Fractional topological insulators

Christopher Mudry[1] Titus Neupert[1] Claudio Chamon[2] Luiz Santos[3] Shinsei Ryu[4]

[1]Paul Scherrer Institut, Switzerland

[2]Boston University, USA

[3]Harvard University, USA

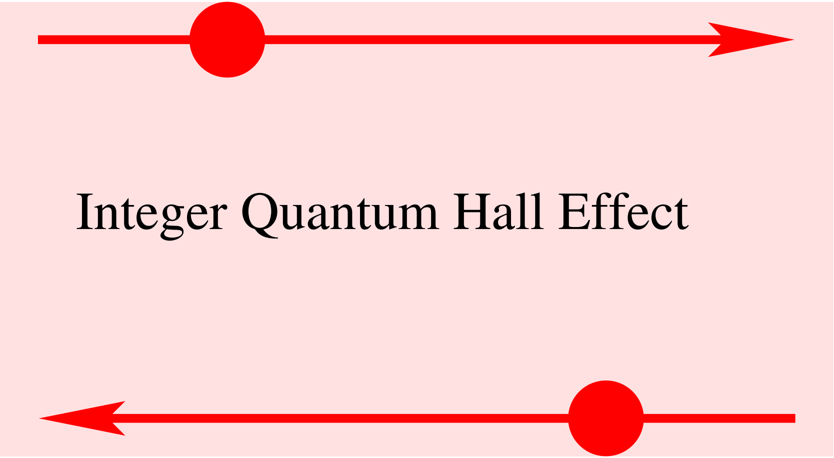
[4]University of Illinois, Urbana-Champaign, USA

KITP, 14 October 2011

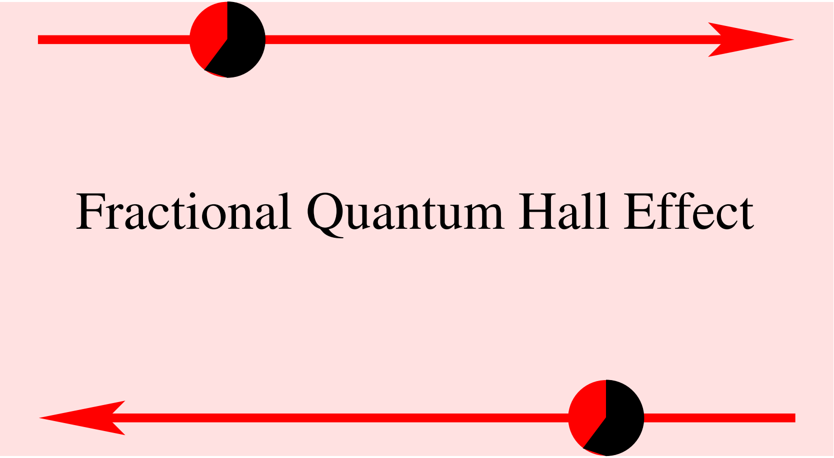
Many thanks to the organizers!!!

Outline

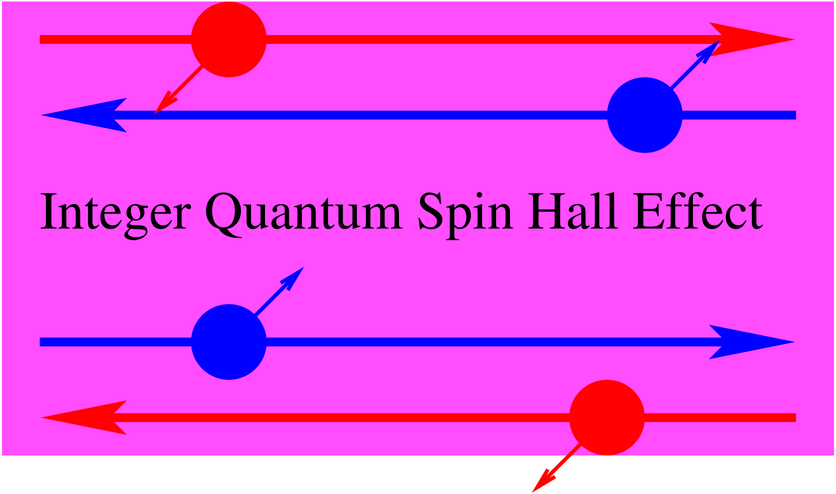
Two-dimensional electrons in a strong magnetic field: IQHE



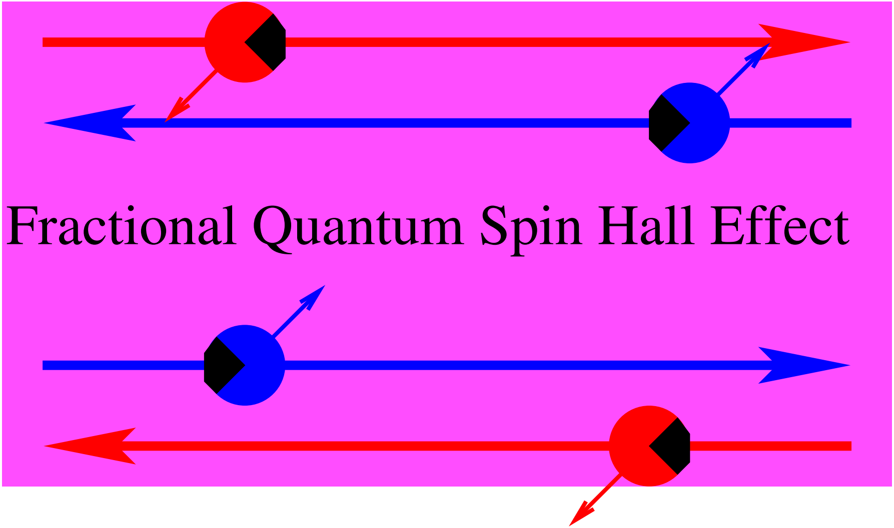
Two-dimensional electrons in a strong magnetic field: FQHE



Two-dimensional electrons with strong spin-orbit coupling: IQSHE



Is there a FQSHE?



