Dynamics of a New Topology – the Euler Class

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KITP, 7 June 2021
Interacting Topological Matter: Atomic, Molecular And Optical Systems

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FNÜ, Bouhon, Slager, PRL 125, 053601 (2020)

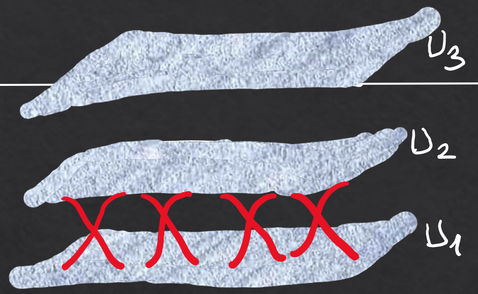
THE ROYAL SOCIETY

Newton International Fellowship

EPSRC
Engineering and Physical Sciences Research Council

DesOEQ

Why Euler Class?



2 ξ nodes

- Remarkable progress in characterising topological phases; crystalline insulators, higher order topology, fragile invariants…

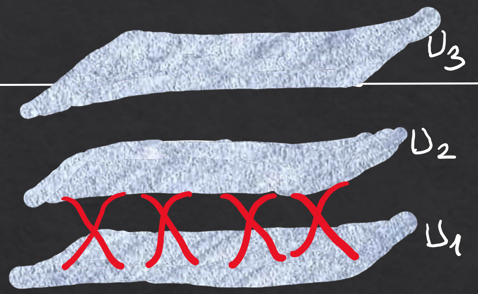
- Exotic new invariant that falls outside symmetry eigenvalue indicated phases: Euler Class

$$\xi = \frac{1}{2\pi}\oint d^2k \ \left\langle \partial_{k_x} u_1 \middle| \partial_{k_y} u_2 \right\rangle - \left\langle \partial_{k_y} u_1 \middle| \partial_{k_x} u_2 \right\rangle$$

  - Analogue of Chern number in systems with $\mathcal{C}_2\mathcal{T}$ or $\mathcal{PT}$
  - Requires minimum 3 bands
  - Band nodes host non-Abelian braiding properties!
  - Fragile topology

Bouhon, Wu, Slager et al, **Nat Phys 16**, 1137 (2020); Jiang, Slager et al. arXiv:2104.13397; Wu et al. **Science 365**, 1273 (2019); Ahn et al., **PRX 9**, 021013 (2019) …

Why Euler Class?



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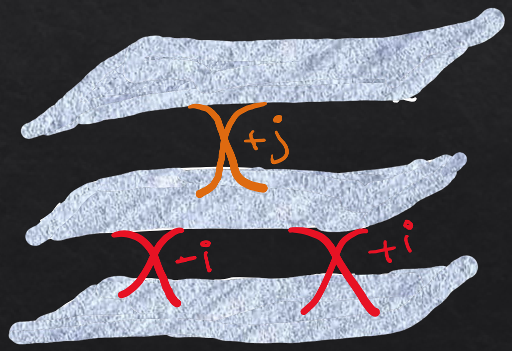
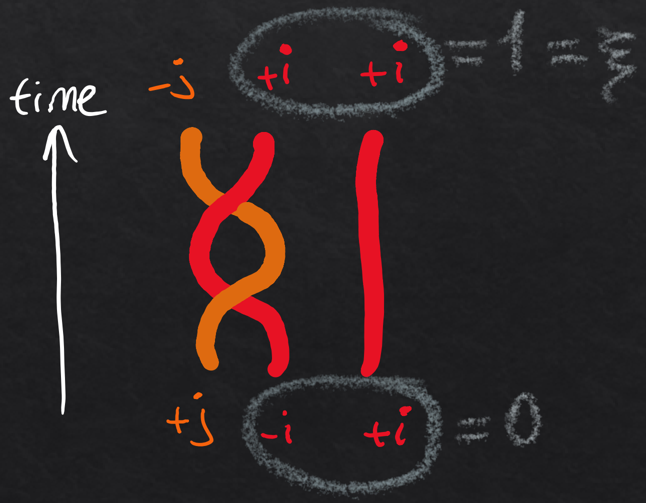
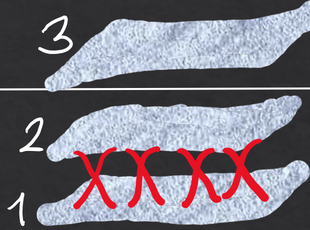




Concrete experimental signatures for Euler class?

