Bulk-edge corresponding in topological pumping

A simple reason of quantization of pumped charge

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“Bulk-edge correspondence in a topological pumping”,
Plan

★ Topological pumping
★ Back to Thouless
★ Time as a synthetic dimension of QHE
★ Experimental realizations after 30 years
★ Edge states ?

★★ Pumped charge & Berry connection
★ Pumped charge, Berry connection
★ & Temporal gauge

★★ Pumped charge & edge states
★ Temporal gauge & center of mass (CM)
★ Singular motion of CM
★ The Chern number & BEC

★★ Observations
★ Adiabatic & non-adiabatic
★ Direct simulations
Adiabatic pump (Thouless ’83)

Periodically driven 1D charge transport

Many-body but non-interacting as IQHE

\[ \text{i} \hbar \frac{\partial}{\partial t} |G(t)\rangle = H(t)|G(t)\rangle \quad |G(t)\rangle = T e^{-\left(\frac{i}{\hbar}\right) \int_{t_0}^{t} d\tau H(\tau)} |G(t_0)\rangle \]

\[ H(t) = \sum_{j}^{L} \left[ -t_x c_{j+1}^\dagger c_j + h.c. + v_j(t) c_{j+1}^\dagger c_j \right] \]

ex. \[ v_j(t + T) = v_j(t) \quad \text{period} \quad T \]

\[ \text{ex.} \quad v_j(t) = -2t_y \cos\left(2\pi \frac{t}{T} - 2\pi \phi j\right) \]

\[ \phi = \frac{p}{q} \]

Adiabatic: ground state is gapped & slow pumping

\[ \Delta E \gg 1/\tau \]

Topological!

Pumped charge is quantized as an integer
Back to Thouless '83

Time dependent 1D charge transport

Quantization of particle transport

D. J. Thouless

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(Received 4 February 1983)

1+1=2

Time as a synthetic dimension

2D Integer quantum Hall effect

TKNN '82 Hall conductance by the Chern number

Brouwer '98 Wang-Troyer-Dai '13

... Marra-Citro-Ortix '15

Experimentally realized in cold atoms after 30+ years in '15

Y. Takahashi, Kyoto
Nakajima et al.,
Nature Phys. 12, 296 (2016)

I. Bloch, Munich
Lohse et al.,

Topological Thouless Pumping of Ultracold Fermions
Shuta Nakajima, Takafulmi Tomita, Shintaro Taie, Tomohiro Ichinose, Hideki Ozawa, Lei Wang,
Matthias Troyer, Yoshiro Takahashi

A Thouless Quantum Pump with Ultracold Bosonic Atoms in an Optical Superlattice
Michael Lohse, Christian Schweizer, Oded Zilberberg, Monika Aidelsburger, Immanuel Bloch
If topological, then edge states?

Not much for the topological pump

Try to revisit the old problem

★ More than reinterpretation
★ New view points even technically
QHE (Hofstadter’s)

\[ \phi = \frac{2}{7} \]
QHE (Hofstadter’s) on 2D cylinder & TP

Non interacting but many body

Adiabatic limit
\[ \Delta E \gg \hbar/T \]

Filling states below \( E_F \) for the snapshot Ham.

Harper eq. (1D for each \( t \))

\[
-t_x (\psi_{m+1} + \psi_{m-1}) - 2t_y \cos(k_y - 2\pi \frac{p}{q}m) \psi_m = E \psi_m
\]

\[
-t_x (\psi_{m+1} + \psi_{m-1}) - 2t_y \cos(2\pi \frac{t}{T} - 2\pi \frac{p}{q}m) \psi_m = E \psi_m
\]
Examples

- $\phi = -1/7$, $t_x = 1$, $t_y = 1$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/7$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$2/7$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$3/7$</td>
<td>$-3$</td>
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<tr>
<td>$4/7$</td>
<td>$+3$</td>
</tr>
<tr>
<td>$5/7$</td>
<td>$+2$</td>
</tr>
<tr>
<td>$6/7$</td>
<td>$+1$</td>
</tr>
</tbody>
</table>

$$v_j(t) = -2t_y \cos\left(2\pi \frac{t}{T} - 2\pi \phi j\right)$$

$\phi = p/q$
The pumping is topological! (Thouless)

$\mathcal{O}(N^0)$ charge is pumped for an insulator with $N$ particles.

Pumped charge is quantized if gapped.