Goal:

- Construct a microscopic realization of a FQHE without a uniform applied magnetic field.
  - One of the known topological field theories that break TRS and parity should capture its universal properties.
- Construct a microscopic realization of a FQSHE.
  - If successful, is it described by a topological field theory?

Related recent works

- E. Tang, J. W. Mei, and X. G. Wen, Phys. Rev. Lett., **106**, 236802 (2011).
- K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Phys. Rev. Lett., **106**, 236803 (2011).
- T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett., **106**, 236804 (2011).

Numerical answer: Yes!

- T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett., **106**, 236803 (2011).
- D. N. Sheng, Z. Gu, K. Sun, and L. Sheng, arXiv:1102.2658.
- Y.-F. Wang, Z.-C. Gu, C.-D. Gong, D. N. Sheng, arXiv:1103.1686.
- N. Regnault and B. A. Bernevig, arXiv:1105.4867.
- T. Neupert, L. Santos, S. Ryu, C. Chamon, and C. Mudry, Phys. Rev. B **84**, 165107 (2011).
- D. Xiao, W. Zhu, Y. Ran, N. Nagaosa, and S. Okamoto, arXiv:1106.4296.
- L. Santos, T. Neupert, S. Ryu, C. Chamon, and C. Mudry, arXiv:1110.1296.

Analytical approaches: Wave functions

- B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
- X.-L. Qi, arXiv:1105.4298.
- L. Santos, T. Neupert, S. Ryu, C. Chamon, and C. Mudry, arXiv:1108.2440.
- Yuan-Ming Lu and Ying Ran, arXiv:1109.0226; John McGreevy, Brian Swingle, Ky-Anh Tran, arXiv:1109.1569.
- L. Santos, T. Neupert, S. Ryu, C. Chamon, and C. Mudry, arXiv:1110.1296.

Analytical approaches: Algebraic

- S. Parameswaran, R. Roy, and S. Sondhi, arXiv:1106.4025.
- M. O. Goerbig, arXiv:1107.1986.
- G. Murthy and R. Shankar, arXiv:1108.5501.

Analytical approaches: Effective quantum field theories for time-reversal symmetric fractional topological insulators

- M. Levin and A. Stern, Phys. Rev. Lett. **103**, 196803 (2009).
- T. Neupert, L. Santos, S. Ryu, C. Chamon, and C. Mudry, Phys. Rev. B **84**, 165107 (2011).
- Abolhassan Vaezi, arXiv:1105.0406; Yuan-Ming Lu and Ying Ran, arXiv:1109.0226.
- L. Santos, T. Neupert, S. Ryu, C. Chamon, and C. Mudry, arXiv:1108.2440.

Definition of the noninteracting lattice models

Let $\Lambda = \mathrm{A} \cup \mathrm{B}$ be a bipartite 2-dimensional lattice.

Example 1: Honeycomb lattice

Example 2: Square lattice

If spinless electrons are hopping so as to preserve the point group sublattice symmetry of sublattice A, then

$$H_0 := \sum_{\boldsymbol{k}\in\mathrm{BZ}} \psi^\dagger_{\boldsymbol{k}} \mathcal{H}_{\boldsymbol{k}} \psi_{\boldsymbol{k}}, \qquad \mathcal{H}_{\boldsymbol{k}} := B_{0,\boldsymbol{k}}\sigma_0 + \boldsymbol{B}_{\boldsymbol{k}}\cdot\boldsymbol{\sigma}, \qquad \psi_{\boldsymbol{k}} := \begin{pmatrix} c_{\boldsymbol{k},\mathrm{A}} \\ c_{\boldsymbol{k},\mathrm{B}} \end{pmatrix}$$

where BZ stands for the Brillouin zone of sublattice A.

Chern numbers

If we define

$$\widehat{\boldsymbol{B}}_{\boldsymbol{k}} := \frac{\boldsymbol{B}_{\boldsymbol{k}}}{|\boldsymbol{B}_{\boldsymbol{k}}|}, \quad \tan\phi_{\boldsymbol{k}} := \frac{\widehat{B}_{2,\boldsymbol{k}}}{\widehat{B}_{1,\boldsymbol{k}}}, \quad \cos\theta_{\boldsymbol{k}} := \widehat{B}_{3,\boldsymbol{k}},$$

then eigenvalues and eigenvectors of Hamiltonian $\mathcal{H}_{\boldsymbol{k}}$ are

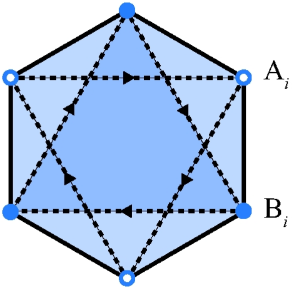
$$\varepsilon_{\pm,\boldsymbol{k}} = B_{0,\boldsymbol{k}} \pm |\boldsymbol{B}_{\boldsymbol{k}}|, \quad \chi_{+,\boldsymbol{k}} = \begin{pmatrix} e^{-\mathrm{i}\phi_{\boldsymbol{k}}/2}\cos\frac{\theta_{\boldsymbol{k}}}{2} \\ e^{+\mathrm{i}\phi_{\boldsymbol{k}}/2}\sin\frac{\theta_{\boldsymbol{k}}}{2} \end{pmatrix}, \quad \chi_{-,\boldsymbol{k}} = \begin{pmatrix} e^{-\mathrm{i}\phi_{\boldsymbol{k}}/2}\sin\frac{\theta_{\boldsymbol{k}}}{2} \\ -e^{+\mathrm{i}\phi_{\boldsymbol{k}}/2}\cos\frac{\theta_{\boldsymbol{k}}}{2} \end{pmatrix}.$$

The first Chern-numbers for the bands labeled by $\pm$ are

$$C_{\pm} = \mp \int_{\boldsymbol{k}\in\mathrm{BZ}} \frac{\mathrm{d}^2\boldsymbol{k}}{4\pi}\, \epsilon_{\mu\nu} \left[\partial_{k_\mu} \cos\theta(\boldsymbol{k})\right] \left[\partial_{k_\nu} \phi(\boldsymbol{k})\right].$$

They have opposite signs if non-zero. All the information about the topology of the Bloch bands of a gaped system is encoded in the occupied single-particle Bloch wave functions.

