Solvable model for a dynamical quantum phase transition from fast to slow scrambling

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Designer Quantum Systems Out of Equilibrium, KITP
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Work done with Ehud Altman (UC Berkeley)

SB & E. Altman, arXiv:1610.04619

Sachdev-Ye-Kitaev (SYK) model

Sachdev & Ye, PRL (1993)
Kitaev, KITP (2015)
Sachdev, PRX (2015)

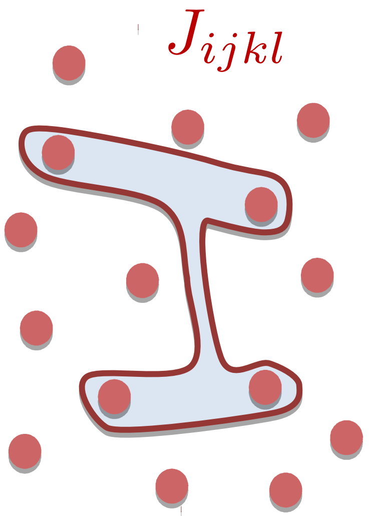
$$\mathcal{H}_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l - \mu \sum_{i=1}^{N} c_i^\dagger c_i$$

- Solvable for large *N*

$$P(J_{ijkl}) \sim e^{-\frac{|J_{ijkl}|^2}{J^2}}$$

- Emergent conformal symmetry at low-energy
- 'Maximally chaotic'
  Quantum chaos or 'scrambling' with Lyapunov exponent,

$$\lambda_L = 2\pi T$$



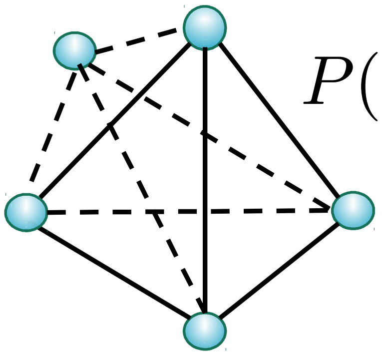
'Upper bound' to quantum chaos as in a black hole

Maldacena, Shenker & Stanford (2016)

A. Kitaev → Solvable model for holography

Contrast with quadratic infinite range model (model for quantum dot)

$$\mathcal{H} = \frac{1}{\sqrt{N}} \sum_{ij} t_{ij} c_i^\dagger c_j$$



$$P(t_{ij}) \sim e^{-\frac{|t_{ij}|^2}{t^2}}$$

- Fermions occupying states of a *N X N* random matrix.
- No thermalization or chaos in the many-body sense.

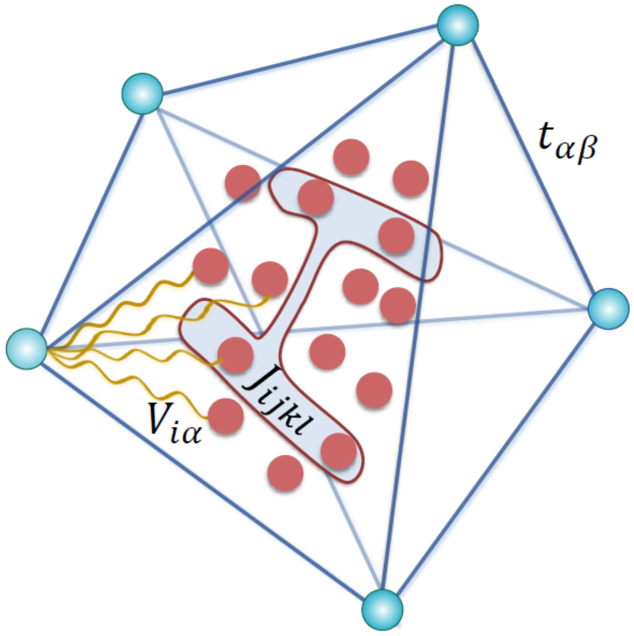
Add weak interaction ➡

Fermi liquid state at infinite *N*. Quasi-particle lifetime $\tau \sim 1/T^2$

Expectation, Lyapunov exponent $\lambda_L \sim T^2$

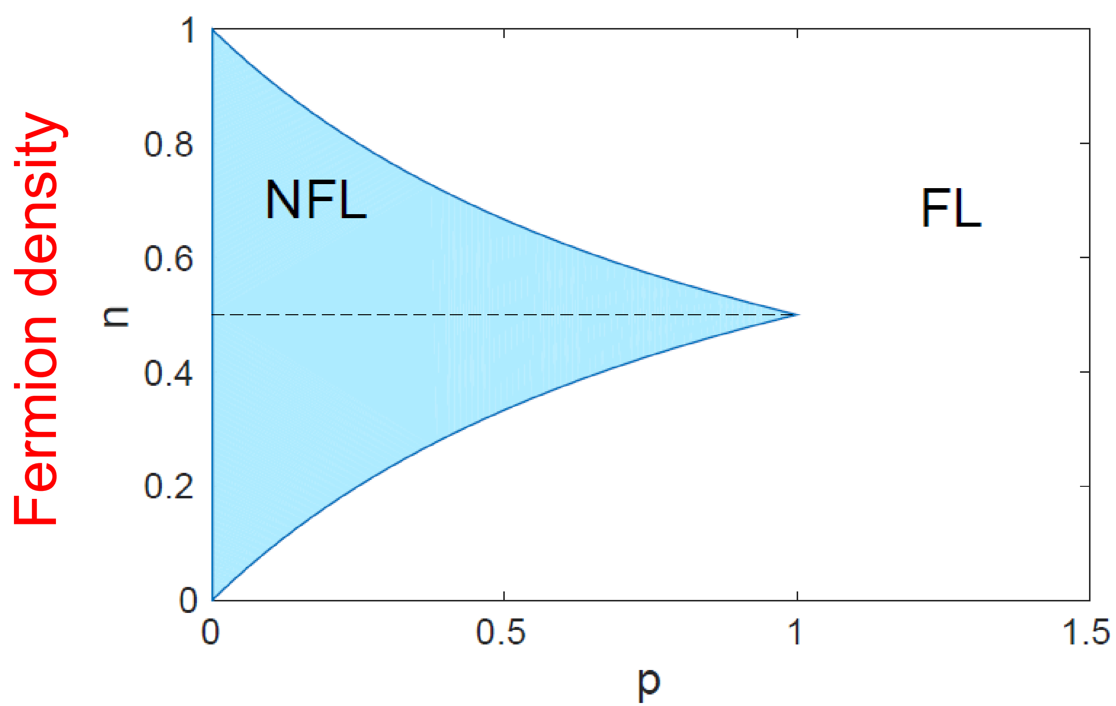
This talk: Solvable model with a quantum critical point two distinct quantum chaotic fixed points, SYK and Fermi-liquid.

Classifying phases and phase transitions in terms of quantum chaos?



Two-species fermion model

How spectrum and quantum chaos evolve?