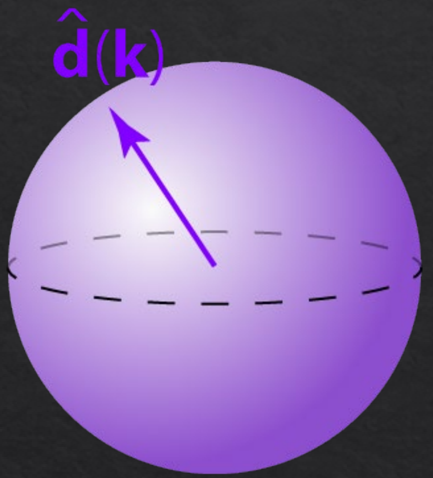
Bouhon, Wu, Slager et al, **Nat Phys 16**, 1137 (2020); Jiang, Slager et al. arXiv:2104.13397; Wu et al. **Science 365**, 1273 (2019); Ahn et al., **PRX 9**, 021013 (2019) …

Euler Class



Chern Number

2 bands: $H_{\mathcal{C}}(\boldsymbol{k}) = \boldsymbol{d}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$



$\pi_2(S^2) = \mathbf{Z}$

$$\mathcal{C} = \frac{1}{4\pi} \oint d^2k \;\, \hat{\boldsymbol{d}} \cdot \left(\partial_{k_x} \hat{\boldsymbol{d}} \times \partial_{k_y} \hat{\boldsymbol{d}} \right)$$

solid angle

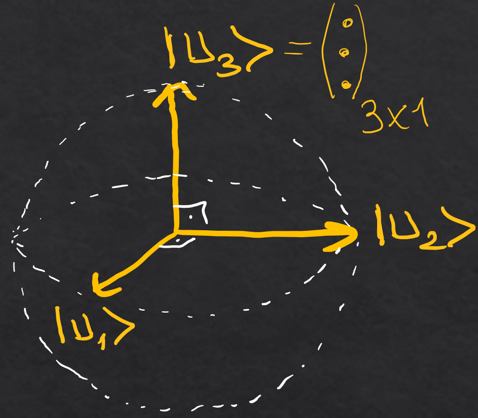
Euler Invariant

Gell-Mann

3 bands: $\mathcal{H}(\boldsymbol{k}) = \boldsymbol{h}(\boldsymbol{k}) \cdot \boldsymbol{\lambda}$

Real!

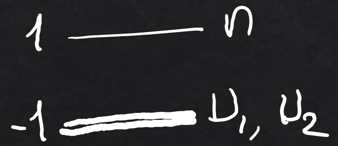
Eigenstates $\{|u_j(\boldsymbol{k})\rangle\}$ form a *dreibein:*



$$\boldsymbol{n}(\boldsymbol{k}) \equiv u_3(\boldsymbol{k}) = u_1(\boldsymbol{k}) \times u_2(\boldsymbol{k})$$

$$\mathcal{H}(\boldsymbol{k}) = 2\, \boldsymbol{n}(\boldsymbol{k}) \cdot \boldsymbol{n}(\boldsymbol{k})^{\mathrm{T}} - \mathbb{I}_3$$

Spectrally flattened



$$\xi = \frac{1}{2\pi} \oint d^2k \;\, \boldsymbol{n} \cdot \left(\partial_{k_x} \boldsymbol{n} \times \partial_{k_y} \boldsymbol{n} \right)$$

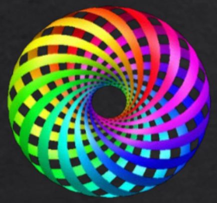
Bouhon, Bzdušek, Slager, PRB'20

Euler Class – dynamics on S^2

FNÜ, Bouhon, Slager, PRL 125, 053601 (2020)

1) $\psi_0(\boldsymbol{k}) = \begin{pmatrix}1\\0\\0\end{pmatrix}$: Trivial initial state, naturally corresponds to S^2

