar \; dp \; \frac{\partial k}{\partial p} = \hbar \; dp \; e^{-L_{\rm min}^2 p^2}$$

- Can but need not have a varying speed of light $d\omega / dk \neq 1$.

Quantisation with a Minimal Length

Parametrization in $p_\nu = f_\nu(k)$ – input from underlying theory.

- Quantize via $k \to -\mathrm{i}\partial$, $p \to f(-\mathrm{i}\partial) := F(\partial)$
- The Klein-Gordon equation, alias modified dispersion relation

$$E^2 - p^2 = m^2 \quad \Rightarrow \quad F^\nu(\partial) F_\nu(\partial)\psi = m^2\psi$$

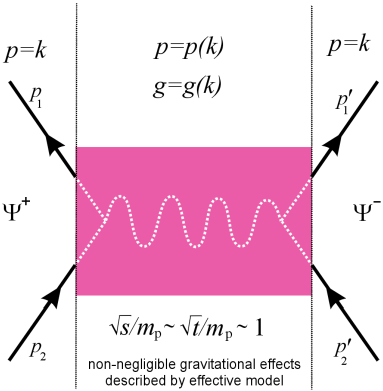
- The Dirac equation

$$(\not{F}(\partial) - m)\psi \quad = \quad 0$$

- (Anti)-commutation relations

$$\left[\hat{a}^\dagger(p), \hat{a}(p')\right]_\pm \quad = \quad \delta(\vec{p} - \vec{p}\,')\left|\frac{\partial p}{\partial k}\right|$$

The Collision Region



Quantisation with a Minimal Length

- Lagrangian for free fermions

$$\mathcal{L}_f = \mathrm{i}\overline{\Psi}(\not{F}(k) - m)\Psi \qquad \mathcal{L}_f = \mathrm{i}\overline{\Psi}(g^{\nu\kappa}(k)\gamma_\nu k_\nu - m)\Psi$$

- Coupling of the gauge field via $\partial_\nu \to D_\nu := \partial_\nu - \mathrm{i}eA_\nu$ yields the gauge- and Lorentz-invariant higher order derivative interaction

$$\mathcal{L} = \bar{\Psi}\not{F}(D)\Psi \qquad \mathcal{L} = \bar{\Psi}\gamma_\nu g^{\nu\kappa}(D)D_\kappa\Psi$$

- To first order one finds the usual $\mathcal{L} = \mathcal{L}_f - e\overline{\Psi}\eta^{\kappa\nu}\gamma_\kappa A_\nu\Psi + \mathcal{O}(eL_{\mathrm{min}}^2)$ and the dominant modification comes from the propagators

$$(\not{F}(k) - m)^{-1} \qquad (g^{\nu\kappa}(k)\gamma_\nu k_\kappa - m)^{-1}$$

$$(F^\nu(k)F_\nu(k) - m^2)^{-1} \qquad (g^{\nu\kappa}(k)k_\nu k_\kappa - m^2)^{-1}$$

Applications of the Model

The model is useful to examine effects of a minimal length scale

- Modified quantum mechanics:

$\longrightarrow$ Schrödinger's equation, levels in hydrogen atom, g-2, Casimir-effect

- Derivation of modified Feynmann-rules:

$\longrightarrow$ General prescription for calculations

- Tree-level cross-sections (e.g. $e^+e^- \rightarrow f^+f^-$):

$\longrightarrow$ Show overall suppression relative to SM-result

- Loop-contributions (e.g. running coupling):

$\longrightarrow$ Finite, minimal length acts as UV-regulator

Relation to Deformed Special Relativity

- Minimal length $L_{\rm min}$ requires new Lorentz-transformations
- New transformations have 2 invariants: c and $L_{\rm min}$
- Generalized Uncertainty $\Longleftrightarrow$ Deformed Special Relativity
  - When relation $k(p)$ is known and p's (usual) transformation, then also the transformation of k is known.
  - When the new transformation on k is known, then one gets $k(p)$ by boosting in and out of the restframe where $k = p$.

SH, Class. Quantum Grav. 23 (2006) 1815.

Deformed, Non-linear Action on Momentum Space

- Lorentz-algebra remains unmodified

$$[J^i, K^j] = \varepsilon^{ijk} K_k \ , \ [K^i, K^j] = \varepsilon^{ijk} K_k \ , \ [J^i, J^j] = \varepsilon^{ijk} J_k$$

- But it acts non-linearly on momentum space, e.g.*

$$e^{-iL_{ab}\omega^{ab}} \to U^{-1}(p_0) e^{-iL_{ab}\omega^{ab}} U(p_0) \quad \text{with} \quad U(p_0) = e^{L_{\min} p_0 p_a \partial p^a}$$

- Leads to Lorentz-boost (z-direction)

$$p'_0 = \frac{\gamma(p_0 - vp_z)}{1 + L_{\min}(\gamma - 1)p_0 - L_{\min}\gamma v p_z}$$
$$p'_z = \frac{\gamma(p_z - vp_0)}{1 + L_{\min}(\gamma - 1)p_0 - L_{\min}\gamma v p_z}$$

which transforms $(1/L_{\min}, 1/L_{\min}) \to (1/L_{\min}, 1/L_{\min})$

*Magueijo and Smolin, Phys. Rev. Lett. **88**, 190403 (2002).

Interpretation of an Invariant Minimal Length

Besides c there is a second invariant L_{min} for all observers

- Standard DSR approach
  - * DSR applies for each observer to agree on minimal-ness ... ?
  - * Therefore deformed transformation applies to free particles
  - * If caused by quantum gravity effects what sets the scale?
- My DSR approach
  - * Two observers can not compare lengths without interaction
  - * The strength of gravitational effects sets the scale for the importance of quantum gravity
  - * Free particles do not experience any quantum gravity or DSR
  - * Effects apply for particles in the interaction region only

$\longrightarrow$ Propagator of exchange particles is modified

Problems of DSR

The Soccer-Ball-Problem

* Usual DSR: A particle has an upper bound on its energy. There better were no such bound for composite objects. How to get the correct multi-particle limit for DSR-trafos?
* My DSR: Composite objects don't experience anything funny as long as the gravitational interaction among the components is weak.

The Conservation-Law Problem

* Usual DSR: the physical momentum p is the one that experiences DSR. But then p is not additive and $f(p_1 + p_2) \neq f(p_1) + f(p_2)$. Which quantity is to be conserved in multi-particle interactions[*]?
* My DSR: the physical momenta p_1, p_2 are the asymptotic momenta, they transform, are added, and are conserved in the usual way[†].

[*] Judes and Visser, Phys. Rev. D **68**, 045001 (2003) [arXiv:gr-qc/0205067].
[†] SH, Phys. Rev. D **73** 105013 (2006) [arXiv:hep-th/0603032]

Summary

- Effective theory that allows to examine phenomenology
- Related to DSR but different interpretation of free particles
- Allows formulation of QFT
- With more investigation, it should be possible to further classify and constrain the model