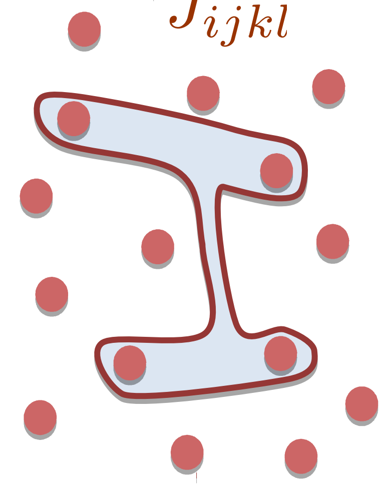


Ratio of number of sites of two species

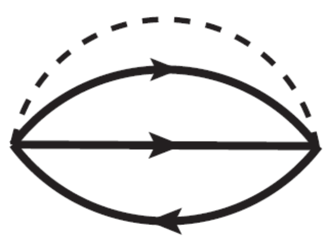
Review of SYK model

Sachdev & Ye PRL (1993), Kitaev, KITP (2015), Georges & Parcollet PRB (1999)

$$\mathcal{H}_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l - \mu \sum_i c_i^\dagger c_i$$



- Large-*N* (disorder averaged) saddle point



$$G^{-1}(\omega) = \cancel{\omega} + \mu - \Sigma(\omega)$$

$$\Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$

$$\hat{\Sigma}(\omega) = \Sigma(\omega) - \mu$$

- Conformal symmetry at low energy $(\omega, T < J)$

$$\int_0^\beta d\tau G(\tau, \tau_1) \hat{\Sigma}(\tau_1, \tau') = -\delta(\tau - \tau')$$

$$\tau = f(\sigma)$$

$$f'(\sigma) = \frac{\partial f}{\partial \sigma}$$

$$\tilde{G}(\sigma_1, \sigma_2) = [f'(\sigma_1) f'(\sigma_2)]^{1/4} G(f(\sigma_1), f(\sigma_2))$$

Sachdev, PRX (2015)

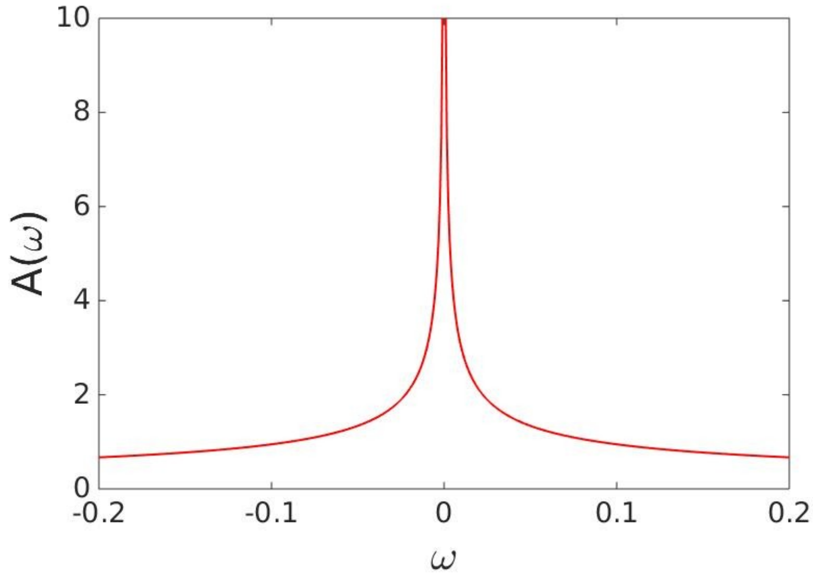
- Diverging DOS for $\omega \to 0$ at $T=0$

$$G_R(\omega) = \Lambda \frac{e^{-i(\pi/4+\theta)}}{\sqrt{J\omega}}$$

$$\Sigma_R(\omega) \sim -\Lambda^3 e^{i(\pi/4+\theta)} \cos 2\theta \sqrt{J\omega}$$

- Spectral asymmetry,
  $-\pi/4 \leq \theta \leq \pi/4$
- $\Lambda = \left(\frac{\pi}{\cos 2\theta}\right)^{1/4}$

Half filling, $\theta=0$



- Non-Fermi liquid fixed point.

- Extensive $T=0$ residual entropy (for $T \to 0$, $N \to \infty$)
  – dense many-body spectra near ground state,
  level spacing ~ e^{-N}

→ Quantum chaos and thermalization in SYK model.

Quantum chaos in SYK model

$$\mathcal{H}_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$

Kitaev, KITP (2015)
Polchinski & Rosenhaus (2016)
Maldacena & Stanford (2016)

- Out-of-time-order correlation

$$\langle c_i^\dagger(t) c_i(0) c_j^\dagger(t) c_j(0) \sim 1 - \left(\frac{\beta J}{N}\right) e^{\lambda_L t}$$

$$\lambda_L = 2\pi T$$

→ Scrambling time

$$t^* \sim \frac{1}{\lambda_L} \ln N$$

Fastest scrambler! Maximally chaotic. Like a black hole.

Upper bound to quantum chaos

Maldacena, Shenker & Stanford (2015)

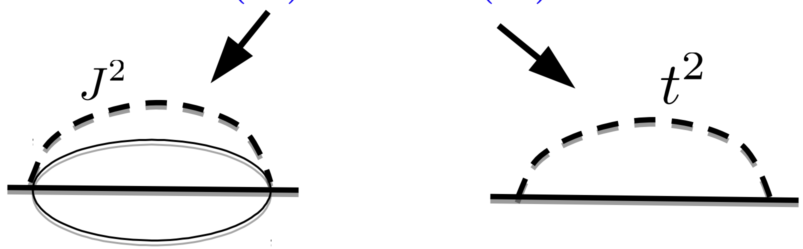
How to drive a phase transition out of this maximally chaotic non-Fermi liquid fixed point?

Naive attempt: add a quadratic term

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{N}} \sum_{ij} t_{ij} c_i^\dagger c_j$$

$$G^{-1}(\omega) = \omega - \Sigma_J(\omega) - t^2 G(\omega)$$

$$\Sigma_J(\tau) = -J^2 G^2(\tau) G(-\tau) =$$



- The ansatz $G_R(\omega) \sim 1/\omega^{1/2}$ is not self-consistent in the limit $\omega \to 0$

$$G_R^{-1}(\omega) \sim \omega - \sqrt{J\omega} - \frac{t^2}{\sqrt{J\omega}}$$

The free fermion ansatz of constant DOS is self consistent:

$$G_R(\omega) \sim -i/t$$

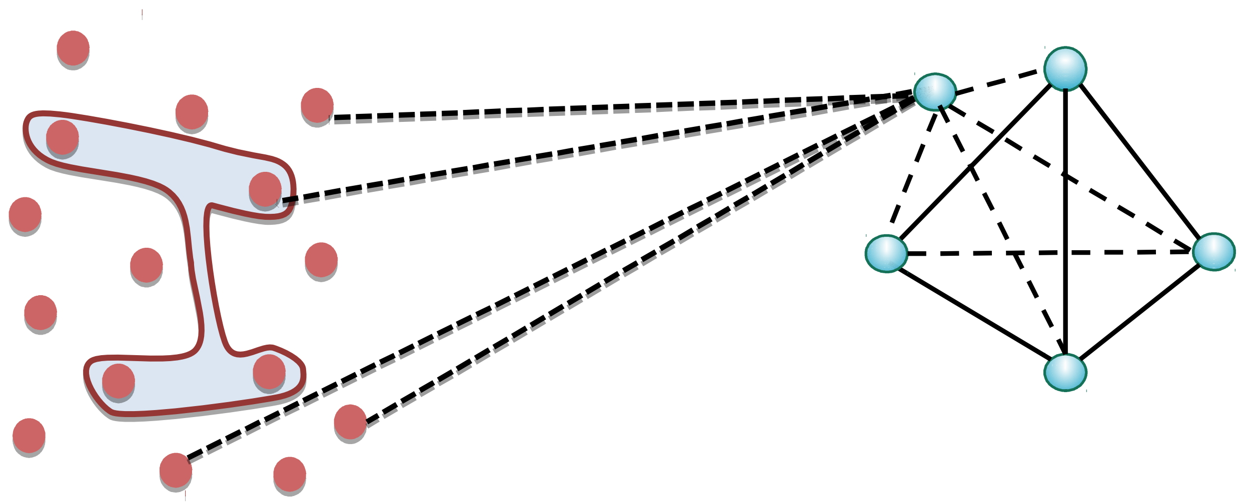
Quadratic term is relevant. Always a Fermi liquid.
No transition!