Example 1: Honeycomb lattice (Haldane 1988)



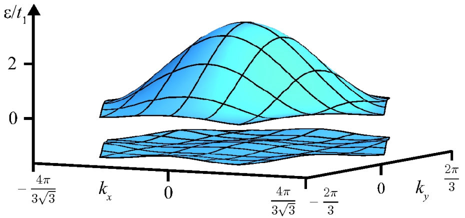
If the NN hopping amplitude, $t_1 > 0$, is positive (solid lines)the NNN hopping amplitude are $t_2 e^{i2\pi\Phi/\Phi_0}$, with $t_2 \geq 0$, in the direction of the arrow (dotted lines),

then

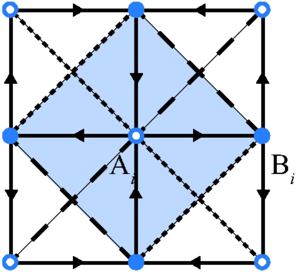
$(\cos\Phi = t_1/(4t_2) = 3\sqrt{3/43}$ with the lower-band flatness ratio 1/7)

$$B_{0,\boldsymbol{k}} := 2t_2 \cos\Phi \sum_{i=1}^{3} \cos \boldsymbol{k}\cdot\boldsymbol{b_i},$$

$$\boldsymbol{B_k} := \sum_{i=1}^{3} \begin{pmatrix} t_1 \cos \boldsymbol{k}\cdot\boldsymbol{a_i} \\ t_1 \sin \boldsymbol{k}\cdot\boldsymbol{a_i} \\ -2t_2 \sin\Phi \sin \boldsymbol{k}\cdot\boldsymbol{b_i} \end{pmatrix}$$



Example 2: Square lattice (Wen, Wilczek, and Zee 1989)



If the NN hopping amplitudes are $t_1\, e^{\mathrm{i}\pi/4}$, with $t_1 > 0$, in the direction of the arrow (solid lines) the NNN hopping amplitudes are $t_2 \geq 0$ and $-t_2$ along the dashed and dotted lines, respectively.

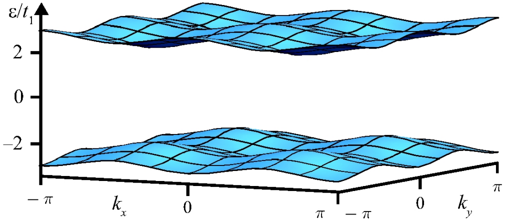
then

$$B_{0,\boldsymbol{k}} := 0,$$

$$B_{1,\boldsymbol{k}} + \mathrm{i}B_{2,\boldsymbol{k}} := t_1\, e^{-\mathrm{i}\pi/4}\left[1 + e^{+\mathrm{i}\left(k_{\mathrm{y}} - k_{\mathrm{x}}\right)}\right] + t_1\, e^{+\mathrm{i}\pi/4}\left[e^{-\mathrm{i}k_{\mathrm{x}}} + e^{+\mathrm{i}k_{\mathrm{y}}}\right],$$

$$B_{3,\boldsymbol{k}} := 2t_2\left(\cos k_{\mathrm{x}} - \cos k_{\mathrm{y}}\right),$$

($t_1/t_2 = \sqrt{2}$ with the flatness ratio 1/5)



Band flattening

Band-flattening is defined by

$$\mathcal{H}^{\text{flat}}_{\boldsymbol{k}} := \frac{\mathcal{H}_{\boldsymbol{k}}}{\varepsilon_{-,\boldsymbol{k}}}.$$

Let there be N sites on sublattice A and N sites on sublattice B.

We fix the number N_{f} of spinless fermions to be $N_{\mathrm{f}} = N$.

Before band-flattening, the $N_{\mathrm{f}} = N$ groundstate is

$$\langle \boldsymbol{r}_1, \cdots, \boldsymbol{r}_N | \boldsymbol{k}_1, \cdots, \boldsymbol{k}_N \rangle = \det \begin{pmatrix} e^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}\chi_{-,\boldsymbol{k}_1} & \cdots & e^{\mathrm{i}\boldsymbol{k}_N\cdot\boldsymbol{r}_1}\chi_{-,\boldsymbol{k}_N} \\ \vdots & \vdots & \vdots \\ e^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_N}\chi_{-,\boldsymbol{k}_1}, & \cdots & e^{\mathrm{i}\boldsymbol{k}_N\cdot\boldsymbol{r}_N}\chi_{-,\boldsymbol{k}_N} \end{pmatrix}.$$

After band-flattening, the $N_{\mathrm{f}} = N$ groundstate has not changed, for all single-particle Bloch states are unchanged under band flattening.

Band flattening preserves locality

Let

$$\mathcal{O}_n(\boldsymbol{x}) := \sum_{i\in\Lambda} a_{n,i}\,\delta(\boldsymbol{x}-\boldsymbol{r}_i), \qquad n=1,2,$$

be any pair of two Hermitean local operators.