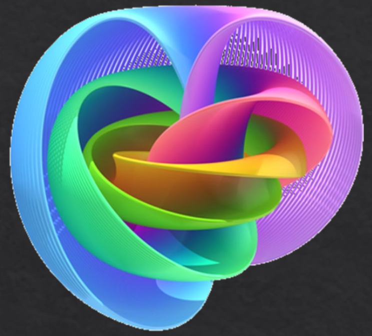
2) Time-evolution traces a circle,
$(k_x, k_y, t) \equiv T^3 \rightarrow S^3$



Hopf

Linking

2D momentum+time



$T^3 \rightarrow S^3$

Hopf Map

$\pi_3(S^2) = \mathbf{Z}$

Bloch Sphere

S^2

(Image: Wikipedia)

Euler Class – dynamics on S^2

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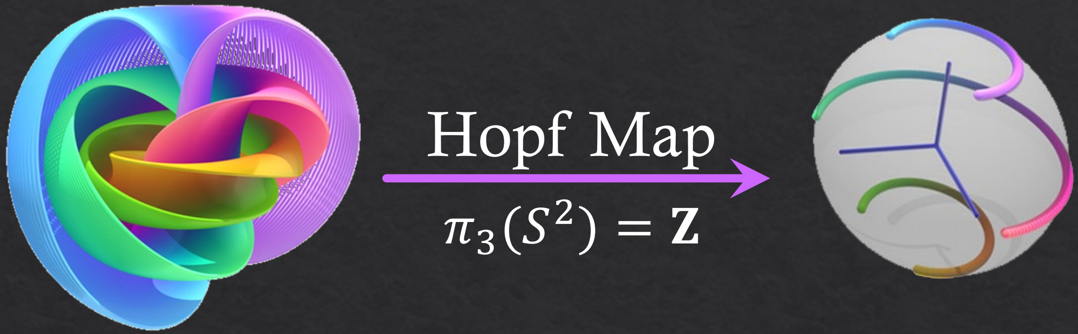


(Image: Wikipedia)

2) Time-evolution traces a circle,
$(k_x, k_y, t) \equiv T^3 \rightarrow S^3$

$$\psi(\boldsymbol{k}, t) = e^{-itH(\boldsymbol{k})}\, \psi_0(\boldsymbol{k})$$

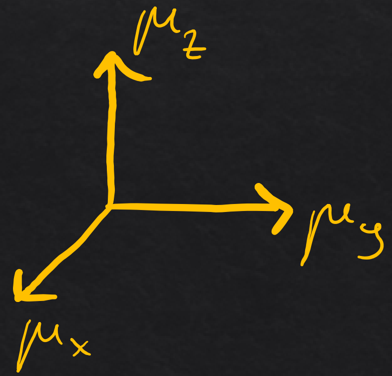
$$\Big\downarrow H^2 = \mathbb{I}_3$$

$$= [\cos(t) - \mathbf{i}\sin(t) H(k)]\, \psi_0(\boldsymbol{k})$$

$\psi(k,t)$! complex ✗

$$\mu_x = \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

"Bloch sphere S^2"



$$\mu_y = \begin{pmatrix} 0 & 1 & -i \\ 1 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\mu_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\widehat{\boldsymbol{p}}(\boldsymbol{k}, \boldsymbol{t}) = \psi^\dagger(\boldsymbol{k}, t)\, \boldsymbol{\mu}\, \psi(\boldsymbol{k}, t)$$

Hopf Map

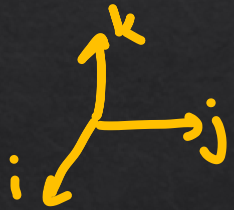
Q: How to analytically establish $S^3 \rightarrow S^2$?