Consider a model with two species of fermions

N SYK sites:
SYK coupling

M "peripheral" sites:
Random hopping

$$\overline{J_{ijkl}^2} = J^2 \qquad \overline{V_{i\alpha}^2} = V^2 \qquad \overline{t_{\alpha\beta}^2} = t^2$$

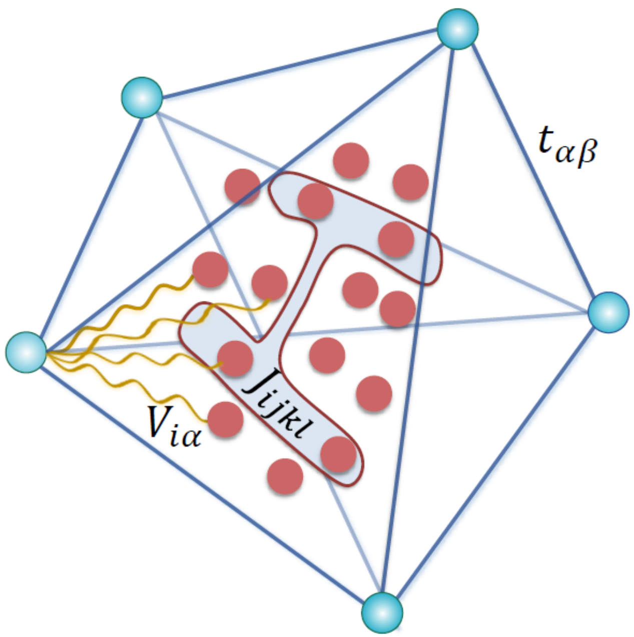


$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{M}} \sum_{\alpha\beta} t_{\alpha\beta} \psi_\alpha^\dagger \psi_\beta - \mu \Big(\sum_i c_i^\dagger c_i + \sum_\alpha \psi_\alpha^\dagger \psi_\alpha \Big) + \frac{1}{(NM)^{1/4}} \sum_{i\alpha} \big(V_{i\alpha} c_i^\dagger \psi_\alpha + h.c. \big)$$

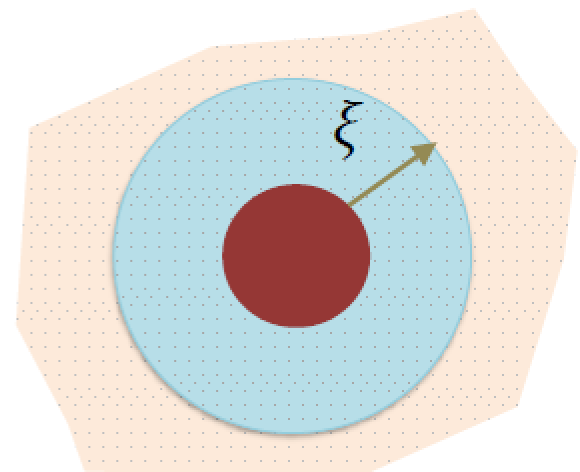
Physical motivation

N SYK sites

M peripheral sites



Ergodic bubble
in an Anderson insulator



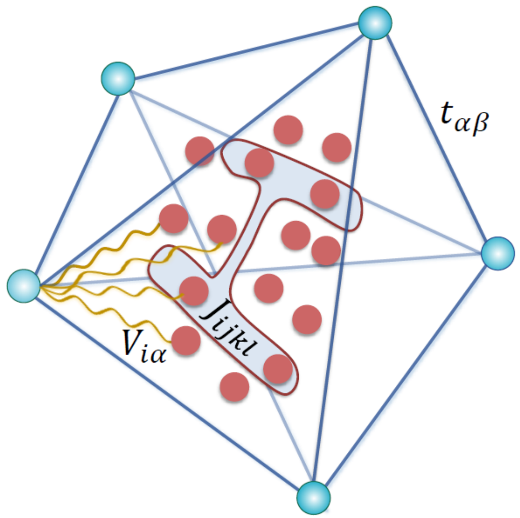
$M \sim \xi^D$

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{M}} \sum_{\alpha\beta} t_{\alpha\beta} \psi_\alpha^\dagger \psi_\beta - \mu\Big(\sum_i c_i^\dagger c_i + \sum_\alpha \psi_\alpha^\dagger \psi_\alpha\Big)$$

$$+ \frac{1}{(NM)^{1/4}} \sum_{i\alpha} (V_{i\alpha} c_i^\dagger \psi_\alpha + h.c.)$$

Saddle point equations at large N

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{M}} \sum_{\alpha\beta} t_{\alpha\beta} \psi_\alpha^\dagger \psi_\beta + \frac{1}{(NM)^{1/4}} \sum_{i\alpha} (V_{i\alpha} c_i^\dagger \psi_\alpha + h.c.)$$
$$-\mu\left(\sum_i c_i^\dagger c_i + \sum_\alpha \psi_\alpha^\dagger \psi_\alpha\right)$$

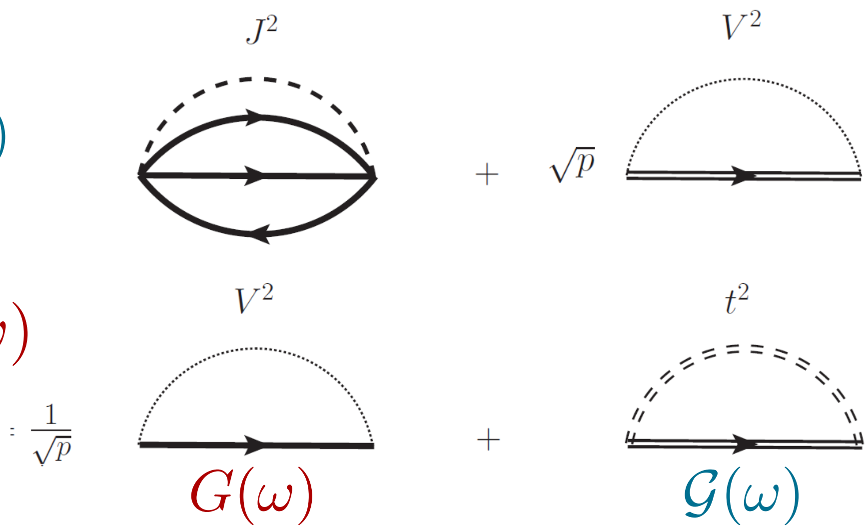


$$p = M/N$$

$$G^{-1}(\omega) = \omega + \mu - \Sigma_J(\omega) - V^2\sqrt{p}\,\mathcal{G}(\omega)$$

$$\mathcal{G}^{-1}(\omega) = \omega + \mu - t^2\mathcal{G}(\omega) - \frac{V^2}{\sqrt{p}} G(\omega)$$