Define

$$C^{(1,2)}_{\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N}(\boldsymbol{x},\mathbf{y}) := \langle \boldsymbol{k}_1,\cdots,\boldsymbol{k}_N\,|\mathcal{O}_1(\boldsymbol{x})\mathcal{O}_2(\mathbf{y})|\,\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N\rangle\,.$$

The correlation function

$$C^{(1,2)}(\boldsymbol{x},\mathbf{y}) \propto e^{-\Delta|\boldsymbol{x}-\mathbf{y}|}$$

must decay exponentially before and after band flattening, for neither the existence of the single-particle gap Δ nor the eigenfunctions are affected by the band flattening.

Definition of the lattice model supporting the FQHE

Let

$$H_0 := \sum_{\boldsymbol{k}\in\mathrm{BZ}} \psi^\dagger_{\boldsymbol{k}} \frac{\boldsymbol{B}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{|\boldsymbol{B}_{\boldsymbol{k}}|} \psi_{\boldsymbol{k}}.$$

This kinetic energy supports the integer quantization $\sigma^{\mathrm{charge}}_{\mathrm{xy}} = \pm\frac{e^2}{h}$ of its filled bands.

We then choose the interaction

$$H_{\mathrm{int}} := \frac{1}{2}\sum_{i,j\in\Lambda} \rho_i V_{i,j} \rho_j \equiv V \sum_{\langle ij\rangle} \rho_i \rho_j, \qquad V>0,$$

where ρ_i is the occupation number on the site $i \in \Lambda := \mathrm{A}\cup\mathrm{B}$ of the square lattice.

Define the filling fraction ν to be the ratio

$$\nu := \frac{N_{\mathrm{f}}}{N}$$

where N_{f} is the number of spinless fermions and N the number of sites in sublattice A of the square lattice.

Fractional quantum Hall ground state

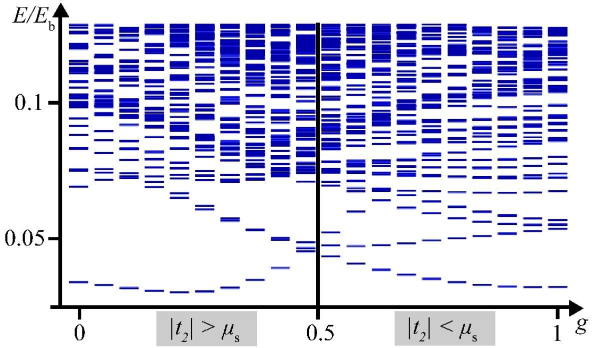
Three distinctive properties of a fractional quantum Hall ground state at filling fraction $\nu < 1$ (where ν^{-1} is an odd integer) and with periodic boundary conditions (toroidal geometry) are

- the existence of a spectral gap above the ground state manifold,
- the ν^{-1}–fold topological degeneracy of the ground state manifold in the thermodynamic limit,
- and the quantization ν_{c} of the Hall conductance $\sigma_{\mathrm{xy}}^{\mathrm{charge}}$ in units of e^2/h.

Spectral gap if $N = 3 \times 6$ and $N_{\mathrm{f}} = 6$, i.e., $\nu = 1/3$

Add a sublattice-staggered chemical potential $4\mu_{\mathrm{s}}$ to the single-particle Hamiltonian by replacing $B_{3,\boldsymbol{k}} \to B_{3,\boldsymbol{k}} + 4\mu_{\mathrm{s}}$.

The parameters t_2 and μ_{s} of H_0^{flat} interpolate between topological ($|t_2| > |\mu_{\mathrm{s}}|$) and non-topological ($|t_2| < |\mu_{\mathrm{s}}|$) single-particle bands.



Here, $g := (2/\pi)\arctan|\mu_{\mathrm{s}}/t_2|$ and all energies are measured relative to the interacting band width E_b. The gap is of order V when $g = 0$.

Topological degeneracy if $N = 3 \times 6$ and $N_\mathrm{f} = 6$

Impose the twisted boundary conditions

$$|\Psi_\gamma(\boldsymbol{r} + N_x \boldsymbol{x})\rangle = e^{\mathrm{i}\gamma_x}|\Psi_\gamma(\boldsymbol{r})\rangle, \qquad |\Psi_\gamma(\boldsymbol{r} + N_y \mathbf{y})\rangle = e^{\mathrm{i}\gamma_y}|\Psi_\gamma(\boldsymbol{r})\rangle$$

where $\gamma^t = (\gamma_x, \gamma_y)$ are the twisting angles and $L_\mathrm{x} \times L_\mathrm{y} = N$ the number of unit cells.

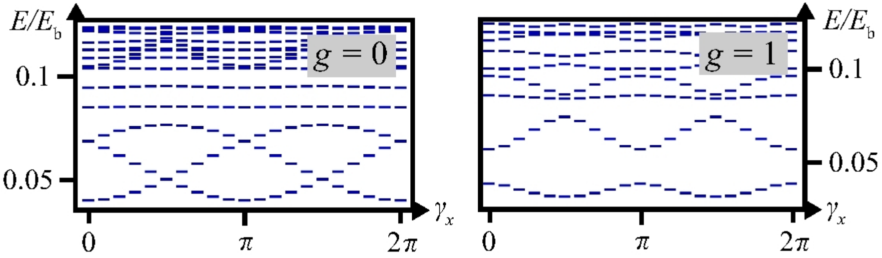
Due to translational invariance, the Hamiltonian does not couple states with different center of mass momenta $\boldsymbol{Q} := \boldsymbol{k}_1 + \ldots + \boldsymbol{k}_{N_\mathrm{f}}$, where $\boldsymbol{k}_i,\ i = 1, \cdots, N_\mathrm{f}$ are the single-particle momenta of an N_f-particle state.

