Local criticality, diffusion, and chaos in generalized Sachdev-Ye-Kitaev model

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Based on [Yingfei Gu, XLQ, Douglas Stanford, arXiv: 1609.07832]

Motivation

- Generic quantum many-body systems are chaotic
- Understanding quantum chaos in many-body systems may be useful for understanding strongly correlated systems
- Quantum chaos is related to holographic duality
- SYK model is a *solvable chaotic* model in $(0+1)$-d. We would like to generalize it to higher dimensions

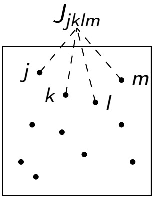
Outline

- The Sachdev-Ye-Kitaev (SYK) model
- Quantum chaos
- Generalized SYK model
  - Effective action
  - Two-point functions and four-point functions
  - Diffusion and chaos
- Summary and discussion

A brief review of Sachdev-Ye-Kitaev model

Quantum mechanics $(0+1\text{-d})$ of N Majorana fermions:

$$H = \sum_{1\leqslant j<k<l<m\leqslant N} J_{jklm}\chi_j\chi_k\chi_l\chi_m, \quad \{\chi_j,\chi_k\} = \delta_{jk}$$



All to all random interaction $\overline{J_{jklm}} = 0$, $\overline{J_{jklm}^2} = \frac{3!J^2}{N^3}$.
[Sachdev, Ye, 1993; Kitaev 2015; Maldacena, Stanford 2016.]

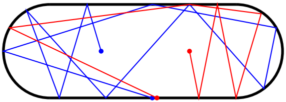
- Solvable in the limit $N \gg \beta J \gg 1$
- Two-point function shows local criticality [Parcollet, Georges 1999]

$$G^s(\tau) = \left(\frac{1}{4\pi}\right)^{\Delta}\left(\frac{\beta J}{\pi}\sin\frac{\pi\tau}{\beta}\right)^{-2\Delta}, \ \Delta = \frac{1}{4}$$

- Four-point function characterizes chaos

From classical chaos to quantum chaos

Chaos: exponential sensitivity to initial conditions



Particles in a stadium

- Classical chaos: Poisson bracket $\{q(t), p(0)\}_{\rm PB} = \frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t}$,
  λ_L: Lyapunov exponent.

- From classical to quantum:
  $\{q(t), p(0)\}_{\rm PB} \to [\hat{q}(t), \hat{p}(0)] \to [W(t), V(0)]$
  [Larkin, Ovchinnikov 1969].

- $W(t)$, $V(0)$ -generic (hermitian) Heisenberg operators in many-body system

Out-of-time-ordered correlator

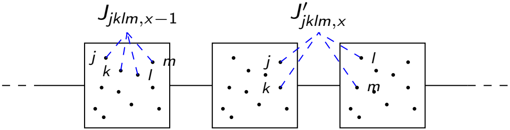
- Diagnostics of chaos: $C(t) = -\langle [W(t), V(0)]^2 \rangle_\beta$
- Important term in $C(t)$: $f(t) = \langle W(t)V(0)W(t)V(0) \rangle_\beta$, out-of-time-ordered correlator (OTOC)
- Regularized OTOC $\widetilde{f}(t) = \mathrm{tr}\left(W(t)\rho^{1/4}V(0)\rho^{1/4}W(t)\rho^{1/4}V(0)\rho^{1/4} \right)$ with $\rho = Z^{-1}e^{-\beta H}$ the thermal density matrix.
- Example: 0+1 SYK model.

$$\widetilde{f}(t) = \left\langle \chi_i \left(t + i\frac{3\beta}{4} \right) \chi_j \left(i\frac{\beta}{2} \right) \chi_i \left(t + i\frac{\beta}{4} \right) \chi_j(0) \right\rangle_\beta$$

  with $\widetilde{f}(t) - \widetilde{f}(0) \propto -\frac{1}{N} e^{\lambda_L t}$.
- $\lambda_L = \frac{2\pi}{\beta}$ saturates the Maldacena-Shenker-Stanford bound

Generalized SYK model in higher dimensions

- Random quartic interaction between SYK islands. For the example of 1D chain



$$H = \sum_{x=1}^{M} \left[\sum_{j<k<l<m} \underbrace{J_{jklm,x}\ \chi_{j,x}\chi_{k,x}\chi_{l,x}\chi_{m,x}}_{\text{SYK term}} + \sum_{j<k;l<m} \underbrace{J'_{jklm,x}\ \chi_{j,x}\chi_{k,x}\chi_{l,x+1}\chi_{m,x+1}}_{\text{Nearest neighbour coupling}} \right]$$