$$\Sigma_J(\tau) = -J^2 G^2(\tau) G(-\tau)$$

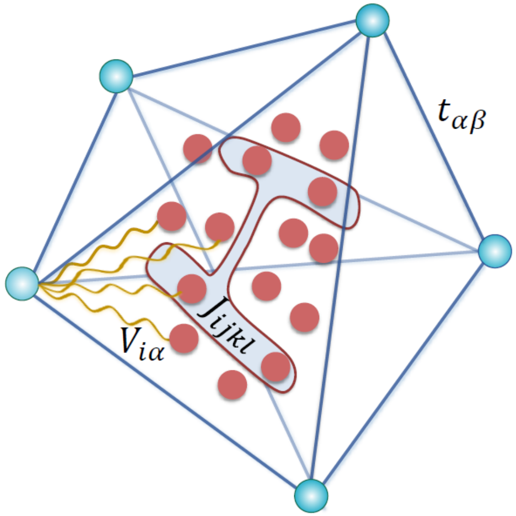


Non-Fermi liquid fixed point

$$G^{-1}(\omega) = \omega + \mu - \Sigma_J(\omega) - V^2\sqrt{p}\mathcal{G}(\omega)$$

$$\mathcal{G}^{-1}(\omega) = \omega + \mu - t^2\mathcal{G}(\omega) - \frac{V^2}{\sqrt{p}}G(\omega)$$

$$\Sigma_J(\tau) = -J^2G^2(\tau)G(-\tau)$$



→ Emergent conformal symmetry

Scaling dimension

$$\tau = f(\sigma)$$

$$\tilde{G}(\sigma_1, \sigma_2) = [f'(\sigma_1)f'(\sigma_2)]^{\Delta_c}G(f(\sigma_1), f(\sigma_2))$$

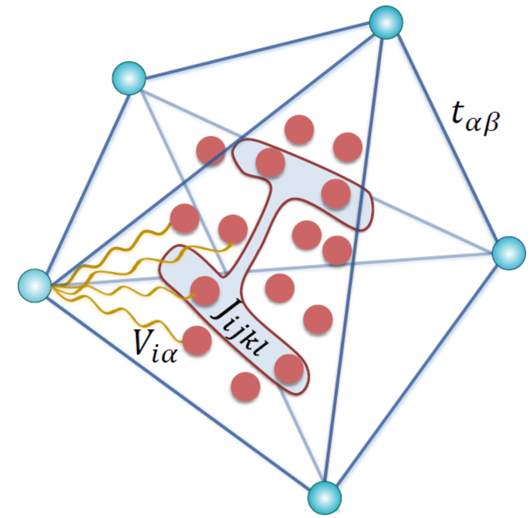
$$\Delta_c = \frac{1}{4}$$

$$\tilde{\mathcal{G}}(\sigma_1, \sigma_2) = [f'(\sigma_1)f'(\sigma_2)]^{\Delta_\psi}\mathcal{G}(f(\sigma_1), f(\sigma_2))$$

$$\Delta_\psi = \frac{3}{4}$$

→ Power-law solution

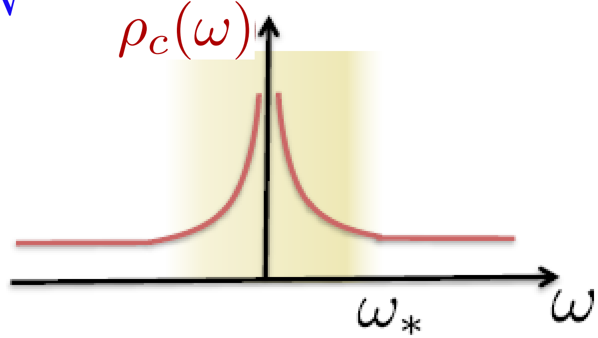
Half filling: Non-Fermi liquid



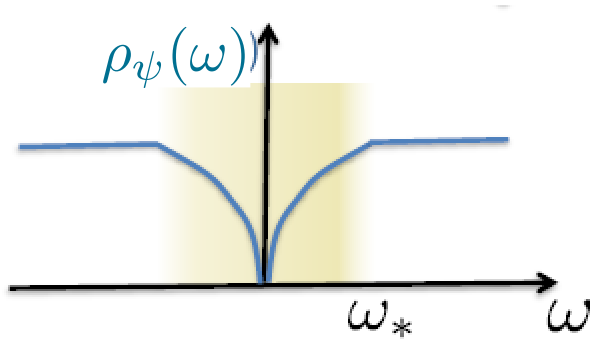
$\mu = 0$

→ Solution at *T=0*, for $p = \frac{M}{N} < 1$

$$G_R(\omega) \sim \frac{(1-p)^{1/4}}{\sqrt{J\omega}} e^{-i\frac{\pi}{4}}$$



$$\mathcal{G}_R(\omega) \sim -\frac{\sqrt{p}}{(1-p)^{1/4}} \frac{\sqrt{J\omega}}{V^2} e^{i\pi/4}$$



$$\omega^* \sim \frac{V^4}{t^2 J} \frac{(1-p)^{1/2}}{p}$$

Weight and bandwidth of the singularity in $G_R(\omega)$ vanishes continuously as $p \to p_c = 1$

- Conformal Green's functions at finite temperature

From $T=0$ $\quad G_R(\omega),\ \mathcal{G}_R(\omega) \quad \Rightarrow \quad G(\tau) \sim -\frac{\mathrm{sgn}(\tau)}{\sqrt{J|\tau|}},\ \mathcal{G}(\tau) \sim -\frac{\mathrm{sgn}(\tau)}{(J|\tau|)^{3/2}}$

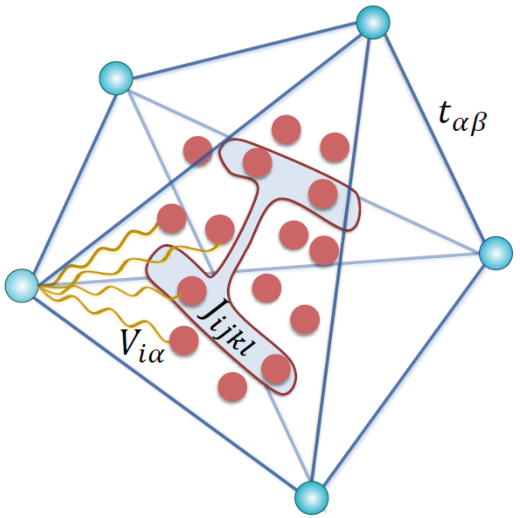
→ Finite temperature Green's function obtained by conformal transformation $\quad \tau = (\beta/\pi)\tan(\pi\sigma/\beta)$

$$G(\tau) \sim -\frac{\mathrm{sgn}(\tau)}{(\beta J \sin(\pi|\tau|/\beta))^{1/2}},\ \mathcal{G}(\tau) \sim -\frac{\mathrm{sgn}(\tau)}{(\beta J \sin(\pi|\tau|/\beta))^{3/2}}$$

$$G_R^{\Delta}(\omega) \sim \frac{T^{2\Delta-1}}{\Gamma(2\Delta)\sin(2\pi\Delta)} \frac{\Gamma(\Delta - i\frac{\omega}{2\pi T})}{\Gamma(1-\Delta-i\frac{\omega}{2\pi T})}$$

$$\Delta = \frac{1}{4} \to G_R(\omega) \qquad\qquad \Delta = \frac{3}{4} \to \mathcal{G}_R(\omega)$$

Half filling: Fermi-liquid



- Solution for $p = \dfrac{M}{N} > 1$

$$G_R(\omega) = -i\frac{1}{\sqrt{p(p-1)}}\frac{t}{V^2}$$

$$\mathcal{G}_R(\omega) = -i\sqrt{\frac{p-1}{p}}\frac{1}{t}$$

$\rightarrow$ Constant DOS for $\omega < \omega^* \sim \dfrac{V^2}{t}\sqrt{p(p-1)}$

