Geometry of Response for open quantum systems

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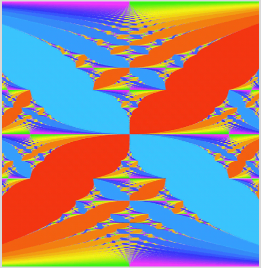
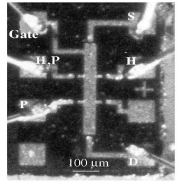


Motivation: Quantized transport

- God created all atoms equal;
  all devices created by man are different
- Hall conductance: Topological
- What if there is no H and no $|\psi\rangle$?
- Is response in open system geometric? Topological?



Krauss maps

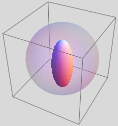
Evolution of subsystems

$$\begin{array}{ccc} |\psi\rangle_{s+b} & \xrightarrow{Unitary} & |\psi'\rangle_{s+b} \\ Tr_b \downarrow & & \downarrow Tr_b \\ \rho_s & \xrightarrow{Krauss} & \rho'_s \end{array}$$

Trace and Positivity preserving

Krauss (CP) maps

$$\rho \mapsto \sum \underbrace{A_j \rho A_j^\dagger}_{positive}, \quad \underbrace{\sum A_j^\dagger A_j = 1}_{trace\ \ preserving}$$



Bloch sphere:
Unitary:Rigid rotation

Krauss: Contraction

Lindbladians

$$\delta\rho = \mathcal{L}(\rho)\,\delta t$$

Lindblad: generator of Krauss maps

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum [\Gamma_a, \rho\Gamma_a^*] + [\Gamma_a\rho, \Gamma_a^*]\,, \quad H = H^*$$

Example (Thermalizaiton & Dephasing)

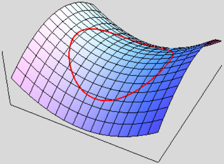
Gauge freedom: $\mathcal{L}$ invariant under

$$\delta\Gamma = g\,\mathbb{1}, \quad \delta H = i(g\Gamma^* - g^*\Gamma)$$

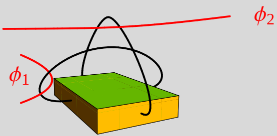
- Energy of system ambiguous;
- Fuzzy bdry

Adiabatic Lindbladians

- $\phi \in$ Control space
- $H(\phi), \Gamma_a(\phi)$: Controlled Lindbladians
- $\dot{\phi}(s)$ Adiabatic drivers; $s = \epsilon t$ slow time.
- $\epsilon \dot{\rho} = \mathcal{L}_\phi(\rho)$: adiabatic evolution



Closed adiabatic path in control space



quantum Hall effect as control problem

(Formal) Adiabatic expansion

- Anzatz: $\rho = \rho_0 + \epsilon\rho_1 + \dots$
- Substitute in $\epsilon\dot\rho = \mathcal{L}(\rho)$
- To $O(\epsilon)$: $\epsilon\dot\rho_0 = \mathcal{L}(\rho_0) + \epsilon\mathcal{L}(\rho_1) =$

$$\begin{cases} \mathcal{L}(\rho_0) = 0; & \rho_0 = \textit{stationary} \\ \rho_1 = \mathcal{L}^{-1}(\dot\rho_0); & \rho_1 = \textit{slave} \end{cases}$$

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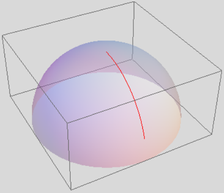
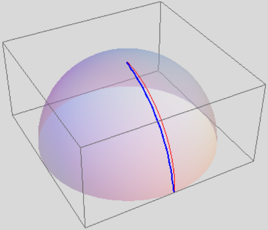


Motion of the inst. stat. state.

Adiabatic expansion slaved to stationary states

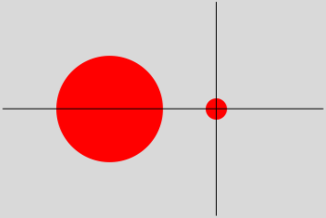
- $\sigma = \rho_0$ inst. stat. state: $\mathcal{L}(\sigma) = 0$
- $\rho = \sigma + \epsilon\mathcal{L}^{-1}(\dot\sigma) + \dots$



adiabatic motion slaved to stat. state.