Independent random coefficients:

$$\overline{J_{jklm,x}^2} = \frac{3! J_0^2}{N^3}, \quad \overline{J'^2_{jklm,x}} = \frac{J_1^2}{N^3}$$

Effective action

Same as 0+1 SYK, our model also self-averages (replicon diagonal):

1. Average over $\{J_{jklm}\} \Rightarrow$ disorder averaged partition function;
2. Introduce new fields G and Σ: Σ is Lagrange multiplier enforces $G_x(\tau_1, \tau_2) = \frac{1}{N} \sum_j \chi_{j,x}(\tau_1) \chi_{j,x}(\tau_2)$

$$
\begin{aligned}
\overline{Z} &= \int \mathcal{D}\chi \exp\left(-S[\chi]\right) \\
&= \int \underbrace{\mathcal{D}G\mathcal{D}\Sigma}_{\text{new fields}} \mathcal{D}\chi \exp\left(-S[\chi, G, \Sigma]\right)
\end{aligned}
$$

where $S[\chi, G, \Sigma]$ contains "$\chi \partial_\tau \chi$" + "$\Sigma(G - \chi\chi)$" + "G^4"

Effective action

3. Integrate over fermions:

$$S_{\text{eff}}[G,\Sigma] = \sum_{x=1}^{M} S_0[G_x,\Sigma_x] + \sum_{x=1}^{M} \frac{J_1^2}{16} \int d^2\tau \left(G_x(\tau_1,\tau_2)^2 - G_{x+1}(\tau_1,\tau_2)^2 \right)^2$$

$$\underbrace{S_0[G_x,\Sigma_x]}_{\text{SYK action}} = \underbrace{-\frac{1}{2}\log\det\left(\partial_\tau - \Sigma_x\right)}_{\text{From } \frac{1}{2}\chi(\partial_\tau - \Sigma)\chi} + \frac{1}{2}\int_0^\beta d^2\tau \left(\underbrace{\Sigma_x G_x}_{\text{Lagrange multiplier}} \underbrace{- \frac{J_0^2 + J_1^2}{4} G_x(\tau_1,\tau_2)^4}_{\text{intra-site coupling}} \right)$$

- Large N limit $\Rightarrow$ semiclassical limit of Σ, G
- Analog: spin chain with external field and nearest neighbor coupling

Two-point function

Large N saddle point analysis:

1. Saddle point equation from

$$\frac{\delta S_{\mathrm{eff}}}{\delta G_x} = 0, \quad \frac{\delta S_{\mathrm{eff}}}{\delta \Sigma_x} = 0, \quad \forall x \in \text{lattice}$$

2. Using averaged translational symmetry: $G_x^s(\tau_1, \tau_2) = G^s(\tau_1, \tau_2)$

$$G^s(i\omega_n) = \frac{1}{-i\omega_n - \Sigma^s(i\omega_n)}, \quad \Sigma^s(\tau) = J^2 G^s(\tau)^3$$

   $J^2 = J_0^2 + J_1^2$: effective coupling.

3. Saddle point equation same as in the $(0+1)$-d SYK model.

Two-point function continued

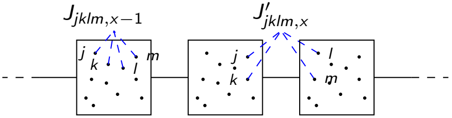
1. Analytic solution at strong coupling $N \gg \beta J \gg 1$: [Sachdev,Ye; Parcollet, Georges]

$$G^s(\tau) = \left(\frac{1}{4\pi}\right)^{\Delta} \left(\frac{\beta J}{\pi} \sin \frac{\pi\tau}{\beta}\right)^{-2\Delta}, \ \Delta = \frac{1}{4}$$

2. At $T \to 0$, power law correlation for fermion on the same site:

$$\langle \mathcal{T}_\tau \chi_{j,x}(\tau) \chi_{j,x}(0) \rangle \propto \text{sgn}(\tau) |J\tau|^{-2\Delta}$$

3. Fermion correlation functions between different sites is zero, due to on-site $\mathbb{Z}_2$ fermion parity symmetry, and $SO(N)$ after average.