Independent of the parameters.
Examples 2

\[ \phi = -1/7 \quad t_x = 1, \quad t_y = 0.1 \]

\[
\begin{align*}
\rho &= 1/7 & C &= -1 \\
\rho &= 2/7 & C &= -2 \\
\rho &= 3/7 & C &= -3 \\
\rho &= 4/7 & C &= +3 \\
\rho &= 5/7 & C &= +2 \\
\rho &= 6/7 & C &= +1 \\
\end{align*}
\]

\[ t_x \gg t_y \quad \text{wave dynamics: quantum (weak potential)} \]
Examples 3

\[ \phi = -\frac{1}{7}, \quad t_x = 0.01, \quad t_y = 1 \]

\begin{align*}
\rho = 1/7 & \quad C = -1 \\
\rho = 2/7 & \quad C = -2 \\
\rho = 3/7 & \quad C = -3 \\
\rho = 4/7 & \quad C = +3 \\
\rho = 5/7 & \quad C = +2 \\
\rho = 6/7 & \quad C = +1 \\
\end{align*}

\[ t_x \ll t_y \quad \text{(deep potential)} \quad \text{quantum tunneling: semi-classical} \quad \text{easy to count charge!} \]
(quantum tunneling) \( t_x/t_y \ll 1 \) \( t_x \ll |v_j| \)

Diophantine eq. (TKNN)

\[ \phi = p/q = 2/7 \]
\[ \rho = r/q = 3/7 \]

Count in the tunneling limit if topological

\[ \Delta j = t = C = -2 \]

**Tunneling condition**

\[ \epsilon_j(t) = \epsilon'_j(t) \]

\[ \epsilon_j(t) = -2t_y \cos(2\pi \phi j - 2\pi \frac{t}{T}) \]

\[ \star 1 \quad 2\pi \phi j - 2\pi \frac{t}{T} = -(2\pi \phi j' - 2\pi \frac{t}{T}) + 2\pi s \quad s \in \mathbb{Z} \]

**Tunneling at the filling \( r \) per unit**

\[ \epsilon_j(t) = -2t_y \cos(\frac{r}{q}) \]

\[ \star 2 \quad \pi \frac{r}{q} = 2\pi \phi j - 2\pi \frac{t}{T} \]

\[ r : \text{even} \]

\[ r = 4 \]

\[ -2t_y \cos(\frac{4}{q}) \]

\[ r = 5 \]

\[ -2t_y \cos(\frac{5}{q}) \]

\[ \star 1 \& \star 2 \]

\[ p = 2, q = 7 \]

\[ (r, t, s) = (1, -3, 1) \]
\[ (2, +1, 0) \]
\[ (3, -2, 1) \]
\[ (4, +2, 0) \]
\[ (5, -1, 1) \]
\[ (6, +3, 0) \]

Diophantine eq. (TKNN)

\[ r = pt + qs \equiv pt \quad \text{(mod } q) \]

\[ t = \Delta j = j - j' \quad |t| \leq q/2 \]

= Chern number \( C \)

\[ \text{Magic!} \]

Streda formula:

\[ C = \frac{dN}{dB} = \frac{d(r/q)}{d\phi} = \frac{r/q}{p/q} = t \]

: algebraic

: analytic (integral)
Diophantine eq. (TKNN) \( t_x/t_y \ll 1 \) (quantum tunneling)

\[
\phi = p/q = 2/7 \\
\rho = r/q = 3/7
\]

\[
\Delta j = t = C = -2
\]

\[
2\pi\phi j - 2\pi \frac{t}{T} = -(2\pi\phi j' - 2\pi \frac{t}{T}) + 2\pi s \\
\frac{p}{q} j = -\frac{p}{q} j' + \frac{2t}{T} + s
\]

\[
\pi \frac{r}{q} = 2\pi\phi j - 2\pi \frac{t}{T} \\
\frac{r}{q} = \frac{2p}{q} j - \frac{2t}{T}
\]

\[
\frac{r}{q} = \frac{p}{q} (j - j') + s
\]

\[
\begin{align*}
p &= 2, q = 7 \\
(r, t, s) &= (1, -3, 1) \\
&(2, +1, 0) \\
&(3, -2, 1) \\
&(4, +2, 0) \\
&(5, -1, 1) \\
&(6, +3, 0)
\end{align*}
\]

Streda formula: \( C = \frac{dN}{dB} = \frac{d(r/q)}{d\phi} = \frac{r/q}{p/q} = t \)

Magic!
Diophantine eq. (TKNN) \( t_r / t_n \ll 1 \) (quantum tunneling)

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

\[ r = \begin{cases} 
\text{even} & \text{if } r = 4, 6, 8, \ldots \\
\text{odd} & \text{if } r = 5, 7, 9, \ldots 
\end{cases} \] 

\[ r = pt + sq \pmod{q} \]

\[ |t| \leq q/2 \]

Sreda formula: \[ C = \frac{dN}{dB} = \frac{d(r/q)}{d\phi} = \frac{r/q}{p/q} = t \]

\[ \Delta j = t = \]

\[ j = j_0 \]

