A “Lieb-Schultz-Mattis" constraint for space-groups

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Adrian Po, et al., arXiv:161x.xxxxx


Fractional filling & fractional excitations

Fractional Quantum Hall Effect

\[ p \text{ electrons per } q \text{ flux quanta ( } \nu = \frac{p}{q} \text{ )} \]

Fractional excitations: charge \( \frac{p}{q} \)

Frustrated magnetism

\[ S = \frac{1}{2} \text{ moment per unit cell} \]

Fractional excitations: spinons with \( S = \frac{1}{2} \)

From L. Balents, Nature 2010
Lieb-Schultz-Mattis-type contraints

[Lieb, Schultz, Mattis 1963]

A constraint between the symmetry implementation in the UV (e.g. electron filling, spin of magnetic moments) and the emergent theory in the IR.

IR: fractional excitations
UV: fractional filling

\[ s_{ab} \sim a \begin{pmatrix} b \end{pmatrix} a b \]

\[ H = \frac{1}{2} \sum_{i<j} V(r_i - r_j) + H_K \]
Lieb-Schultz-Mattis theorem in two dimensions

A magnet with half-integral spin / unit cell is either:

1. Symmetry broken

2. Gapless

Spin liquid

3. Topologically ordered: anyonic excitations

Symmetric & short-range entangled (e.g. product state or SPT-phase)

LSM is a “no-go” for sym-SRE phases

[Liub, Schultz, Mattis 1963]
[Oshikawa, 1999; Hastings, 2005]
Wave 1: Translation only: “spin per unit cell”
[Lieb, Schultz, Mattis 1961; LSM - Affleck; Oshikawa, 1999; Misguich 2002; Hastings, 2003]

Wave 2: Non-symmorphic (glides and screws): “spin per reduced unit cell”
[Parameswaran et al. 2013; Watanabe, Po, Vishwanath & Zaletel PNAS 2015]

A magnet with integer spin per unit cell may still have a LSM-type no-go if there is half-integer spin in a “reduced” unit cell defined by glides & screws

Point group symmetries (rotation, reflection, inversion) were never used

This talk: No-gos from the full space group
Starting point: the lattice of spins

Consider a magnet of spins symmetric under $S \times SO(3)$ space group spin rotation.

Each site carries either integer (○) or half-integer (●) spin.

Forget the other microscopic details: e.g. $S = 1/2$ is “same” as $S = 3/2$.

Question: can the lattice $\Lambda$ ever have a symmetric, short-range-entangled ground state?
Conjecture

A symmetric, short-range entangled ground state is possible only if $\Lambda$ is “equivalent” to a “trivial” lattice.

need to define
“Trivial” lattice

A lattice is “trivial” if it only contains integer spins.

Example: spin-1 square lattice with $C_4$ & mirror symmetries

There is no “obvious” symmetric SRE state here. AKLT construction requires $S = \text{[coordination number]} / 2 = 2$.

Haldane’s analysis of NLSM suggest monopoles see staggered flux $(i)^{2S}$

[References: Haldane 88; Read & Sachdev 90]

Nevertheless, a generalized AKLT construction, “tensor networks,” allow us to propose manifestly sym-SRE phases  [CM Jian & Zaletel, PRB 93 2016]
“Equivalence” relation: $\Lambda \sim \Lambda'$

**Rule 1**: You can move spins around, so long as the symmetry is preserved throughout.

Example:
- Translations
- $C_3$ rotations
- Mirrors

**Rule 2**: When spins collide, they “fuse” ($\mathbb{Z}_2$)

(coarse grained!)

\begin{align*}
(a) \quad \bigcirc + \bullet &= \bullet \\
(b) \quad \bullet + \bullet &= \bigcirc \\
(c) \quad \bullet &\leftrightarrow \bullet = \bigcirc \\
(d) \quad \bullet \quad \bigcirc &= \bullet
\end{align*}
Example: $S=1/2$ Honeycomb lattice

Half-integer honeycomb $\sim$ integer kagome $\sim$ “trivial”

No “obvious” AKLT-like state, recently shown there is a trivial $S=1/2$ honeycomb magnet

Implies that graphene at charge neutrality could be in a completely insulating, symmetric, short-range entangled phase

[Kim, et al. PRB 94 2016]
“Lattice classification”

Given a space group $\mathcal{S}$, compute the inequivalent lattice types $\left[\Lambda\right]_{\mathcal{S},\mathbb{Z}_2}$.

