Thanks to...

Hao Song  
(Boulder → Madrid)

Sheng-Jie Huang  
(Boulder)

Liang Fu (MIT)

Funding: Department of Energy Basic Energy Sciences, 
Grant # DE-SC0014415
Symmetry protected topological (SPT) phases

**$T=0$ phases of matter characterized by:**

1. Energy gap
2. Symmetry $G$ not spontaneously broken
3. Ground state becomes trivial if $G$ explicitly broken

No intrinsic topological order in the bulk, i.e., no non-trivial braiding statistics or ground state degeneracy on torus

**Classic examples:**

- **$d=1$**
  - Haldane $S=1$ chain
  - Symmetry: time reversal, or SO(3) spin rotation, or reflection

- **$d=2$**
  - Quantum spin-hall insulator
  - Symmetry: charge conservation + time reversal

- **$d=3$**
  - Topological band insulator
  - Symmetry: charge conservation + time reversal
What SPT phases to study?

Some guidance (paraphrase):
“*The life without reflection is not worth living.*” - Socrates

- Most theory of SPT phases focuses on internal symmetries, or on non-interacting fermions
- Discrete symmetries of crystal lattices, including reflection, are pervasive in solids, we should not ignore them when studying SPT phases
- For crystalline SPT phases with strong interactions, some examples and case studies, but no general theory
Focus on SPT phases protected by point group symmetry (= pgSPT phases)

A surprise: “pgSPT phases are easier than SPT phases protected by internal symmetry in the same spatial dimension.”

There is a mapping between pgSPT states in spatial dimension $d$ and certain lower-dimensional topological states with internal symmetry
• Classification and characterization of pgSPT phases in terms of lower-dimensional topological states with internal symmetry

• pgSPT phase \(\simeq\) stack/array of lower-dimensional topological phases with internal symmetry

• For simplicity, this talk will mostly focus on reflection symmetry, but the approach applies to any point group.

• Remark: reflection pgSPT’s are related to time-reversal SPT’s if one assumes a Lorentz-invariant field theory description. I will not make this assumption (more comments at the end).
Prior work on interacting point group SPT phases

- **d=1 (inversion symmetry):**
  Z.-C. Gu & X.-G. Wen
  Pollmann, Turner, Berg, Oshikawa
  X. Chen, Z.-C. Gu, X.-G. Wen
  Schuch, Perez-Garcia, Cirac
  Fuji, Pollmann, Oshikawa

- **Higher dimensions:**
  Y. Qi & L. Fu; Isobe & L. Fu
  G.-Y. Cho, C.-T. Hsieh, R. Leigh, T. Morimoto S. Ryu, O. Sule
  A. Furusaki, T. Morimoto, C. Mudry, T. Yoshida
  Ware, Kimchi, Parameswaran, Bauer
  Lapa, Teo & Hughes
  Y.-Z. You & C. Xu
  Kapustin, Thorngren, Turzillo & Zitao Wang
  MH & X. Chen
1. Bosonic mirror SPT phases in d=3
2. Electronic topological crystalline insulators with interactions
3. Point groups beyond reflection
4. Odds and ends, outlook
Consider bosonic system in \(d=3\) with \textit{only} mirror (reflection) symmetry \(\sigma : (x,y,z) \rightarrow (-x,y,z)\). (Ignore any other symmetry present.)

The mirror symmetry becomes an internal \(Z_2\) symmetry.

\textit{The \(d=3\) point group SPT state is equivalent to a \(d=2\) state on the mirror plane, with \(Z_2\) internal symmetry.}
Why can we dimensionally reduce?

\[ |\psi\rangle \rightarrow |T\rangle_{r_1} \otimes |\tilde{\psi}\rangle_{r_0} \otimes |T\rangle_{\sigma r_1} \]

- Quick argument: can locally trivialize any patch away from the mirror plane

Hamiltonian density here can be changed arbitrarily...as long as corresponding changes made here
Why can we dimensionally reduce?

\[ |\psi\rangle \rightarrow |T\rangle_{r_1} \otimes |\tilde{\psi}\rangle_{r_0} \otimes |T\rangle_{\sigma r_1} \]

- More detailed argument based on “cutting” a finite-depth quantum circuit

\[ U^{loc} |\psi\rangle = |\text{product}\rangle \]

\( U^{loc} \) breaks symmetry

\( U^{loc} \) can be represented as a finite-depth quantum circuit:

We can “cut” \( U^{loc} \), to get a new quantum circuit \( U^{loc}_L \) acting only in region \( r_1 \)

Act here with \( U^{loc}_L \)

\[ U^{loc}_L U^{loc}_R |\psi\rangle = |T\rangle_{r_1} \otimes |\tilde{\psi}\rangle_{r_0} \otimes |T\rangle_{\sigma r_1} \]

Width \( w \gg \xi \), correlation length
Proceed in two steps:

First: What $d=2$ quantum phases can occur on mirror plane?