At 1/3-filling of the 3×6 sublattice A, the particle number $N_\mathrm{f} = 6$ is commensurate with the lattice dimensions and all three topological states have the same $\boldsymbol{Q}$.

As a consequence, their topological degeneracy is lifted and a unique ground state appears.

We can now use twisted boundary conditions to probe the topological nature of the ground state: varying γ_x between 0 and 2π is equivalent to the adiabatic insertion of a flux quantum in the system.

During this process, a topological ground state with $\sigma_{\rm xy}^{\rm charge} \times h/e^2 = 1/3$ should undergo two level crossings with the other two gaped topological states (Thouless 1989).



Hall conductance if $N = 3 \times 6$ and $N_{\rm f} = 6$

The Hall conductance $\sigma_{\rm xy}^{\rm charge}$ is related to the Chern-number C of the many-body ground state $|\Psi\rangle$ as

$$\sigma_{\rm xy}^{\rm charge} = C\, e^2/h$$

where (Niu and Thouless 1984)

$$C := \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma\in[0,2\pi]^2} \mathrm{d}^2\gamma\; \boldsymbol{\nabla}_{\gamma} \wedge \left\langle \Psi_{\gamma} \left| \boldsymbol{\nabla}_{\gamma} \right| \Psi_{\gamma} \right\rangle .$$

Alternatively, we introduce

$$\widetilde{C} = \frac{1}{2\pi \mathrm{i}} \int\limits_{\boldsymbol{k}\in \mathrm{BZ}} \mathrm{d}^2\boldsymbol{k}\; n_{-,\boldsymbol{k}} \left[\boldsymbol{\nabla}_{\boldsymbol{k}} \wedge \left(\chi^{\dagger}_{-,\boldsymbol{k}} \boldsymbol{\nabla}_{\boldsymbol{k}} \chi_{-,\boldsymbol{k}} \right) \right]$$

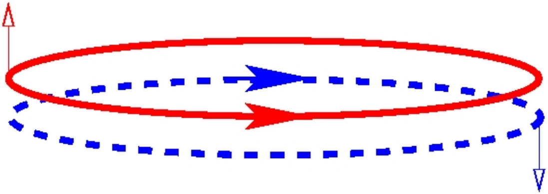
where $n_{-,\boldsymbol{k}} = \langle\Psi| c^{\dagger}_{-,\boldsymbol{k}} c_{-,\boldsymbol{k}} |\Psi\rangle$ is the occupation number of the single-particle Bloch state in the lower $(-)$ band with wave vector $\boldsymbol{k}$ evaluated in the many-body ground state.

It can be shown that $C = \widetilde{C}$.

When $\mu_{\mathrm{s}} = 0$, $t_2 = t_1/\sqrt{2}$, we find $C = 0.29$ and $\widetilde{C} = 0.30$ and attribute the deviations from $C = 1/3$ to finite-size effects.

When $\mu_{\mathrm{s}} = t_1/\sqrt{2}$, $t_2 = 0$, we find that C and $\widetilde{C}$ vanish to a precision of 10^{-6} and 10^{-3}, respectively.

Definition of the lattice model supporting the FQSHE



Bernevig and Zhang 2006

Let

$$H_0 := \sum_{\boldsymbol{k}\in\mathrm{BZ}} \left(\psi^\dagger_{\boldsymbol{k},\uparrow} \frac{\boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{\tau}}{|\boldsymbol{B}_{\boldsymbol{k}}|} \psi_{\boldsymbol{k},\uparrow} + \psi^\dagger_{\boldsymbol{k},\downarrow} \frac{\boldsymbol{B}_{-\boldsymbol{k}} \cdot \boldsymbol{\tau}^{\mathrm{t}}}{|\boldsymbol{B}_{-\boldsymbol{k}}|} \psi_{\boldsymbol{k},\downarrow} \right).$$

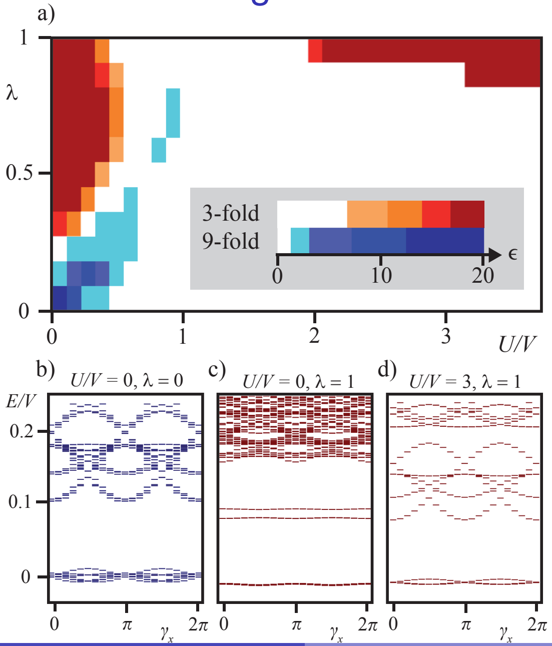
This kinetic energy supports the integer QSH quantization $\sigma^{\mathrm{spin}}_{\mathrm{xy}} = \pm 2 \times \frac{e}{4\pi}$.

We then choose the interaction

$$H_{\mathrm{int}} := U \sum_{i\in\Lambda} \rho_{i,\uparrow}\rho_{i,\downarrow} + V \sum_{\langle ij\rangle\in\Lambda} \left(\rho_{i,\uparrow}\rho_{j,\uparrow} + \rho_{i,\downarrow}\rho_{j,\downarrow} + 2\lambda \rho_{i,\uparrow}\rho_{j,\downarrow} \right),$$

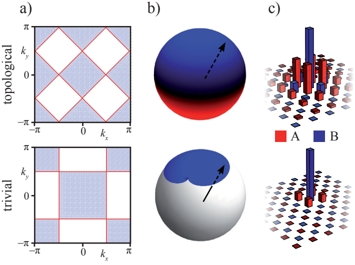
$U, V \geq 0$.