- Self energy, $\mathrm{Im}\Sigma_J(\omega) \sim -\left(\dfrac{J^2 t^3}{V^6}\right)\dfrac{1}{(p(p-1))^{3/2}}\omega^2$

- Free fermion fixed point, emergent conformal symmetry

$$\tilde{G}(\sigma_1, \sigma_2) = [f'(\sigma_1)f'(\sigma_2)]^{1/2} G(f(\sigma_1), f(\sigma_2)) \sim \tilde{\mathcal{G}}(\omega)$$

$\rightarrow$Critical point at *p=M/N=1* separates NFL and FL fixed points

Away from half filling

$$G_R(\omega) = \Lambda \frac{e^{-i(\pi/4+\theta)}}{\sqrt{J\omega}}$$

$$\mathcal{G}_R(\omega) = -\frac{\sqrt{p}}{V^2\Lambda}\sqrt{J\omega}\, e^{i(\pi/4+\theta)}$$

$$\Lambda = \left(\frac{\pi(1-p)}{\cos 2\theta}\right)^{1/4}$$

$-\pi/4 \leq \theta \leq \pi/4$ ←Spectral asymmetry

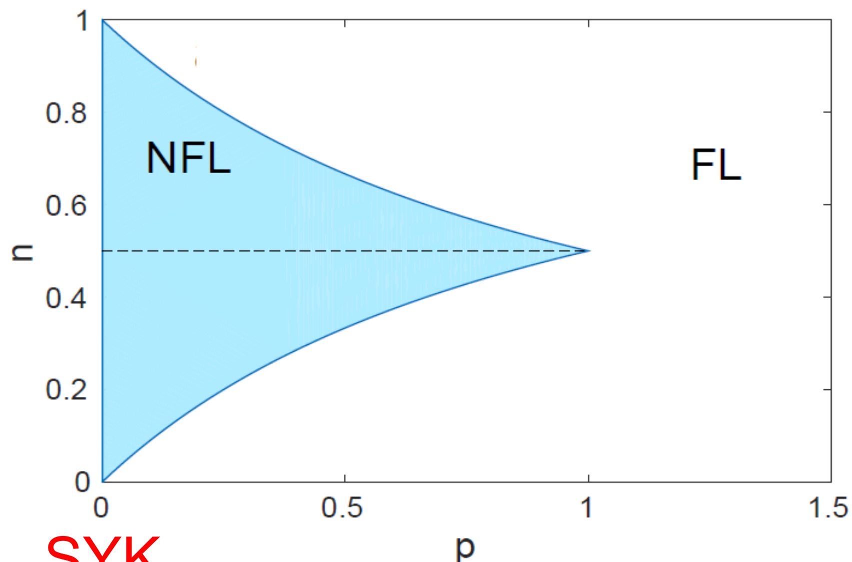
Luttinger theorem for the NFL

$$n = \frac{1}{1+p}\left[\left(\frac{1}{2}-\frac{\theta}{4}\right) + p\left(\frac{1}{2}+\frac{\theta}{4}\right) - (1-p)\frac{\sin 2\theta}{4}\right]$$

→ Allowed density range

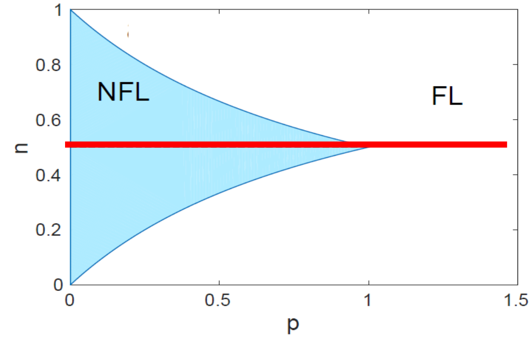
$$\frac{p}{1+p} \leq n \leq \frac{1}{1+p}$$

→ NFL-FL Phase boundary

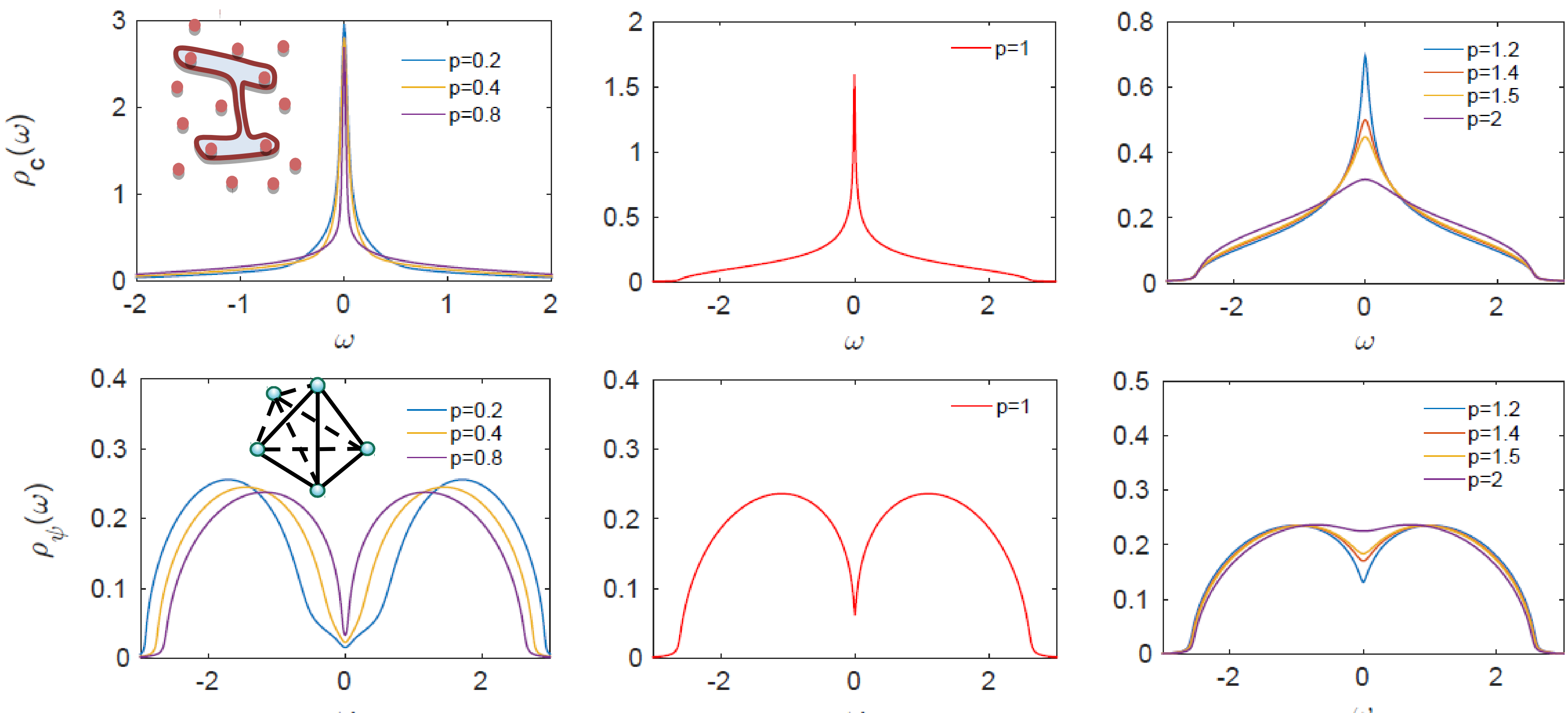


SYK

Numerical results at half filling



- Spectral function across QCP



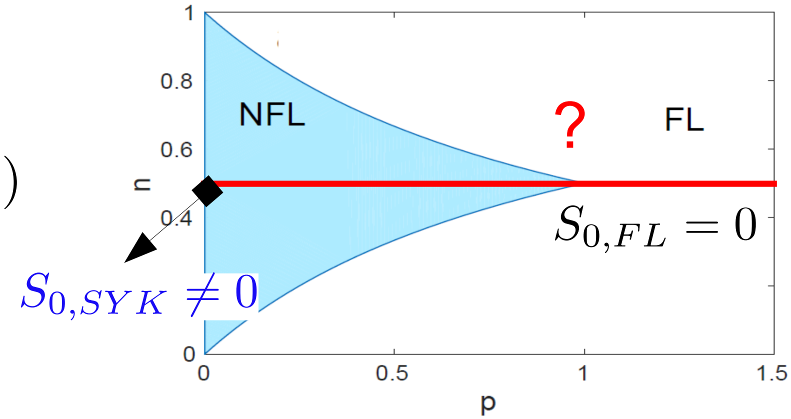
$$J = t = V = 1, T = 0.025$$

How else is the transition manifested?