Second: How to group $d=2$ states on mirror plane into equivalence classes of $d=3$ quantum phases?
First: What $d=2$ quantum phases can occur on mirror plane?

“Integer” topological phases (gap, no anyons), preserving $Z_2$ symmetry

Two possibilities…

A. Non-trivial $d=2$ SPT phase with $Z_2$ symmetry

(Levin & Gu; X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen)

Domain wall picture

$$|\psi\rangle = \sum_{D} (-1)^{N(D)} |D\rangle$$

- Gapless edge modes protected by $Z_2$ symmetry
- Stack to get non-trivial pgSPT phase
First: What $d=2$ quantum phases can occur on mirror plane?

B. $E_8$ state (Kitaev)

- This state has 8 co-propagating edge modes $\Rightarrow$ quantized thermal Hall conductance

\[ K_H = 8 \frac{\pi^2}{3} \frac{k_B^2}{h} T \]

- No fractional excitations (anyons) in bulk

- Like IQH state, but in bosonic system, not a SPT phase

To get a $d=3$ pgSPT state, make alternating-chirality stack of $E_8$ states
Second: How to group $d=2$ states on mirror plane into equivalence classes of $d=3$ quantum phases?

- Naively, classification of $d=2$ phases directly gives a classification of $d=3$ pgSPT phases.
- In this case, classification of $d=2$ phases is $\mathbb{Z}_2 \times \mathbb{Z}$

  From $\mathbb{Z}_2$ SPT
  From $E_8$ states

- But this is not the correct classification of $d=3$ phases, instead it collapses to a coarser classification
Second: How to group $d=2$ states on mirror plane into equivalence classes of $d=3$ quantum phases?

**Three equivalence operations**

A. Adiabatic continuity (preserving symmetry)

B. Stable equivalence (adding trivial degrees of freedom)

C. “Adjoining layers”

- Corresponds to making region surrounding the mirror plane wider
- Adjoined layers can be $E_8$ states

\[ |\psi\rangle_{r_0} \rightarrow |L\rangle \otimes |\psi\rangle_{r_0} \otimes |R\rangle \]
Second: How to group $d=2$ states on mirror plane into equivalence classes of $d=3$ quantum phases?

- $d=2$ classification is $\mathbb{Z}_2 \times \mathbb{Z}$

- Study effect of adjoining layers on two E8 states:

Two E8 states ($n=2$ state of the $\mathbb{Z}$), reflection acts trivially

Reflection exchanges adjoined layers $\sigma$

Adjoined E8 layers of opposite chirality.

Resulting state is non-chiral, two possibilities:

1. Trivial $\rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$
2. $d=2$ Z2 SPT state $\rightarrow \mathbb{Z}_4$

Can show it’s trivial by analyzing edge theory

Classification collapses to $\mathbb{Z}_2 \times \mathbb{Z}_2$
1. Bosonic mirror SPT phases in $d=3$
2. Electronic topological crystalline insulators with interactions
3. Point groups beyond reflection
4. Odds and ends, outlook
Topological crystalline insulators (mirror reflection)

Consider electrons with U(1) charge conservation and mirror reflection $\sigma : (x,y,z) \rightarrow (-x,y,z)$

SPT phases = topological crystalline insulators (TCIs)

Non-interacting electrons: $Z$ classification, “mirror Chern number”

TCI predicted and observed in Pb$_{1-x}$Sn$_x$Te

Theory: Teo, Fu & Kane; T. Hsieh, H. Lin, J. Liu, W. Duan, A. Bansil, L. Fu; …
Experiment: Tanaka, …, Y. Ando; P. Dziawa, …, T. Story; S.-Y. Xu, …, M. Z. Hasan; …

Interacting electrons:

$Z$ is reduced to $Z_8$. Isobe & Fu showed this by:

1. Adding spatially varying Dirac mass terms to “dimensionally reduce” the surface theory to 1d lines
2. Using bosonization to show $n=8$ surface can be gapped
Q: Is the \( \mathbb{Z}_8 \) classification complete?

A: Full classification is \( \mathbb{Z}_8 \times \mathbb{Z}_2 \)

Intrinsically strongly-interacting electron TCI
First, reproduce non-interacting $Z$ classification using dimensional reduction

1. What can go on mirror plane?
   
   A. Integer quantum Hall state ($\nu=1$) with $\sigma=+1$
   
   B. Integer quantum Hall state ($\nu=1$) with $\sigma=-1$

   Naively gives $Z \times Z$ classification, which is too big.