Forms an Abelian group under “stacking,” with the empty lattice the identity.

Mike: “the way we were taught not to define equivariant homology in grad school.”

**Conjecture**: short-range-entangled only if $\left[\Lambda\right] = 1$

Example: 1D lattices with Translation & Reflection

Four inequivalent lattices:

$$\left[\Lambda\right] \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

Invariant: is there half-integer spin on reflection plane?

Beyond LSM
Internal symmetry groups other than SO(3)

If the internal symmetry group is $G$, the role of "integer vs. half-integer" played by the *projective representation* of a site: $\alpha, \beta \in \mathcal{H}^2[G, U(1)]$

Fusion = abelian multiplication of projective classes
Lattice classification: The 17 2D space groups

For projective $\mathcal{H}^2 [G, U(1)] = \mathbb{Z}_n$ reps.

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Non-trivial indices when the order of point group is commensurate with order of projective rep.
Lattice classification: 3D

230 space groups: luckily, can be automated using tables of “Wyckoff positions”
Space group No 227: Electron count is “trivial” for all previous “LSM”-bounds

\[
\left[ \Lambda \right]_{227, \mathbb{Z}_2} \in \mathbb{Z}_2^4
\]

Lattice of Pr: non-trivial element, (presumably) explains inevitable degeneracy

[caveat: spin-orbit coupled & we haven’t yet proved for pure time-reversal]
Proof in 2D

**Conjecture**: short-range-entangled only if \([\Lambda] = 1\)

Step 1: A set of physical arguments which show a sym-SRE impossible if either:

1. Half-integer spin per "reduced unit cell" (LSM-like) [Watanabe, P, V, Z 2015]
2. Half-integer spin per "unit length of mirror line" [Watanabe, P, V, Z 2015]
3. Half-integer spin on a site with \(C_2\) symmetry

Step 2: Prove that the above physical arguments allow SRE phase if and only if \([\Lambda] = 1\)
No-go for half-integer spin with $C_2$-site symmetry

Focus on the $\pi$-rotations, $1, X, Y, Z \in SO(3)$

Half-integer spin: $XZ = -ZX$

Tool: pair of $C_2$ related spin-flux on a torus

$X_i X_j + Y_i Y_j + Z_i Z_j \rightarrow X_i X_j - Y_i Y_j - Z_i Z_j$
What symmetries are there?

1) Rotation: \( \hat{C}_2^{(X)} \)

\[
\prod_{j \in \text{green}} X_j \quad \text{“gauge transformation”}
\]

2) Internal symmetry:
\[
Z = \prod_j Z_j \quad X_j Z_j = -Z_j X_j
\]

\[
\hat{C}_2^{(X)} Z = (-1)^{\text{Vol}} Z \hat{C}_2^{(X)} = -Z \hat{C}_2^{(X)}
\]

ground state degeneracy
Why does this ground state degeneracy rule out SRE?

In some SPT phases, fluxes bind degeneracies - *GSD itself not a contradiction.*

“Decorated domain walls” [Chen, Lu, Vishwanath 2013]

Principle: in an SRE phase, all degeneracies are “localizable”

\[ d_1 d_2 = d \text{-fold ground states: } \{ |\alpha_1, \alpha_2 \rangle : \alpha_1 = 1, \ldots, d_1; \alpha_2 = 1, \ldots, d_2 \} \]

Density matrix near flux:

\[ \rho_{12}^{\alpha_1 \alpha_2} = \rho_1^{(\alpha_1)} \otimes \rho_2^{(\alpha_2)} + \mathcal{O}(e^{-R/\xi}) \]

Density matrix away from flux:

\[ \rho_{12}^{\alpha_1 \alpha_2} = \rho_{12} \]

Very different than non-local “topological degeneracy” of non-Abelian anyons
Why does this ground state degeneracy rule out SRE?