Zero-temperature entropy

- Thermodynamic integration

$$\left(\frac{\partial S}{\partial n}\right)_{T=0} = -\left(\frac{\partial \mu}{\partial T}\right)_n = -\ln\left(\tan(\pi/4+\theta)\right)$$

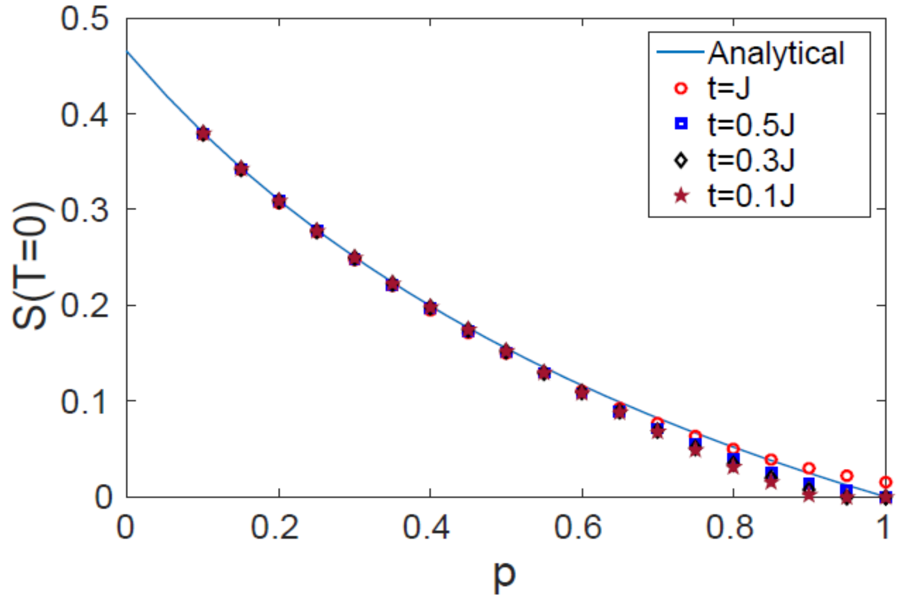


$$S_0(n) = S(n_0) + \int_{n_0}^{n} dn \ln(\tan(\pi/4+\theta(n)))$$

0

Luttinger theorem → *θ(n)*

$$S_0(n=1/2) = \frac{1-p}{1+p} S_{0,SYK}$$

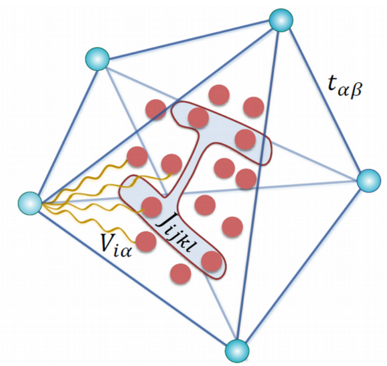


Zero-T entropy vanishes continuously at the transition

→ Change of geometry in dual gravity across QCP?

Fast to slow scrambling

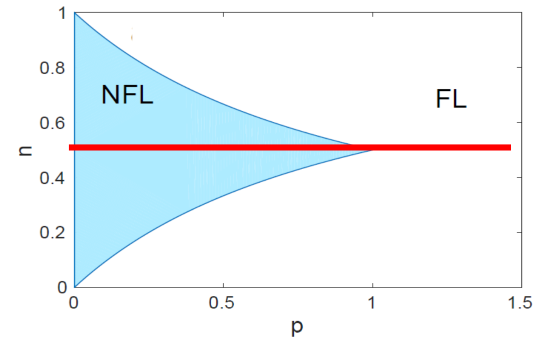
$$\mathcal{H} = \frac{1}{4!}\sum_{ijkl} J_{ijkl}\chi_i\chi_j\chi_k\chi_l + \frac{i}{2!}\sum_{\alpha\beta} t_{\alpha\beta}\eta_\alpha\eta_\beta + i\sum_{i\alpha} V_{i\alpha}\chi_i\eta_\alpha$$



$c_i \to \chi_i$

$\psi_\alpha \to \eta_\alpha$

$p = M/N$



Quantum chaos across QCP?

Out-of-time-order (OTO) correlation

Kitaev, KITP (2015)

p=0 $\overline{\langle \chi_i(t)\chi_j(0)\chi_i(t)\chi_j(0)\rangle} \simeq f_0 - \frac{f_1}{N}e^{\lambda_L t} + \dots$

$t \sim t^* = (1/\lambda_L)\ln N$

$\lambda_L = 2\pi T$

- Two OTO correlators

$$F_1(t_1,t_2) \sim \overline{\langle \chi_i(t)\chi_j(0)\chi_i(t)\chi_j(0)\rangle}$$

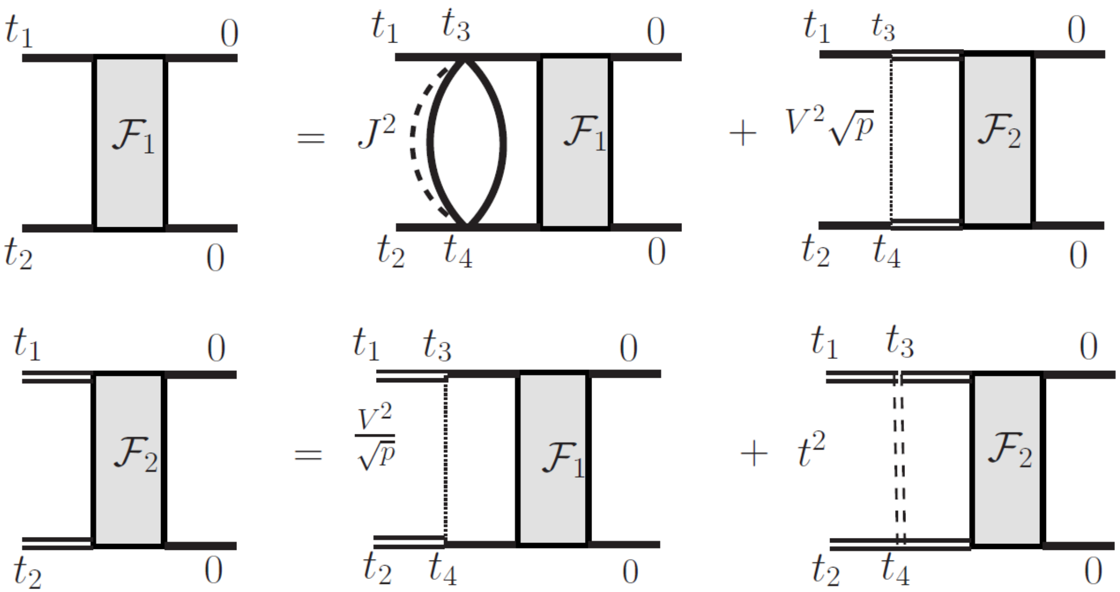
$$F_2(t_1,t_2) \sim \overline{\langle \eta_\alpha(t)\chi_i(0)\eta_\alpha(t)\chi_i(0)\rangle}$$