A: Quaternions*

*Extends Complex numbers; $\mathbf{R}^4$ with $\mathbf{C}^2$ ($\mathbf{C}$ relates to $\mathbf{R}^2$)

$$q = x_0 + x_1 i + x_2 j + x_3 k, \qquad q \in \mathbf{R}^4$$

3D vector



$$ij = k, jk = i, ki = j$$

$$i^2 = j^2 = k^2 = -1$$

Generates rotations in 3D! $R_v \mathbf{t} = \mathbf{t}'$

The matrix of a proper rotation R by angle θ around the axis $\mathbf{u} = (u_x, u_y, u_z)$, a unit vector with $u_x^2 + u_y^2 + u_z^2 = 1$, is given by:

$$R = \begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}.$$

Hopf Map

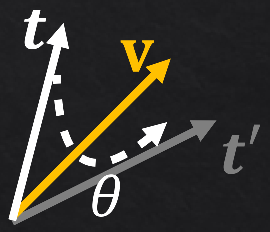
- Quaternion, $q = x_0 + x_1 i + x_2 j + x_3 k, \quad q \in \mathbf{R}^4$
- Versor, $|v| = \sqrt{\sum_n x_n^2} = 1$, spans $\boldsymbol{S}^3 \subset \mathbf{R}^4$
- Pure quaternion: a vector $\boldsymbol{t} \in \mathbf{R}^3$

➢ Multiplying a pure quaternion with an arbitrary versor results in another pure quaternion

$$R_v : \mathbf{t} \mapsto \mathbf{t}' = v\mathbf{t}v^{-1} \qquad\qquad v^{-1} = \frac{v^*}{\|v\|^2}$$

So this norm preserving action generates Rotations in 3D!

$$v = \cos(\theta/2) + \sin(\theta/2)\mathbf{v}$$



$$R_v \mathbf{t} = \mathbf{t}'$$

$$R_v = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2x_1x_2 - 2x_0x_3 & 2x_0x_2 + 2x_1x_3 \\ 2x_1x_2 + 2x_0x_3 & x_0^2 - x_1^2 + x_2^2 - x_3^2 & -2x_0x_1 + 2x_2x_3 \\ -2x_0x_2 + 2x_1x_3 & 2x_0x_1 + 2x_2x_3 & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}$$

Hopf Map

How does this define the Hopf Map?

- Versor, $|v| = \sqrt{\sum_n x_n^2} = 1$, spans $S^3 \subset \mathbf{R}^4$
- Pure quaternion: a vector $t \in \mathbf{R}^3$ unit => $t \in S^2$

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$$R_v : \mathbf{t} \mapsto \mathbf{t}' = v\mathbf{t}v^{-1}$$

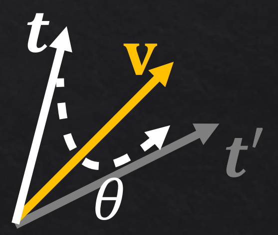
$S^3 \to S^2$

Hopf Invariant

$$H = \frac{1}{V} \oint d^3p \, \epsilon^{ijk\ell} x_i \, \partial_{p_x} x_j \, \partial_{p_y} x_k \, \partial_{p_z} x_\ell$$

$$R_v \mathbf{t} = \mathbf{t}'$$

$$v = \cos(\theta/2) + \sin(\theta/2)\mathbf{v}$$



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Key Insight:

$$\psi(\boldsymbol{k}, t) = [\cos(t) - \mathbf{i}\sin(t) H(k)]\, \psi_0(\boldsymbol{k})$$

$$\{x_0, \boldsymbol{x}\} = \{\cos t\,, -\sin(t)\, H\psi_0\}$$

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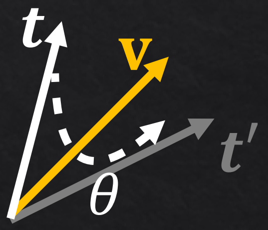

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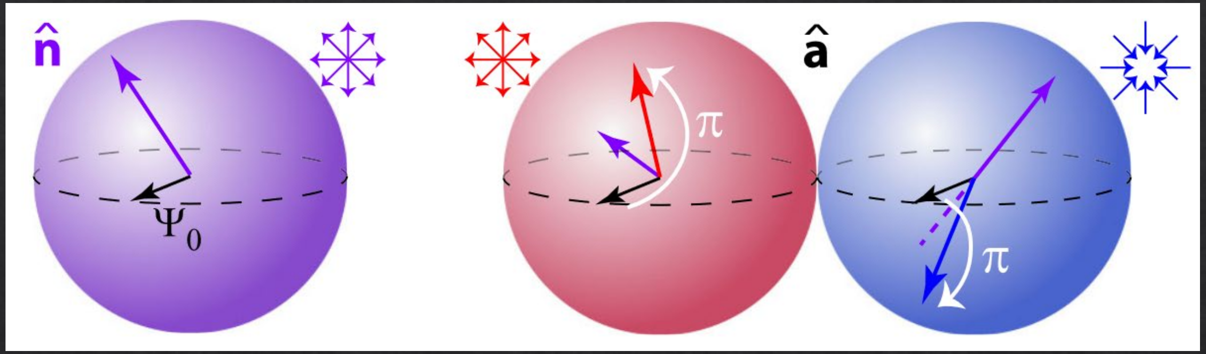
Monopole – anti-monopole pairs