Numerical diagonalization at 4/3 filling of sublattice A

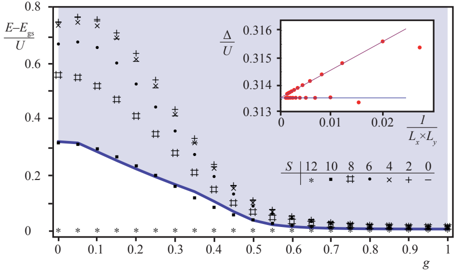


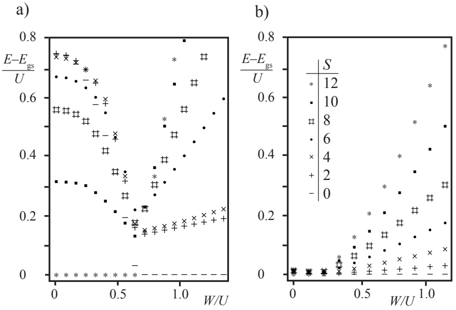
Numerical diagonalization results for 16 electrons when sublattice A is made of 3×4 sites and with $t_2/t_1 = 0.4$. (a) Ground state degeneracies. Denote with E_n the n-th lowest energy eigenvalue of the many-body spectrum where E_1 is the many-body ground state, i.e., $E_{n+1} \geq E_n$ for $n = 1, 2, \cdots$. Define the parameter ϵ by $\epsilon_n := (E_{n+1} - E_n)/(E_n - E_1)$. If a large gap $E_{n+1} - E_n$ opens up between two consecutive levels E_{n+1} and E_n compared to the cumulative level splitting $E_n - E_1$ between the first n many-body eigenstates induced by finite-size effects, then the parameter ϵ_n is much larger than unity. The parameter ϵ_n has been evaluated for $n = 3$ and $n = 9$, yielding the blue and red regions, respectively. For all other $n \neq 1$, no regions with $\epsilon_n \gtrsim O(1)$ of significant size were found. Within the limited range of available system sizes, it is thus not possible to decide on whether and how the level-splitting above the ground state in the white regions of the parameter space extrapolates in the thermodynamic limit. (b)-(d) The lowest eigenvalues with spin-dependent twisted boundary conditions as a function of the twisting angle γ_x. The number of low-lying states that are energetically separated from the other states is 9, 3, and 3, respectively. In panel (c), it is the lowest band parametrized by γ_x that is 3-fold degenerate.

Spontaneous IQHE when $N_e = N = 3 \times 4$



For $N_e = N = 3 \times 4$, the ground state is an Ising ferromagnet with $|\sigma_{xy}^{\mathrm{charge}}| = \frac{e^2}{h}$ when the flat band is topologically non-trivial:





Bulk time-reversal symmetric effective theory (Abelian)

Define $S := S_0 + S_e + S_s$ with

$$S_0 := -\int \mathrm{d}t\, \mathrm{d}^2\boldsymbol{x}\; \epsilon^{\mu\nu\rho}\, \frac{1}{4\pi} K_{ij}\; a^i_\mu\; \partial_\nu\; a^j_\rho,$$

where $K = \begin{pmatrix} \kappa & \Delta \\ \Delta^{\mathsf{T}} & -\kappa \end{pmatrix}$, $\kappa^{\mathsf{T}} = \kappa \in \mathrm{GL}(N, \mathbb{Z})$, $\Delta^{\mathsf{T}} = -\Delta \in \mathrm{GL}(N, \mathbb{Z})$;

$$S_e := +\int \mathrm{d}t\, \mathrm{d}^2\boldsymbol{x}\; \epsilon^{\mu\nu\rho}\, \frac{e}{2\pi}\; Q_i\; A_\mu \partial_\nu\; a^i_\rho,$$

where $Q = \begin{pmatrix} \varrho \\ \varrho \end{pmatrix} \in \mathbb{Z}^{2N}$, $(-)^{Q_i} = (-)^{K_{ii}}$;

$$S_s := +\int \mathrm{d}t\, \mathrm{d}^2\boldsymbol{x}\; \epsilon^{\mu\nu\rho}\, \frac{s}{2\pi}\; S_i\; B_\mu \partial_\nu\; a^i_\rho,$$

where $S = \begin{pmatrix} \varrho \\ -\varrho \end{pmatrix} \in \mathbb{Z}^{2N}$. Then, $|\det K| = (\text{integer})^2$ is the topological degeneracy and

$$\nu_{\mathrm{e}} := Q^{\mathsf{T}}\; K^{-1}\; Q = 0, \qquad \nu_{\mathrm{s}} := \frac{1}{2}\, Q^{\mathsf{T}}\; K^{-1}\; S \neq 0, \qquad \sigma^{\mathrm{spin}}_{\mathrm{xy}} := \frac{e}{2\pi} \times \nu_{\mathrm{s}}\,.$$

Wave function for $N = 1$

If

$$K = \begin{pmatrix} +m & 0 \\ 0 & -m \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}), \qquad Q = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{Z}^2,$$

for some given positive odd integer m, then

$$\nu_{\mathrm{s}} = \frac{1}{m}$$

and [generalization of Laughlin's wavefunctions]

$$\Psi_{1/m}(\{z, \bar{z}\}_n \,|\, \{w, \bar{w}\}_n) = \left[\prod_{i=1}^{n} \prod_{j=i+1}^{n} \left(z_i - z_j\right)^m \left(\bar{w}_i - \bar{w}_j\right)^m\right] \prod_{i=1}^{n} \exp\left(-\frac{\left|z_i\right|^2 + \left|\bar{w}_i\right|^2}{4\ell^2}\right).$$