Projection on stationary states

- Gap condition
- $P_\phi = \frac{1}{2\pi i} \oint \frac{dz}{\mathcal{L}_\phi - z}$:
  projects on inst. stat. state
- Exapmle: If σ_ϕ unique sta. state:
  $$P_\phi(\rho) = \sigma_\phi Tr(\rho)$$



0- Isolated eigenvalue, possibly degenerate

Useful fact about projections

$$P\dot{P}P = 0$$

Differentiate: $P^2 = P$

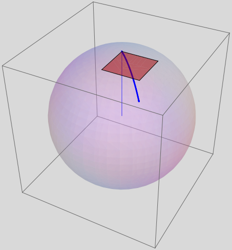
Parallel transport of stationary states

- $0 = \mathcal{L}P$; by definition
- $[P, \mathcal{L}] = 0$; a fact about Lindbladians
- $\epsilon\dot{\rho} = \mathcal{L}(\rho) \Longrightarrow P\dot{\rho} = P\mathcal{L}(\rho) = \mathcal{L}(P\rho) = 0$

Stationary states move by parallel transport

$$P\dot{\sigma} = 0$$

First order ODE: $\sigma = P\,\sigma \to \dot{\sigma} = \dot{P}\sigma$



Parallel transport

Rates and Response

- Rates: $\epsilon\dot{Q} = \mathcal{L}^*(Q)$
- Exapmle: velocity =rate of position
- $\dot{Q}$ depend on $(H, \Gamma) \leftrightarrow \mathcal{L}$

Substitute adiabatic expansion

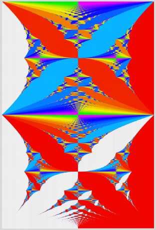
$$\begin{aligned}\left\langle \dot{Q} \right\rangle &= \frac{1}{\epsilon} Tr\big(\rho\mathcal{L}^*(Q)\big) \\ &= \frac{1}{\epsilon} Tr\big(\sigma\mathcal{L}^*(Q)\big) + Tr\big(\mathcal{L}^{-1}(\dot{\sigma})\mathcal{L}^*(Q)\big) \\ &= Tr\big(\underbrace{\mathcal{L}(\sigma)}_{0} Q\big) + Tr\big(\dot{\sigma}\, Q\big)\end{aligned}$$

Linear response: $\left\langle \dot{Q} \right\rangle = Tr(Q\,\dot{\sigma}) + O(\epsilon)$

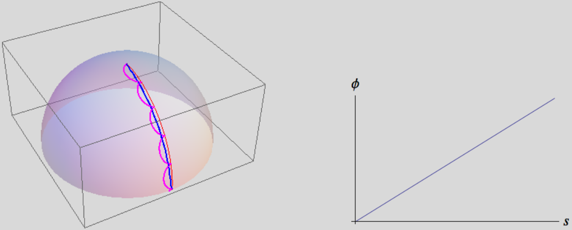
Main results

Geometry & stability

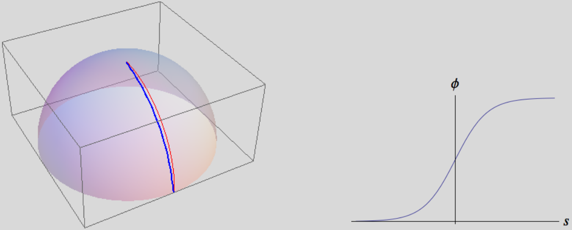
- Motion of stat. states is by parallel transport
- Response of rates depends only on stationary states, not on dynamics
- Lindblad and Hamiltonians can share stationary states (dephasing, decay to ground state)



Jerk, Rapid oscillations, and initial data



Jerked motion that starts on stationary states oscillates



No oscillations when slaved motion is not jerked

More on rates

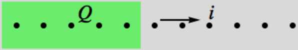
- Rates: Depend on (H, Γ);
- Gauge independent

Example (Dephasing oscillator:)

$$2H = x^2 + p^2, \quad \Gamma = \sqrt{\gamma} H \Longrightarrow \underbrace{f}_{force} = \mathcal{L}^*(p) = \underbrace{-x}_{spring} - \underbrace{\gamma p}_{friction} \;;$$

When charge conserved:

Current $=$ rate of charge in semi-infinite box



Iso-spectral Lindbladians

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$$H(\phi) = U(\phi)HU^*(\phi), \quad \Gamma(\phi) = U(\phi)\Gamma U^*(\phi), \quad U(\phi) = e^{iD_\mu \phi^\mu}$$

Linear response of $\dot{D}_\nu$ is geometric:

The response coefficients are the expectations of the Lie algebra

$$\underbrace{\left\langle \dot{D}_\nu \right\rangle}_{response} = f_{\mu\nu} \underbrace{\dot{\phi}^\mu}_{driving}, \quad f_{\mu\nu} = \underbrace{iTr([D_\nu, D_\mu]\sigma)}_{response\ \ coeff.}$$

Use $\partial_\mu \sigma = i[D_\mu, \sigma]$ and cyclicity of trace.