- Physical consequences of the Hopf construction

$$\psi_0(\boldsymbol{k},0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Quench w/ non-trivial Euler Hamiltonian:

Monopole – anti-monopole pair topologically stable



$$\mathcal{H}(\boldsymbol{k}) = 2\,\boldsymbol{n}(\boldsymbol{k}) \cdot \boldsymbol{n}(\boldsymbol{k})^{\mathrm{T}} - \mathbb{I}_3$$

$$\psi(\boldsymbol{k},t) = [\cos t - i \sin t\, \mathcal{H}(k)]\, \psi_0$$

$$\boldsymbol{a}(\boldsymbol{k}) = \mathcal{H}(\boldsymbol{k})\psi_0$$

Hamiltonian corresponds to a π–rotation around $\boldsymbol{n}(\boldsymbol{k})$.

$$\boldsymbol{n}(\boldsymbol{k}) = \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \end{pmatrix} \Longrightarrow \boldsymbol{a}(\boldsymbol{k}) = \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha\cos\beta \\ \sin 2\alpha\sin\beta \end{pmatrix}$$

$\alpha \in [0,\pi)$ from $+\hat{x}$

Monopole – anti-monopole pairs

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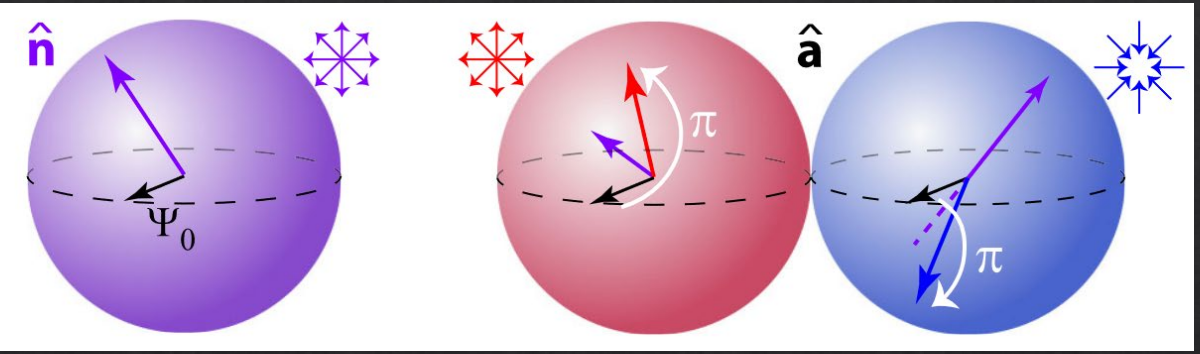
Monopole – anti-monopole pair topologically stable



Hamiltonian corresponds to a π–rotation around $\boldsymbol{n}(\boldsymbol{k})$.

- Analytically $\text{Hopf} = \frac{1}{4\pi} \oint d^2k \;\; \boldsymbol{a} \cdot \left(\partial_{k_x} \boldsymbol{a} \times \partial_{k_y} \boldsymbol{a} \right)$