Thermodynamics

Large N thermodynamics: plug G^s and Σ^s back to the effective action:

$$\begin{aligned}\frac{F}{NM} &= \frac{1}{\beta}\left[-\frac{1}{2}\log\det\left(\partial_\tau - \Sigma^s\right) + \frac{1}{2}\int d\tau_1 d\tau_2 \left(\Sigma^s(\tau_1,\tau_2)G^s(\tau_1,\tau_2) - \frac{J^2}{4}G^s(\tau_1,\tau_2)^4\right)\right]\\ &= U - S_0 T - \frac{\gamma}{2}T^2 + \dots\end{aligned}$$

Same large N free energy density as $0+1$ SYK model:

1. Extensive zero temperature entropy $S_0 = \frac{\mathrm{Catalan}}{2\pi} + \frac{\log 2}{8} = 0.2324...$
2. Specific heat $c_v = \gamma T \approx \frac{0.396}{\beta J}$.

Spatial structure enters at level of quantum fluctuations.

Quantum fluctuations and four-point functions

- Quantum fluctuations around saddle point:

$$G_x = G^s + \delta G^s_x, \quad \Sigma_x = \Sigma^s + \delta \Sigma^s_x$$

- Expands to quadratic order, integrate over $\delta\Sigma \Rightarrow$

$$S_{\text{eff}}[G] = S_{\text{eff}}[G^s] + \int \delta G_x(\tau_1, \tau_2) Q_{xy}(\tau_1, \tau_2; \tau_3, \tau_4) \delta G_y(\tau_3, \tau_4)$$

- Quadratic form has simple dependence on spatial coordinates

$$Q_{xy}(\tau_1, \tau_2; \tau_3, \tau_4) = \underbrace{K^{-1}(\tau_1, \tau_2; \tau_3, \tau_4)}_{\text{same as 0+1 SYK}} \delta_{xy} - \delta(\tau_{13})\delta(\tau_{24}) \underbrace{S_{xy}}_{\text{was } \delta_{xy}}$$

- K: diagonalizable at $\beta J \gg 1$ [Kitaev 2015; Maldacena, Stanford 2016];

- S: $\delta_{xy} \to c_0 \delta_{xy} + c_1 \delta_{x, y\pm 1}$: "band structure" $s(p) = 1 - c_1 p^2 + \ldots$

Symmetry analysis

$$G^s(i\omega_n) = \frac{1}{\underbrace{-i\omega_n}_{\text{IR},\to 0} - \Sigma^s(i\omega_n)}, \quad \Sigma^s(\tau) = J^2 G^s(\tau)^3$$

1. The model at IR has an emergent symmetry — time reparametrization: $f \in \mathrm{Diff}(S^1)$ [symmetry of the model]:

$$G^s_x(\tau_1, \tau_2) \to \left(f'(\tau_1) f'(\tau_2)\right)^{\Delta} G^s_x(f(\tau_1), f(\tau_2))$$

2. Spontaneously broken to $\mathrm{PSL}_2(\mathbb{R})$ [symmetry of the solution]:

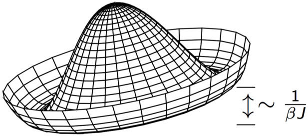
$$(\tau_1 - \tau_2)^{-2\Delta} \to (f'(\tau_1) f'(\tau_2))^{\Delta} (f(\tau_1) - f(\tau_2))^{-2\Delta} = (\tau_1 - \tau_2)^{-2\Delta}$$

$$\forall f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

3. This symmetry is also explicitly broken by UV terms $-i\omega_n$.

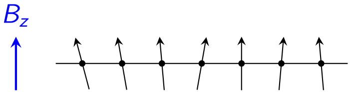
Pseudo-Goldstone mode

- Almost spontaneous symmetry breaking $\Rightarrow$ "nearly flat direction". Pseudo-Goldstone mode: $f_x \in \mathrm{Diff}(S^1)/\,\mathrm{PSL}_2(\mathbb{R})$



Tilted Mexican hat

- Analogy: ferromagnetic spin chain with a small pining field. Order parameter $\in S^2 = \mathrm{SU}(2)/U(1)$. $SU(2)$ symmetry is explicitly broken by a small B_z.