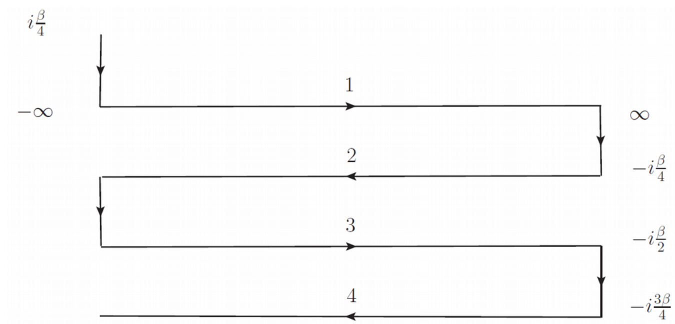
$$F_1(t_1,t_2) = \frac{1}{N^2}\sum_{ij}\overline{\mathrm{Tr}[y\chi_i(t)y\chi_j(0)y\chi_i(t)y\chi_j(0)]}$$

$$F_2(t_1,t_2) = \frac{1}{NM}\sum_{i\alpha}\overline{\mathrm{Tr}[y\eta_\alpha(t)y\chi_i(0)y\eta_\alpha(t)y\chi_i(0)]}$$

$$\simeq F^{(0)}(t_1,t_2) + \frac{1}{N}\mathcal{F}(t_1,t_2) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$y^4 = e^{-\beta\mathcal{H}}/Z$$





$$\mathcal{F}_1(t_1,t_2) = \int dt_3 dt_4 [K_{11}(t_1,t_2,t_3,t_4)\mathcal{F}_1(t_3,t_4) + K_{12}(t_1,t_2,t_3,t_4)\mathcal{F}_2(t_3,t_4)]$$

$$\mathcal{F}_2(t_1,t_2) = \int dt_3 dt_4 [K_{21}(t_1,t_2,t_3,t_4)\mathcal{F}_1(t_3,t_4) + K_{22}(t_1,t_2,t_3,t_4)\mathcal{F}_2(t_3,t_4)]$$

→ Eigenvalue problem

$$\mathcal{K}|\mathcal{F}\rangle = k|\mathcal{F}\rangle$$

$$\mathcal{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \simeq \begin{pmatrix} 3J^2 G_R(t_{13}) G_R(t_{24}) G_{lr}^2(t_{34}) & -V^2\sqrt{p} G_R(t_{13}) G_R(t_{24}) \\ -\frac{V^2}{\sqrt{p}} \mathcal{G}_R(t_{13}) \mathcal{G}_R(t_{24}) & -t^2 \mathcal{G}_R(t_{13}) \mathcal{G}_R(t_{24}) \end{pmatrix}$$

- Wightmann correlator $G_{lr}(t) \equiv iG(it + \beta/2)$

- Chaos ansatz

$$|\mathcal{F}\rangle = \begin{pmatrix} \mathcal{F}_1(t_1, t_2) \\ \mathcal{F}_2(t_1, t_2) \end{pmatrix} = e^{\lambda_L \frac{(t_1+t_2)}{2}} \begin{pmatrix} f_1(t_{12}) \\ f_2(t_{12}) \end{pmatrix}$$

→ Lyapunov exponent

$$k(\lambda_L) = 1 \quad \Rightarrow \lambda_L$$

→ Use conformal Green's functions in NFL to solve the eigenvalue problem for T→0

→ Integral equation

$$\frac{3}{4\pi}\frac{|\Gamma\left(\frac{1}{4}+\frac{h}{2}+iu\right)|^2}{|\Gamma\left(\frac{3}{4}+\frac{h}{2}+iu\right)|^2}\int_{-\infty}^{\infty}du'|\Gamma(\frac{1}{2}+i(u-u'))|^2 f_1(u') = \left(k-\frac{p}{k}\right)f_1(u)$$