Hopf ⟷ Euler

↕

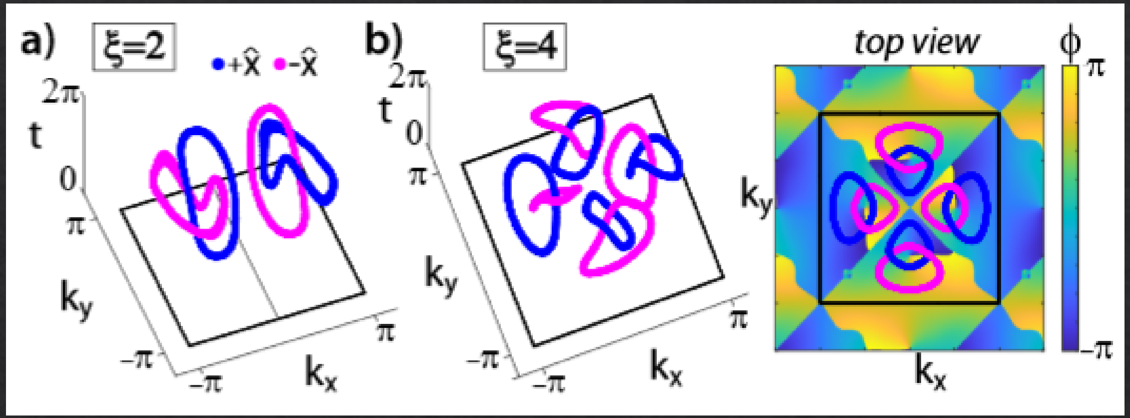
Linking

Euler Class – observables

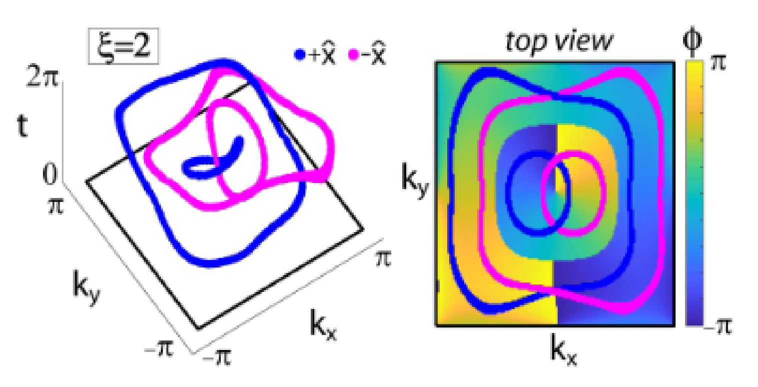
- Linkings

$\psi_0(\boldsymbol{k},0)$

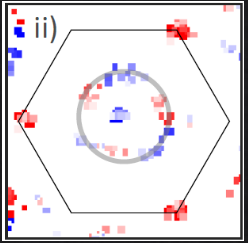
$$= \begin{pmatrix}1\\0\\0\end{pmatrix}$$



$$= \begin{pmatrix}0\\0\\1\end{pmatrix}$$

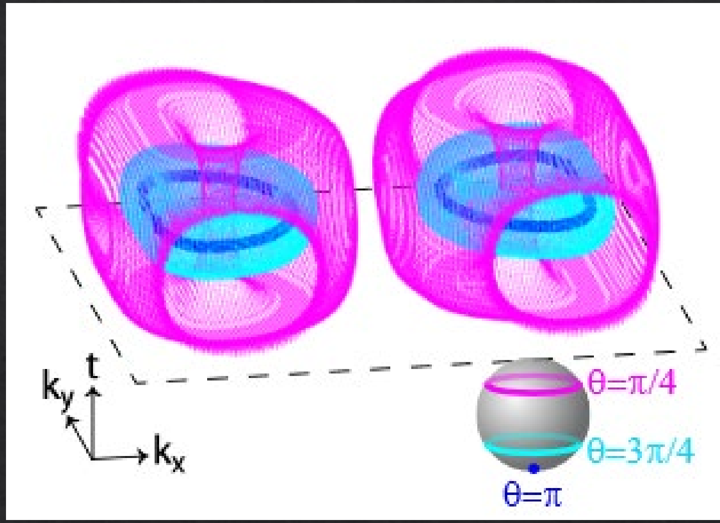


Linking measured in experiment for Chern# !
Tarnowski, FNÜ et al. Nature Comm'19



$\mathcal{C} = 1$

- Hopf Tori $\xi = 2$



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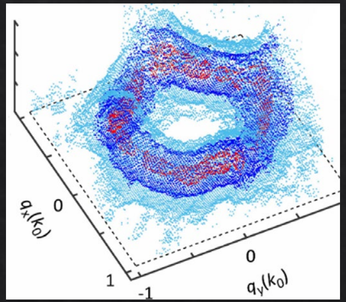