Effective action for pseudo-Goldstone mode

- Effective action for pseudo-Goldstone mode $f_x(\tau) = \tau + \epsilon_x(\tau)$

$$S \simeq \frac{1}{256\pi} \sum_{n,p} \epsilon_{n,p} \left(\underbrace{\frac{\sqrt{2}\alpha_K}{\beta J} n^2(n^2-1)}_{\text{Explicit breaking}} + \underbrace{\frac{J_1^2}{3J^2} p^2 |n| (n^2-1)}_{\text{Kinectic term}} \right) \epsilon_{-n,-p}$$

$$= \frac{\sqrt{2}\alpha_K}{512\pi^2} \sum_{n,p} \epsilon_{n,p} \left(\frac{2\pi|n|}{\beta} + Dp^2 \right) |n|(n^2-1) \epsilon_{-n,-p}$$

- The pseudo-Goldstone modes determine the long-wavelength long time-scale dynamics.

- $D = \frac{\sqrt{2}\pi J_1^2}{3\alpha_K J}$ turns out to be the energy diffusion constant.

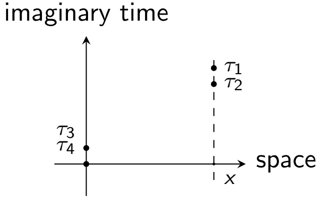
Four-point functions

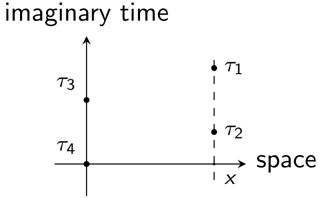
- Connected four-point function determined by quantum fluctuations:

$$\frac{1}{N^2}\sum_{j,k}\langle \chi_{j,x}(\tau_1)\chi_{j,x}(\tau_2)\chi_{k,y}(\tau_3)\chi_{k,y}(\tau_4)\rangle_{\rm conn.} = \langle G_x(\tau_1,\tau_2)G_y(\tau_3,\tau_4)\rangle - \langle G\rangle\langle G\rangle$$

$$= \langle \delta G_x(\tau_1,\tau_2)\delta G_y(\tau_3,\tau_4)\rangle = \frac{1}{N}Q^{-1}_{xy}(\tau_1,\tau_2;\tau_3,\tau_4)$$

- Next: physical consequence
  1. OPE: collective modes and energy transport;
  2. Out-of-time-ordered correlation function: characterization of chaos.

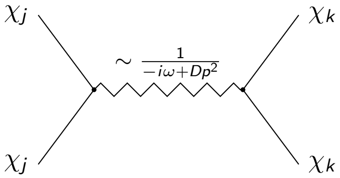




OPE and energy diffusion

OPE region: $\tau_1 \approx \tau_2 \gg \tau_3 \approx \tau_4$: the four-point function $\sim$ two-point function of collective modes.

1. Leading contribution: energy momentum tensor.



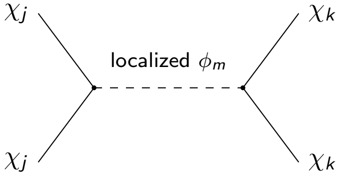
2. A diffusion pole $\frac{1}{-i\omega+Dp^2}$ with diffusion constant:

$$D \sim \frac{J_1^2}{J}, \text{independent of temperature}$$

J_1: coupling between neighbor sites; J: effective on-site coupling.

Other fields in the OPE

- Subleading contributions: an infinite family of locally critical fields ϕ_m, $m = 1, 2, \ldots$. Short-range correlated in space



Chaos and butterfly velocity

- OTOC

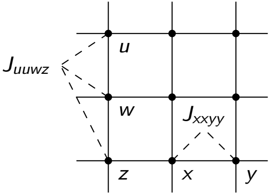
$$\langle \chi_{j,x}(t)\chi_{k,0}(0)\chi_{j,x}(t)\chi_{k,0}(0)\rangle_{\text{conn.}} \sim \frac{1}{N}\exp\frac{2\pi}{\beta}(t-|x|/v_B)$$

- $\lambda_L = \frac{2\pi}{\beta}$ true at least to $\frac{1}{(\beta J)^2}$. (Correction vanishes at $\frac{1}{\beta J}$ order)
- Butterfly velocity: $v_B \sim J_1\sqrt{\frac{T}{J}}$ satisfies $v_B^2 = 2\pi TD$ (agree with incoherent black hole [Blake 2016]. Relevant to incoherent metal [Hartnoll 2014].)
- Intuitive reason: propagator $\propto \frac{1}{-i\omega+Dp^2}\frac{1}{\omega^2+\lambda_L^2}$. The characteristic frequency $\omega = i\lambda_L = 2\pi Ti$ leads to the pole $p^2 = -\frac{\lambda_L}{D}$, $\Rightarrow v_B/\lambda_L = |p|^{-1} = (D/\lambda_L)^{1/2}$.