Wave function in the symmetric representation for $N=2$

If

$$K=\begin{pmatrix} +\begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} & +\begin{pmatrix} 0 & +d \\ -d & 0 \end{pmatrix} \\ -\begin{pmatrix} 0 & +d \\ -d & 0 \end{pmatrix} & -\begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}), \qquad Q=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{Z}^4,$$

with $m_1 m_2 - n^2 > 0$, then

$$\nu_{\mathrm{s}} = \frac{m_1 + m_2 - 2n}{m_1\, m_2 - n^2 + d^2}$$

and [generalization of Halperin's (m_1, m_2, n) bilayer wavefunction]

$$\begin{aligned}
&\Psi^{\mathrm{symm}}_{m_1,m_2,n,d}\left(\{z_1,\bar{z}_1\}_{n_1};\{z_2,\bar{z}_2\}_{n_2}\,|\,\{w_1,\bar{w}_1\}_{n_1};\{w_2,\bar{w}_2\}_{n_2}\right) = \\
&\qquad \Psi_{1/m_1}\left(\{z_1,\bar{z}_1\}_{n_1}\,|\,\{w_1,\bar{w}_1\}_{n_1}\right)\times\Psi_{1/m_2}\left(\{z_2,\bar{z}_2\}_{n_2}\,|\,\{w_2,\bar{w}_2\}_{n_2}\right) \\
&\qquad \times\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\left(z_{1,i}-z_{2,j}\right)^{n}\left(\bar{w}_{1,i}-\bar{w}_{2,j}\right)^{n}\left(z_{1,i}-w_{2,j}\right)^{d}\left(\bar{w}_{1,i}-\bar{z}_{2,j}\right)^{d}.
\end{aligned}$$

Wave function in the hierarchical representation for $N=2$

If

$$K=\begin{pmatrix} +\begin{pmatrix} +m & +1 \\ +1 & -p \end{pmatrix} & +\begin{pmatrix} 0 & +d \\ -d & 0 \end{pmatrix} \\ -\begin{pmatrix} 0 & +d \\ -d & 0 \end{pmatrix} & -\begin{pmatrix} +m & +1 \\ +1 & -p \end{pmatrix} \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}), \qquad Q=\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{Z}^4,$$

with m a positive odd integer and p an even integer then

$$\nu_{\mathrm{s}}=\frac{p}{mp+1-d^2}$$

and [generalization of Halperin's $\nu_{\mathrm{c}}=p/(mp+1)$ single-layer wavefunction]

$$\Psi^{\mathrm{hier}}_{m,-p,1,d}\left(\{z,\bar{z}\}_{pn}\,|\,\{w,\bar{w}\}_{pn}\right)=$$

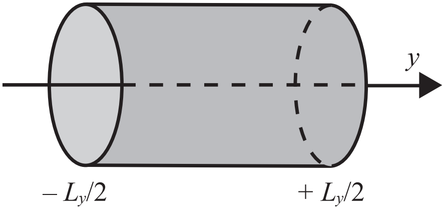
$$\left[\prod_{i=1}^{n}\int_{\Omega}\mathrm{d}^2\,\eta_i\int_{\Omega}\mathrm{d}^2\,\xi_i\right]\times\Psi_{1/m}\left(\{z,\bar{z}\}_{pn}\,|\,\{w,\bar{w}\}_{pn}\right)\times\Psi_{1/p}\left(\{\xi,\bar{\xi}\}_{n}\,|\,\{\eta,\bar{\eta}\}_{n}\right)$$

$$\times\prod_{i=1}^{pn}\prod_{j=1}^{n}\left(z_i-\eta_j\right)\left(\bar{w}_i-\bar{\xi}_j\right)\left(z_i-\xi_j\right)^{d}\left(\bar{w}_i-\bar{\eta}_j\right)^{d}.$$

Edge theory with time-reversal symmetry

The bulk action with a two-body and translation-invariant interaction is equivalent to

$$\hat{H}_0 := \int_0^L \mathrm{d}x\, \frac{1}{4\pi} \partial_x \hat{\Phi}^{\mathsf{T}}\, V\, \partial_x \hat{\Phi}$$



where V is a $2N \times 2N$ symmetric and positive definite matrix and

$$\left[\hat{\Phi}_i(t,x), \hat{\Phi}_j(t,x')\right] = -\mathrm{i}\pi \Big(K^{-1}_{ij}\, \mathrm{sgn}(x-x') + \Theta_{ij} \Big).$$

Here,

$$\Theta_{ij} := K^{-1}_{ik}\, L_{kl}\, K^{-1}_{lj}$$

and the antisymmetric $2N \times 2N$ matrix L is defined by (Haldane 1995)

$$L_{ij} = \mathrm{sgn}(i-j)\, \big(K_{ij} + Q_i\, Q_j\big)\,,$$

where $\mathrm{sgn}(0) = 0$ is understood.

Tunneling of electronic charge among the different edge branches is

$$\hat{H}_{\mathrm{int}} := -\int_0^L \mathrm{d}x \sum_{T\in\mathbb{L}} h_T(x) : \cos\left(T^{\mathsf{T}} K\,\hat{\Phi}(x) + \alpha_T(x)\right) : \;.$$

The real functions $h_T(x) \geq 0$ and $0 \leq \alpha_T(x) \leq 2\pi$ encode information about the disorder along the edge when position dependent. The set

$$\mathbb{L} := \left\{ T \in \mathbb{Z}^{2N} \,\middle|\, T^{\mathsf{T}} Q = 0 \right\}$$

encodes all the possible charge neutral tunneling processes, i.e., those that just rearrange charge among the branches.
At least one pair of Kramers degenerate edge state remains delocalized along the edge described by $\hat{H} := \hat{H}_0 + \hat{H}_{\mathrm{int}}$ if the integer

$$R := r\,\varrho^{\mathsf{T}}\,(\kappa - \Delta)^{-1}\,\varrho$$

is odd. Here, the integer r is the smallest integer such that all the N components of the vector $r\,(\kappa - \Delta)^{-1}\,\varrho$ are integers.