Observed in Raman lattices for Chern# !
Yi et al. arXiv:1904.11656
Sun, Yi et al. PRL'18

Euler Class – experimental protocols

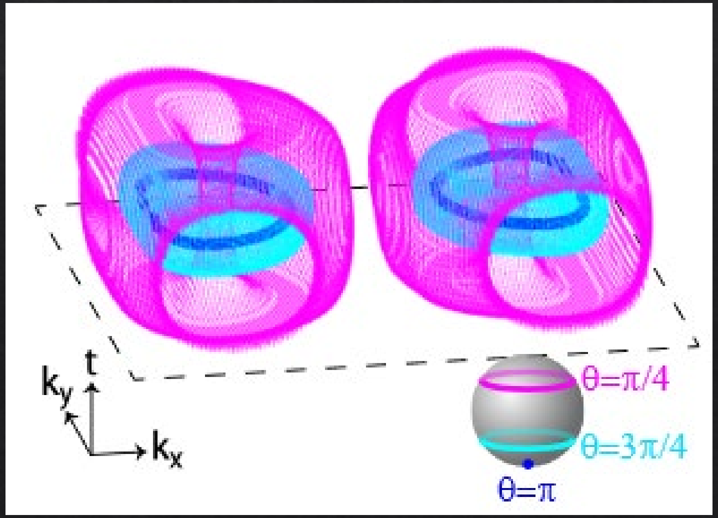
1) Pseudospin ≡ hyperfine states $|A\rangle, |B\rangle, |C\rangle$

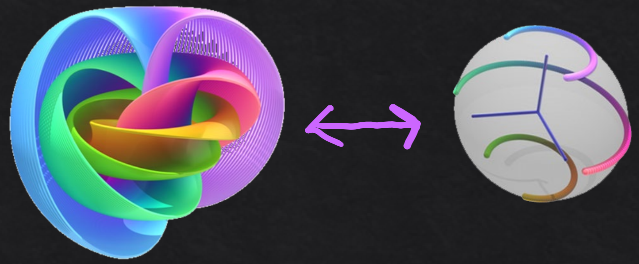
- Quenching $|\psi_0\rangle = |A\rangle$ initiates Raman-induced oscillations
- Spin polarization: $P_z(\boldsymbol{k}, t) = \frac{N_A - N_B - N_C}{N_A + N_B + N_C}$
- Inverse image of $P_z(\boldsymbol{k}, t) = \cos\theta$ traces a closed surface in T^3

$\hat{z}$- direction

$$\mu_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Nested Hopf tori





Euler Class – experimental protocols

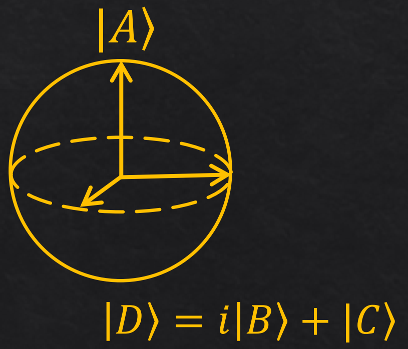
1) Pseudospin ≡ hyperfine states $\Longrightarrow$ $Polarization$

2) Pseudospin ≡ sublattice

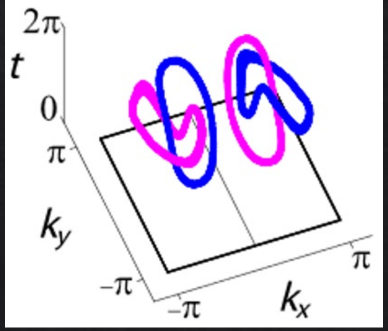
- TOF: $m(k,t) \propto |\, (\langle A| + \langle B| + \langle C|)\, |\psi(\boldsymbol{k},t)\rangle\, |^2$

$$\psi(\boldsymbol{k},t) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \xrightarrow[H_{pulse} \propto diag(1,-1,1)]{\pi/2 \text{ pulse}} \begin{pmatrix} \psi_1 \\ i\psi_2 \\ \psi_3 \end{pmatrix} \xrightarrow[H_{tom} = \omega\mu_z/2]{\text{time } t} m(k,t) \propto [1 + \sin\theta_k \cos(\phi_k + \omega t)]$$

$$\psi(\boldsymbol{k},t) = \begin{pmatrix} x_0 + ix_1 \\ x_2 \\ x_3 \end{pmatrix} \Longrightarrow \zeta = \begin{pmatrix} x_0 + ix_1 \\ x_2 + i\, x_3 \end{pmatrix}$$



