Scattering theory of topological insulators

Anton Akhmerov
with Carlo Beenakker, Jan Dahlhaus, Cosma Fulga,
Fabian Hassler, and Michael Wimmer

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Outline

1. Topological invariant and topological insulators
2. Scattering approach example: Majorana fermions in 1D
3. General case: dimensional reduction
4. Applications

Topological insulators and superconductors

Topological insulator is

- A material with a band gap in the bulk
  (and a certain discrete symmetry)
- It has protected zero energy states at the edge
- Number of these states is a *topological invariant* $\mathcal{Q}[H(\mathbf{k})]$, an integer which does not change under small perturbations.

Classification

Three discrete symmetries (Altland&Zirnbauer):
$\mathcal{T} : H(k) = U_{\mathcal{T}} H^*(-k) U_{\mathcal{T}}^\dagger$, $\mathcal{P} : H(k) = -U_{\mathcal{P}} H^*(-k) U_{\mathcal{P}}^\dagger$,
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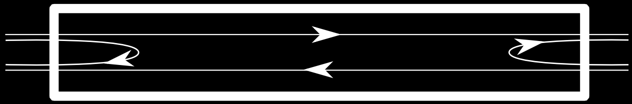
give 10 symmetry classes and a lot of topological insulators (Kitaev):

Symmetry	d							
class	1	2	3	4	5	6	7	8
A		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
AIII	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
AI				$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\mathbb{Z}$				$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$		$\mathbb{Z}_2$
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$	
AII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$
CII	$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
C		$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
CI			$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Scattering matrix



$$\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}_{\text{out}} = S \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}_{\text{in}}$$

- Describes scattering of free particles from the system at the Fermi level.
- Is also constrained by symmetry.
- Easy to tell an insulator from a conductor.

What about $\mathcal{Q}(S)$?

Simple case: Majorana fermions (1D superconductor)

Reflection matrix r has

Current conservation:

$$rr^\dagger = 1 \Rightarrow |\det r| = 1$$

Particle-hole symmetry:

$$r = \begin{pmatrix} r_{ee} & r_{he} \\ r_{eh} & r_{hh} \end{pmatrix} = \begin{pmatrix} r_{ee} & r_{he} \\ r_{he}^* & r_{ee}^* \end{pmatrix} \Rightarrow \mathrm{Im}\det r = 0$$

Together:

$$\det r = \pm 1$$

Simple case: Majorana fermions (1D superconductor)

$\det r = -1 \Rightarrow \det(r-1) = 0 \Leftrightarrow$ bound state at zero energy.
$\Rightarrow$ Superconductor is in topologically nontrivial phase.

Scattering invariant

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Phase transition is accompanied by a single fully transmitted mode.

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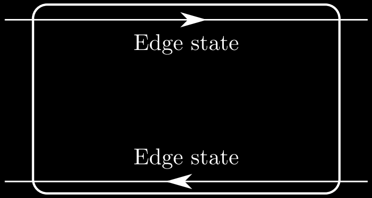
It works!

Symmetry	D	DIII	AIII	BDI	CII
$\mathcal{Q}(r)$	$\operatorname{sign}\det r$	$\operatorname{sign}\operatorname{Pf} r$	$\nu(r)$	$\nu(r)$	$\nu(r)$

Question

What about higher dimensions?

Higher dimensions: QHE



- Not insulating due to edge states?

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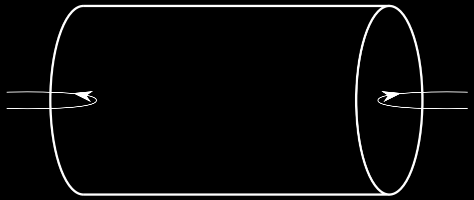

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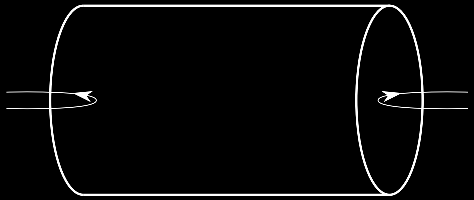

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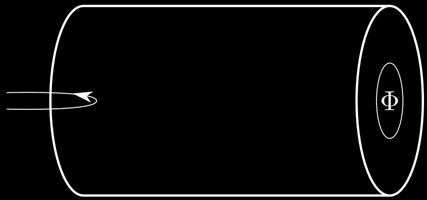

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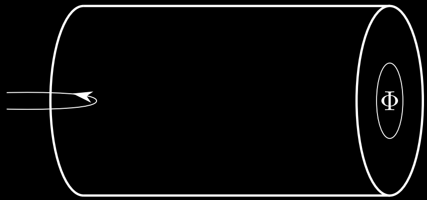

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- Charge pumping is a winding number of $\det r(\Phi)$!