Summary

- We have proposed a simple recipe to deform some non-interacting lattice models so as to obtain flat bands, while preserving locality.
- We flattened the bands of the chiral π-flux phase and then lifted the resulting macroscopic ground state degeneracy with repulsive interactions.
- Via exact diagonalization, we have found signatures for a FQH-like topological ground states at 1/3 and 2/3 fillings of sublattice A.
- We took the same approach to construct a FQSH-state and found microscopic signatures for it as well as spontaneous breaking of TRS and parity via IQH and FQH ground states.
- This opens the door for the realization of the QHE without an applied magnetic field or the QSHE at room temperature.

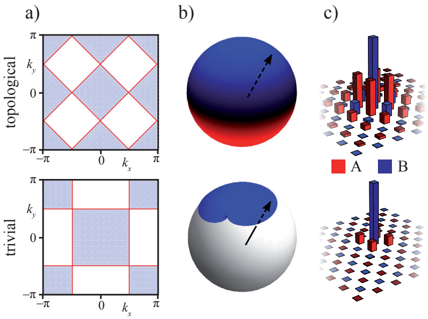


Figure: Comparison of the topological (top; $t_2 = 1/\sqrt{2}$, $\mu_s = 0$) and non-topological (bottom; $t_2 = 0$, $\mu_s = 1/\sqrt{2}$) single-particle model. (a) The shaded area represents the Fermi see of the lower band at the commensurate filling fraction $N_e = N$ when $\kappa < 1$. (b) The eigenspinor $\chi_{\boldsymbol{k},\sigma}$, when interpreted as a point on the surface of the unit sphere, swipes out the full surface of this sphere (a small portion of this sphere near one pole) as $\boldsymbol{k}$ takes values everywhere in the BZ for the topological (non-topological) band structure. (c) The spread of the Wannier states in real space indicates their delocalized (localized) character for the topological (non-topological) band structure.

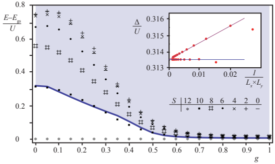


Figure: Numerical exact diagonalization results for flat bands $\kappa = 1$ at the commensurate filling fraction $N_{\mathrm{e}} = N$. Markers show the energy of the lowest state in different sectors of total spin S (in units of $\hbar/2$) measured with respect to the ground state energy for $L_{\mathrm{x}} = 3,\ L_{\mathrm{y}} = 4$. Here, $g := (2/\pi)\mathrm{arctan}|\mu_{\mathrm{s}}/t_2|$ so that $g > 0.5$ and $g < 0.5$ correspond to the trivial and topological single-particle bands, respectively. Since there is only one state in the fully polarized sector $|S| = 12$, the difference between the asterisks and the squares is the many-body excitation gap $\Delta(g)$. The thick blue line shows the extrapolation of $\Delta(g)$ to the thermodynamic limit. In the inset, exact diagonalization in the sector with one spin flipped away from the fully polarized sector is presented for $\mu_{\mathrm{s}} = 0$, $t_2 = 1/\sqrt{2}$ and $L_{\mathrm{x}} = L_{\mathrm{y}}$ ranging from 6 to 30. The straight lines are guide to the eye and make evident an even-odd effect in $L_{\mathrm{x}} = L_{\mathrm{y}}$. Deep in the topologically non-trivial regime $g \ll 0.5$, we observe a sizable $\Delta(g \ll 0.5)$. The topologically trivial regime $g > 0.5$ is also characterized by a gap $\Delta(g > 0.5)$ in the sector with one spin flipped away from the fully polarized sector, however this gap is much smaller than $\Delta(g \ll 0.5)$.

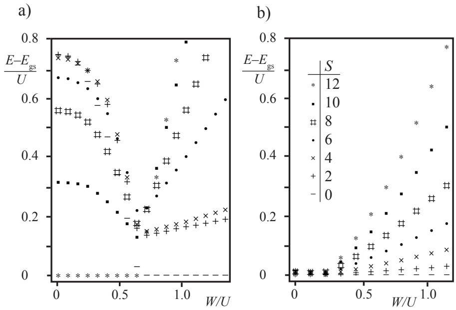


Figure: Numerical exact diagonalization results at the commensurate filling fraction $N_e = N$ as a function of the bandwidth W for $L_x = 3$, $L_y = 4$. Plotted is the energy of the lowest state in different sectors of total spin S (in units of $\hbar/2$) measured with respect to the ground state energy. (a) Topological phase with $\mu_s = 0$, $t_2 = 1/\sqrt{2}$. The ground state is gaped and fully spin-polarized for $W/U < 0.7$, while it is unpolarized for $W/U > 0.7$. (b) Topological trivial phase with $\mu_s = 1/\sqrt{2}$, $t_2 = 0$. The unpolarized ground state appears already for very small values of W/U.

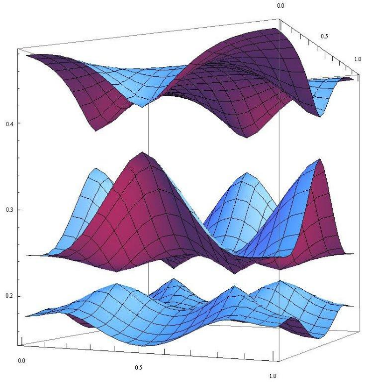


Figure: Low energy spectrum spinless Haldane model with interactions at one third filling of 3×3 sublattice A.