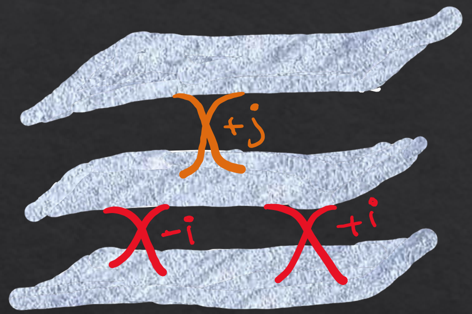
Linking as vortices in ϕ



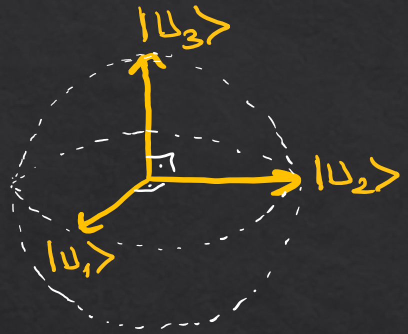
*Tomography in experiments: Fläschner et al. Science’16 ; Li et al. Science’16 …

Conclusion

FNÜ, Bouhon, Slager, **PRL 125**, 053601 (2020)

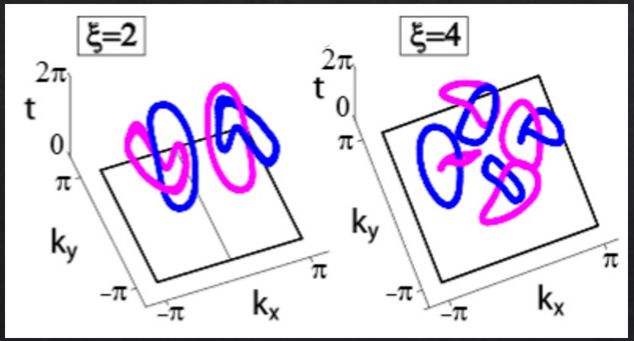


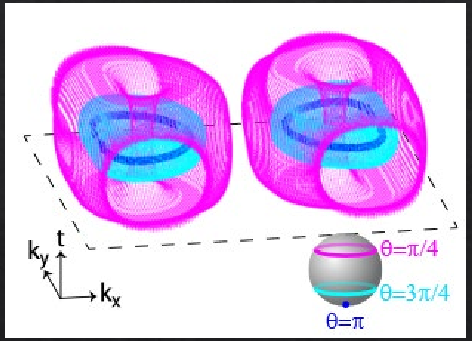
- Euler Class features many intriguing topological properties multi-gap topology, non-Abelian braiding …



- New phenomena in its quench dynamics!

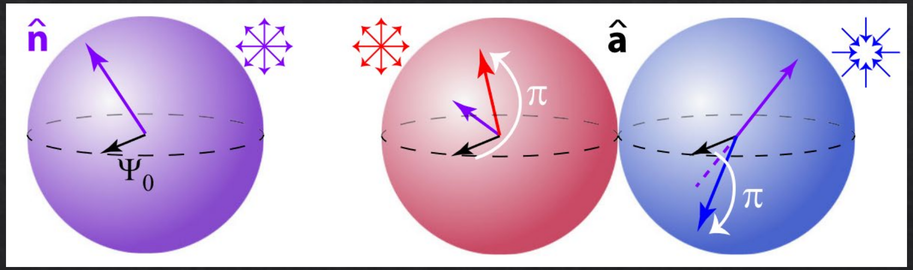
- Naturally embodies a Hopf construction





- Concrete signals/protocols for experiments to observe Euler Class!

- Monopole – anti-monopole pairs



➢ *New crystalline and exotic fragile topologies to explore in ultracold atoms…*

Thank you!