Brief discussion on general construction

- Our model can be defined on arbitrary lattice Γ in any dimensions.

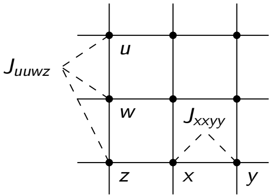
$$H = \sum_{x,y,z,w\in\Gamma} \sum_{j,l,k,m=1}^{N} J_{jklm,xyzw}\chi_{j,x}\chi_{k,y}\chi_{l,z}\chi_{m,w}$$



- Independent random numbers $\overline{J^2_{jklm,xyzw}} = \frac{J^2_{xyzw}}{N^3}$, solvable at large N.

- Locality: J_{xyzw} are "local functions" of $xyzw$.

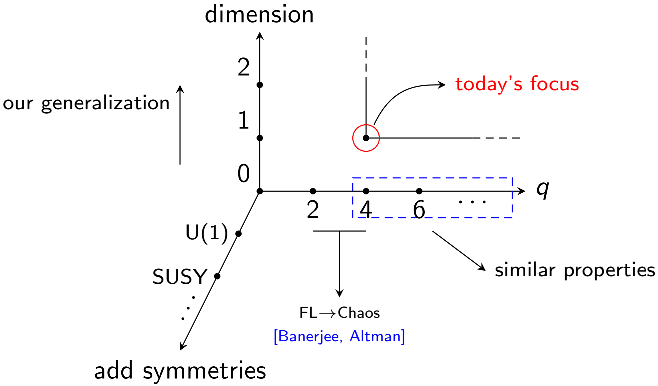
Brief discussion on general construction



1. Emergent $(0+1)$-d conformal symmetry at strong coupling; local $SO(N)$ symmetry on each site after average; maximal chaos;
2. Diffusive energy transport and butterfly velocity. For square lattice $v_{B,j}^2 = 2\pi T D_j$ holds for all directions x_j, $j = 1, 2 \ldots, d$.

Further generalization: adding global symmetries.

Overview of SYK family



- q: random q-body interaction (previous slide $q = 4$)

$$H = \sum J \underbrace{\chi\chi\ldots\chi}_{\text{q Majoranas}},$$ [Kitaev; Maldacena, Stanford]

Summary and discussion

Summary:

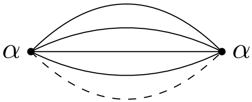
1. Generalized SYK models are diffusive metals without quasiparticles
2. Local criticality and maximal chaos
3. Universal relation between diffusion and butterfly velocity

Discussions:

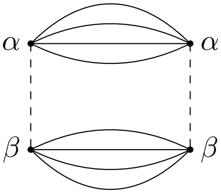
1. A platform to study properties of strongly correlated system **exactly**;
2. Is our model holographic?
3. Further generalizations?
4. Localization transition?

Replicon diagonal effective action (Backup slides)

1. Replica trick: $\overline{\log Z} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}$
2. Start with replicated partition function Z^n;
3. Disorder average $\overline{Z^n}$;
4. Large $N \Rightarrow \overline{Z^n} = \overline{Z}^n$, therefore $\overline{\log Z} = \log \overline{Z}$

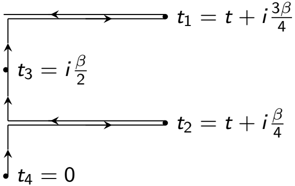


(a) Replicon diagonal $\sim N$



(b) Off-diagonal $\sim 1/N^2$

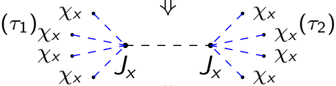
Keldysh-Schwinger contour (Backup slides)



backup

$$J_{jklm,x}\chi_{j,x}\chi_{k,x}\chi_{l,x}\chi_{m,x}$$

$$\Downarrow$$

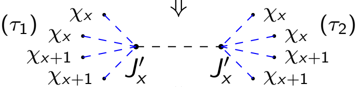


$$\Downarrow$$

$$J_0^2 G_x(\tau_1,\tau_2)^4$$

$$J'_{jklm,x}\chi_{j,x}\chi_{k,x}\chi_{l,x+1}\chi_{m,x+1}$$

$$\Downarrow$$



$$\Downarrow$$

$$J_1^2 G_x(\tau_1,\tau_2)^2 G_{x+1}(\tau_1,\tau_2)^2$$