2. Effect of adjoining layers

   \[ \nu = 1, \sigma = 1 \]
   \[ \nu = -1, \sigma = -1 \]

   $Z \times Z$ collapses to correct $Z$ classification

   $\nu=-1$ layers, related by mirror

\(\sigma\) is the reflection eigenvalue of the fermion field.
1. What can go on mirror plane?

**IQH states** with $\sigma = +1$

$d=2$ classification is: $Z^{IQH} \times Z^{SPT}_4 \times Z^{E_8}$

**E_8 paramagnets:**
Spin sector in E_8 state, trivial action of reflection

---

**2d SPT phases**
protected by $U(1) \times Z_2$

Bilayer of opposite-chirality IQH states:

- $\nu = n$,
- $\sigma = +1$
- $\nu = -n$,
- $\sigma = -1$

$n=4$ state is trivial: can be gapped out at edge ([Isobe & Fu](#))
2. Collapse of \( d=2 \) classification (under adjoining layers operation)

\( d=2 \) classification is: \( \mathbb{Z}_{IQH} \times \mathbb{Z}_{SPT}^4 \times \mathbb{Z}_{E_8} \rightarrow \)

Collapses to \( \mathbb{Z}_2 \), as for bosonic mirror SPTs

Collapses to \( \mathbb{Z}_8 \)... \( \nu = -1 \) layers, related by mirror

Start with \( n=2 \) state of the \( \mathbb{Z}_{IQH} \)

Get root state of the \( \mathbb{Z}_{SPT}^4 \)

\( \nu = 2, \sigma = 1 \)

\( \nu = 1, \sigma = 1 \)

\( \nu = -1, \sigma = -1 \)

Obtain \( \mathbb{Z}_8 \times \mathbb{Z}_2 \) classification
1. Bosonic mirror SPT phases in d=3
2. Electronic topological crystalline insulators with interactions
3. Point groups beyond reflection
4. Odds and ends, outlook
Example: bosonic system with $C_{2v}$ symmetry in $d=3$

$C_{2v}$ is generated by two perpendicular mirror planes

Reduce onto “cross-shaped region:”

Two planes with $Z_2$ internal symmetry

$d=1$ axis with $Z_2 \times Z_2$ symmetry

Root states

- $d=2$ $Z_2$ SPT phase on either mirror plane
- $d=1$ Haldane phase on $d=1$ axis
- $d=2$ $E_8$ states on mirror planes, with chiralities as shown

$(Z_2)^4$ classification
1. Bosonic mirror SPT phases in d=3
2. Electronic topological crystalline insulators with interactions
3. Point groups beyond reflection
4. Odds and ends, outlook
All the $d=3$ bosonic mirror SPT phases admit gapped, topologically ordered surfaces with anomalous implementations of the symmetry.

- Dimensional reduction shows surfaces can be studied in “T-junction” geometry.
- Anomaly of the 2+1 dimensional surface can be canceled by anomaly of a 2+1 dimensional bulk.

- $Z_2$ SPT root state: surface with toric code topological order, mirror squares to (-1) on both bosonic particles “ePmP”

- $E_8$ root state: surface with 3 fermion topological order, preserving reflection (impossible in strict $d=2$)

see also recent work by Ethan Lake, arXiv:1608.02736
**Reflection and time reversal**

- Classifications for reflection and time-reversal are related:

<table>
<thead>
<tr>
<th>d=3 bosonic system, reflection</th>
<th>d=3 bosonic system, time-reversal</th>
<th>$Z_2 \times Z_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=3 fermions, $\sigma^2 = 1$</td>
<td>d=3 fermions, $T^2 = (-1)^F$</td>
<td>$Z_{16}$</td>
</tr>
<tr>
<td>d=3 fermions, $\sigma^2 = (-1)^F$</td>
<td>d=3 fermions, $T^2 = 1$</td>
<td>Trivial</td>
</tr>
<tr>
<td>d=3 fermions, $U(1) \times$ Reflection*</td>
<td>d=3 fermions, $U(1) \times$ Time reversal*</td>
<td>$Z_8 \times Z_2$</td>
</tr>
</tbody>
</table>

- Follows from assuming a Lorentz-invariant field theory description (see e.g. Witten arXiv:1508.0471)

* All fermions carry odd U(1) charge, bosons carry even U(1) charge
Summary

- Point group SPT phases can be classified and studied by a dimensional reduction to lower-dimensional topological phases with internal symmetry
- All point group SPT phases can be constructed as stacks/arrays

Outlook

- Physical realizations, connections to other approaches, etc.
- Formal classification: what is the mathematical structure?
- Space group symmetry
- Dimensional reduction for point group symmetry enriched topological (SET) phases