Claim: it is impossible to realize $Z\hat{C}_2^{(X)} = -Z\hat{C}_2^{(X)}$ on the localized ground states \(\{|\alpha_1, \alpha_2\} : \alpha_1 = 1, \cdots d_1; \alpha_2 = 1, \cdots, d_2\}\).

Proof: Most general symmetry implementation on GSD:

\[
d_1 = d_2 \quad \text{from } C_2
\]

\[
Z = Z_1 \otimes Z_2 \quad \text{from locality}
\]

\[
C_2^{(X)} = \text{SWAP}_{12} \quad \text{without loss of generality: change of basis on ‘2’}
\]

\[
C_2^{(X)} Z C_2^{(X)} \overset{-1}{=} \text{SWAP}_{12}(Z_1 \otimes Z_2) = Z_2 \otimes Z_1 \overset{?}{=} -Z_1 \otimes Z_2
\]

This would require \(Z_1 = \eta Z_2\), but

\[
Z_1 \otimes Z_2 = \eta Z_2 \otimes Z_2 \neq -\eta Z_2 \otimes Z_2 = -Z_2 \otimes Z_1
\]

Impossible to have required degeneracy in a SRE phase.
“Proof” in 2D

**Conjecture**: short-range-entangled only if \([\Lambda] = \mathbb{1}\)

**Step 1**: A set of physical arguments which show a sym-SRE impossible if either:

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**Step 2**: Prove that the above physical arguments allow SRE phase if and only if \([\Lambda] = \mathbb{1}\)
3D Conjecture: no complete proof yet

Manipulate flux-tubes on Bieberbach manifolds…

Many new constraints, but not yet complete
LSM constraints have deep connection to physics of topological insulators / SPT phases

[Meng, Zaletel, Barkeshli, Vishwanath & Bonderson
arXiv:1511.02263]

3D Topo Insulator:

Bulk is insulating

Surface observed to be gapless (odd # Dirac cones)
Robust with U(1) and time-reversal

With interactions

surface of an SPT is either:

[Callan & Harvey 1985; Barkeshli et al. 2014;
X. Chen et al. 2015; Fidkowski Vishwanath 2015;
Wang, Lin, Levin 2015]

symmetry broken
gapless
or
fractionalized insulator
“anomalous surface topological order”
2D $S = 1/2$ magnet:

LSM - Theorem:

- symmetry broken
- gapless
- or
- fractionalized

Surface of 3D SPT:

- symmetry broken
- gapless
- or
- fractionalized

But one is a surface, and one isn't!
How to view the S=1/2 magnet as a surface

Spin-1 chain has
Spin-1/2 edge states
The “Haldane” / “AKLT” phase

Pack together Haldane chains into a 3D crystal:

3D “weak” interacting *bosonic* SPT

[Fu, Kane, Mele 2007; X Chen et al; Song et al 2016]

*Technical trick* to leverage knowledge of SPTs
How to view the S=1/2 magnet as a surface

[Cheng, Zaletel, Barkeshli, Vishwanath & Bonderson, 1511.02263 (2015)]

Bulk-boundary correspondence between 3D SPTs - Anomalous Surface order

\[ \Omega_{\text{weak}} \in \mathcal{H}^4 [\text{translation} \times \text{SO}(3)_{\text{int}}, U(1)] = \emptyset \in \mathcal{H}^4 [\text{translation} \times \text{SO}(3)_{\text{int}}, U(1)] \]

For translation symmetry, this gives the most general constraint.

Example: the Double Semion model forbidden in an S=1/2 magnet [Zaletel, Vishwanath 2015]
Lattice classification: the boundary of M Hermele’s talk?

(with D. Else & M. Hermele)

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Anomaly classes of space-group SETs?
Thanks!

Adrian Po, Chao-Ming Jian, Haruki Watanabe

Adrian Po, et al., arXiv:161x.xxxxx