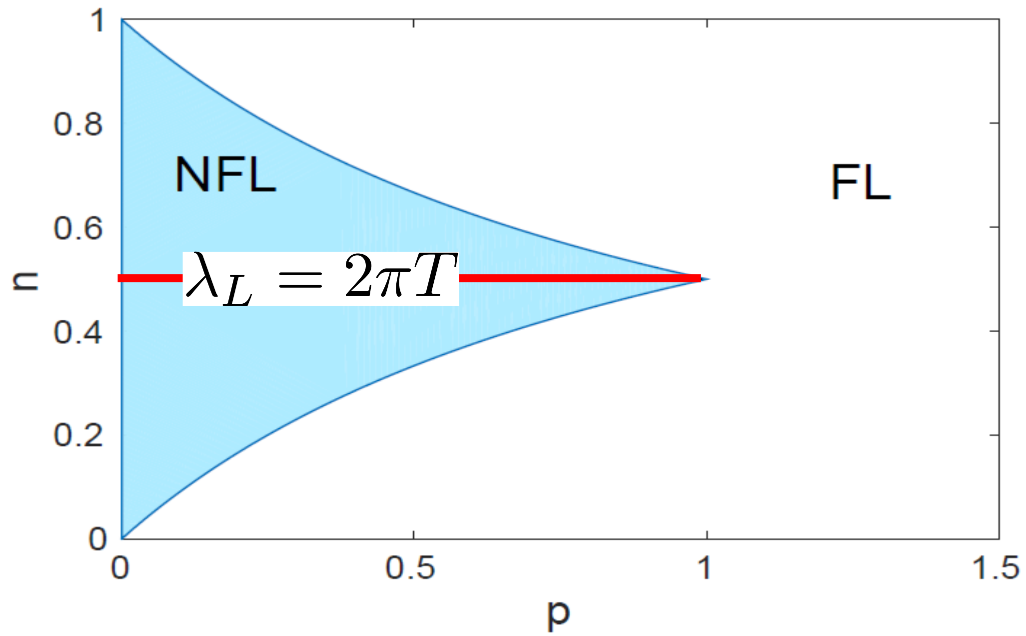
$$h = \frac{\lambda_L}{2\pi T}$$

- Solution $\quad f_1(u) = |\Gamma\left(\frac{1}{4}+\frac{h}{2}+iu\right)|^2$

$$\frac{3(1-p)}{1+2h} = \left(k-\frac{p}{k}\right)$$

$$k=1 \quad \Rightarrow \quad h=1$$

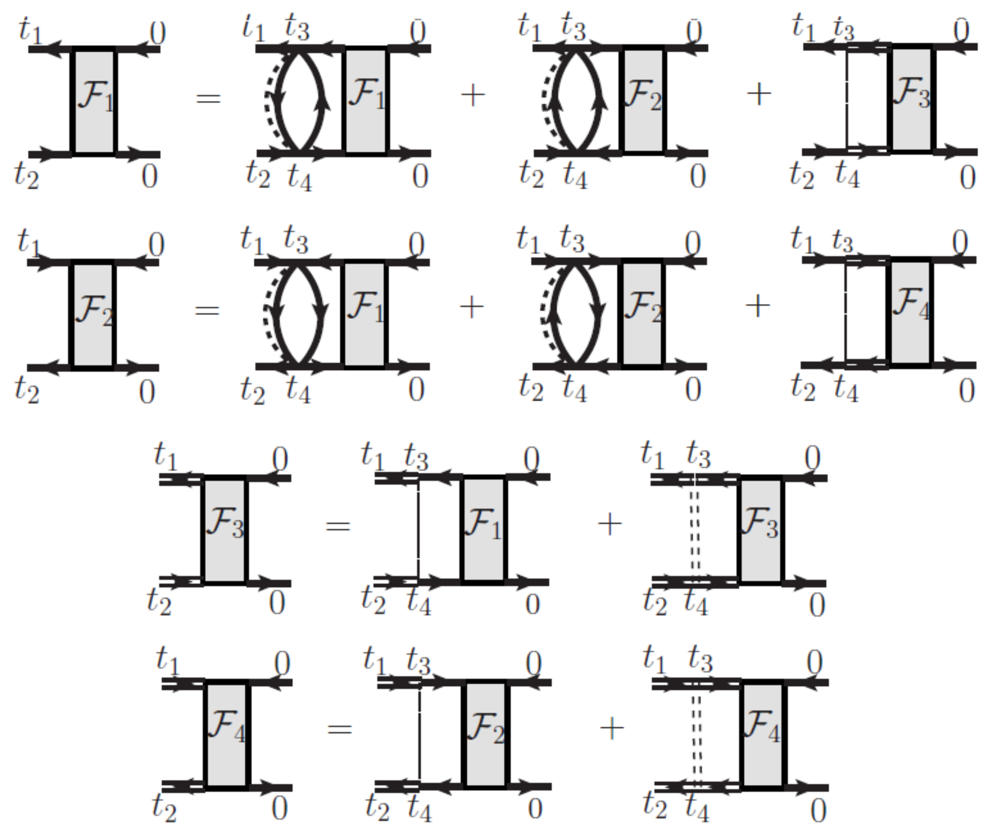
$$\Rightarrow \quad \lambda_L = 2\pi T$$



Lyapunov exponent away from half filling

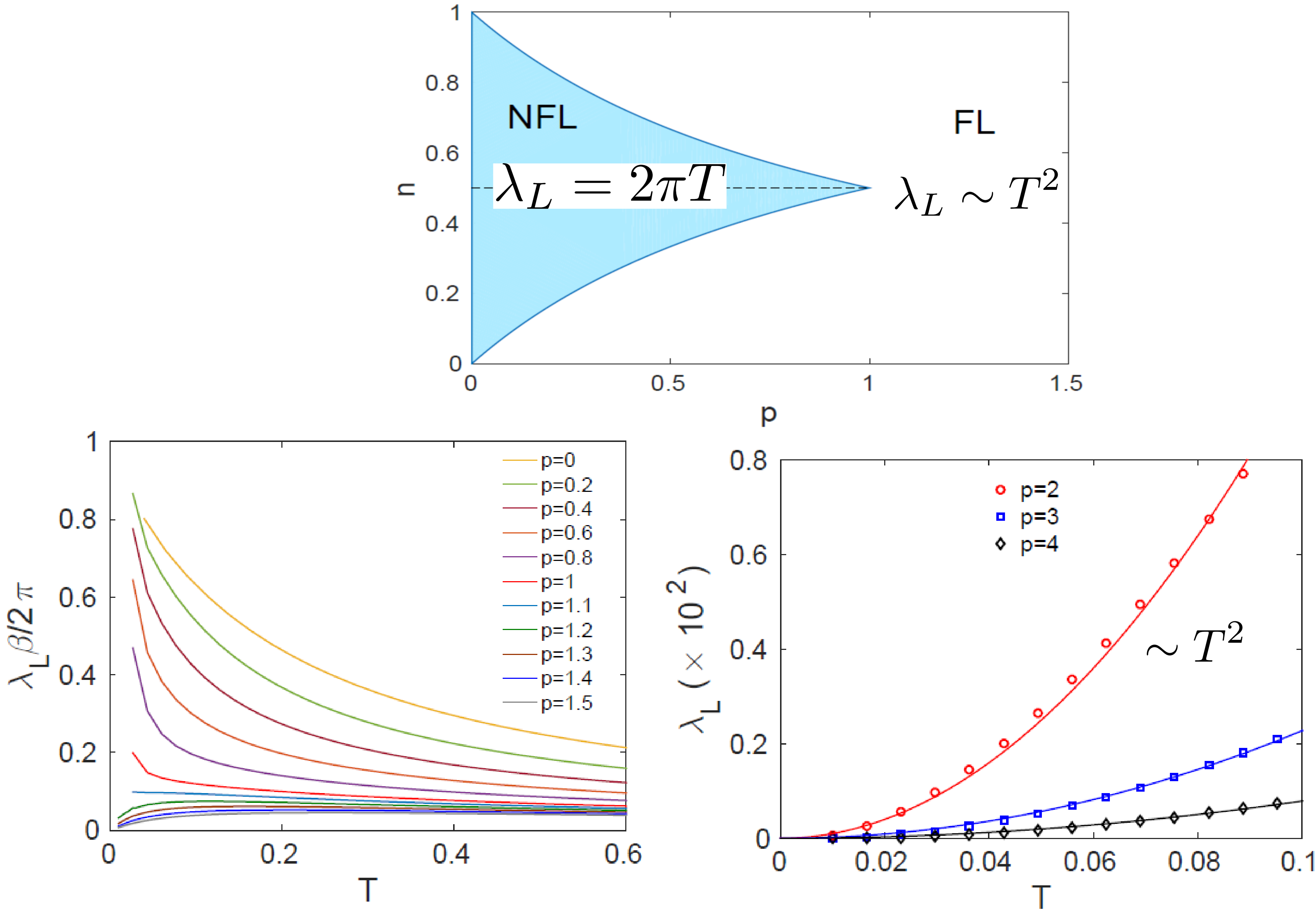
$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{M}} \sum_{\alpha\beta} t_{\alpha\beta} \psi_\alpha^\dagger \psi_\beta + \frac{1}{(NM)^{1/4}} \sum_{i\alpha} (V_{i\alpha} c_i^\dagger \psi_\alpha + h.c.)$$

$$-\mu\left(\sum_i c_i^\dagger c_i + \sum_\alpha \psi_\alpha^\dagger \psi_\alpha\right)$$



$$\lambda_L = 2\pi T$$

arXiv:1610.04619


NFL
FL
$\lambda_L = 2\pi T$
$\lambda_L \sim T^2$
n
p
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
$\lambda_L \beta/2\pi$
T
p=0
p=0.2
p=0.4
p=0.6
p=0.8
p=1
p=1.1
p=1.2
p=1.3
p=1.4
p=1.5
0
0.2
0.4
0.6
0.8
1
0
0.2
0.4
0.6
λ_L ($\times$ 10^2)
T
p=2
p=3
p=4
$\sim T^2$
0
0.2
0.4
0.6
0.8
0
0.02
0.04
0.06
0.08
0.1

Conclusions and outlook

- Solvable model for a non-Fermi liquid to Fermi liquid transition.
  - Spectral function
  - Zero-temperature entropy
  - Many-body quantum chaos,
    fast ($\lambda_L = 2\pi T$) to slow scrambling ($\lambda_L \sim T^2$).

- Theory for the critical point? New chaotic fixed point distinct from either SYK or FL?

- Holographic interpretation? Phase transition involving elimination of black hole?

- Extension to large-N description for MBL and MBL transition? No scrambling or power law scrambling.

Thank you!