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Q: Isn't that a lot of work?

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$$H_{d-1}(\mathbf{k}) = \begin{pmatrix} 0 & r(\mathbf{k}) \\ r^\dagger(\mathbf{k}) & 0 \end{pmatrix}$$

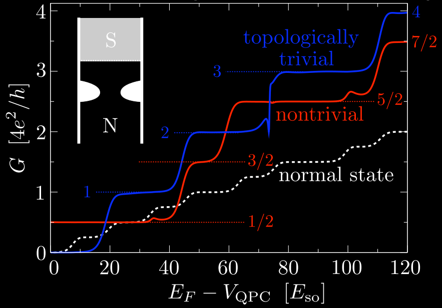
This $H_{d-1}(\mathbf{k})$ has the same topology as $r(\mathbf{k})$,
(Symmetry of H_{d-1} is shifted according to the Kitaev's periodic table.)

Algorithm for $\mathcal{Q}(S)$

1. Start from d-dimensional $H_d(\mathbf{k}_d)$.
2. Close $d-1$ dimensions with twisted boundary conditions.
3. Calculate $r(\mathbf{k}_{d-1})$ and $H_{d-1}(\mathbf{k})$.
4. Finally, look up the expression for $\mathcal{Q}(H_{d-1})$.

Applications I: 1D

1. Half-integer conductance quantization in a topological QPC



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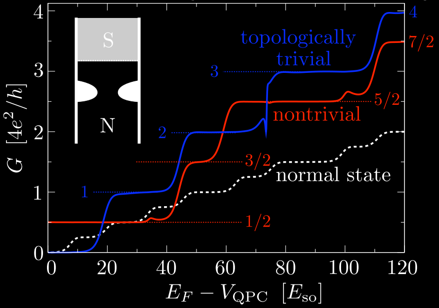

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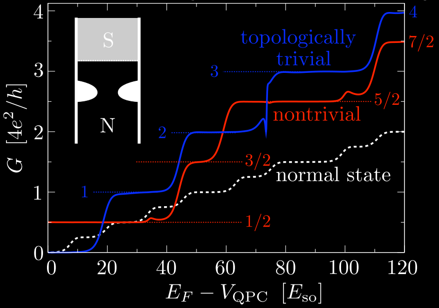
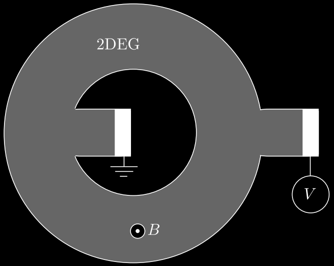
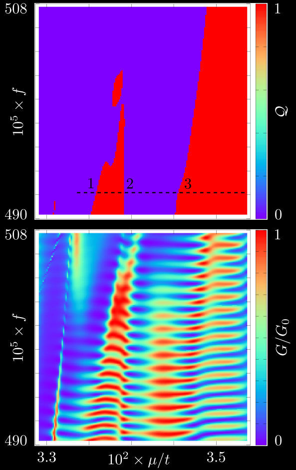


2. Quantized transmission and shot noise at the phase transition.
3. RMT of topological superconductors:
   In N-channel dot $\langle G^N \rangle_{\rm trivial} \neq \langle G^N \rangle_{\rm nontrivial}$

Applications II: QHE

Quantized conductance peaks in a mesoscopic phase transition.





Applications III: uses for $\mathcal{Q}$

1. High numerical efficiency:
   systems of 2000×2000 vs 60×60 in 2D
   and of $50 \times 50 \times 50$ vs $12 \times 12 \times 12$ in 3D
2. A new tool to study phase transitions with disorder.
   Conductance scaling in disordered QHE:
   $$\sigma = \sigma_0 + C_1(\mu - \mu_0)^2 L^{-2/\nu}.$$
   Topological invariant scaling:
   $$\mathcal{Q} = 1/2 + C_2(\mu - \mu_0) L^{-1/\nu}.$$

Summary

- Topological invariant can be calculated from $r(\mathbf{k})$.
- Scattering approach provides a universal framework for studying signatures of topology in transport.

Summary

Thank you all.
The end.