FIG. 1. Motion of electrons in the \( x \) direction under the influence of an electric field in the \( y \) direction for \( V \ll V' \), for the two cases (a) \( \varphi = 5 \) and (b) \( \varphi = \frac{5}{3} \).
Diophantine eq. (TKNN)  \( t_x/t_y \ll 1 \) (quantum tunneling)

\[
\phi = \frac{p}{q} = \frac{2}{7} \quad \rho = \frac{r}{q} = \frac{3}{7}
\]

\[
\varepsilon_j(t) = \varepsilon_{j'}(t)
\]

Tunneling condition

\[
\varepsilon_j(t) = -2t_y \cos\left(2\pi \phi j - 2\pi \frac{t}{T}\right)
\]

\[
\ast \ 2\pi \phi j - 2\pi \frac{t}{T} = -(2\pi \phi j' - 2\pi \frac{t}{T}) + 2\pi s \quad s \in \mathbb{Z}
\]

Tunneling at the filling \( r \) per unit

\[
\varepsilon_j(t) = -2t_y \cos\left(\frac{r}{q}\right)
\]

\[
\ast \ \frac{r}{q} = 2\pi \phi j - 2\pi \frac{t}{T}
\]

\[
\begin{align*}
& \text{r : even} \\
& \text{r : odd}
\end{align*}
\]

Diophantine eq. (TKNN)

\[
(r, t, s) = (1, -3, 1) \\
(2, +1, 0) \\
(3, -2, 1) \\
(4, +2, 0) \\
(5, -1, 1) \\
(6, +3, 0)
\]

\[
p = 2, q = 7
\]

\[
\Delta j = t = C = -2
\]

Streda formula:

\[
C = \frac{dN}{dB} = \frac{d(r/q)}{d\phi} = \frac{r/q}{p/q} = t
\]

Magic!
Adiabatic pump (Thouless '83)

Periodically driven 1D charge transport

\[ i\hbar \frac{\partial}{\partial t} |G(t)\rangle = H(t)|G(t)\rangle \quad |G(t)\rangle = Te^{-i/\hbar} \int_{t_0}^{t} d\tau H(\tau) |G(t_0)\rangle \]

\[ H(t) = \sum_{j}^{L} \left[ -t_x c_{j+1}^{\dagger} c_j + h.c. + v_j(t)c_{j}^{\dagger}c_j \right] \]

\[ v_j(t + T) = v_j(t) \quad \text{period} \quad T \]

ex. \[ v_j(t) = -2t_y \cos\left( \frac{t}{T} + 2\pi \phi j \right) \]

Adiabatic: ground state is gapped & slow pumping

\[ \Delta E \gg \hbar/T \quad \text{Topological!} \]

Pumped charge is quantized as an integer
Pumped charge by adiabatic approximation

\[ j = \langle G | J | G \rangle \]
\[ J = \frac{1}{L_x} \left( +i \frac{t_x}{\hbar} e^{-i \theta / L_x} \right) \sum_j c_{j+1}^\dagger c_j + h.c. \]
\[ = +\hbar^{-1} \partial_\theta H(\theta) \]
\[ H(\theta, t) = \sum_j \left[ -t_x e^{-i \theta / L_x} c_{j+1}^\dagger c_j + h.c. + v_j(t) c_{j+1}^\dagger c_j \right] \]

|\alpha(t)\rangle : Snapshot eigen state

\[ H(t)|\alpha(t)\rangle = E_\alpha(t)|\alpha(t)\rangle, \quad \langle \alpha | \beta \rangle = \delta_{\alpha \beta}. \]

\[ |G\rangle = e^{-(i/\hbar) \int_0^t dt' E_g(t') e^{i \gamma(t)}} \left[ |g\rangle + i\hbar \sum_{\alpha \neq g} \frac{|\alpha\rangle \langle \alpha | \partial t g \rangle}{E_\alpha - E_g} \right] \]

\[ \delta j_x = \langle G | J | G \rangle - \langle g | J | g \rangle = -i B \]

\[ B = \partial_\theta A_t - \partial_t A_\theta, \quad A_\mu = \langle g | \partial_\mu g \rangle, \quad \mu = \theta, t. \]
**Pumped charge & Berry connection**

**Pumped charge in** \( T \)

\[
\Delta Q = \int_0^T dt \, \delta j_x = -i \int_0^T dt \, B
\]

**Berry connection**

\[
B = \partial_\theta A_t - \partial_t A_\theta
\]

\[
A_\mu = \langle g|\partial_\mu g \rangle, \quad \mu = t, \theta
\]

**Snapshot ground state**

\[
|g(t)\rangle : \quad H(t)|g(t)\rangle = E(t)|g(t)\rangle
\]

**Physical observable**

\[
\Delta Q = i \int_0^T dt \partial_t A_\theta^{(t)} = i \left[ A_\theta^{(t)}(T) - A_\theta^{(t)}(0) \right]
\]

**Berry connection (gauge fixed)**

**Adiabatic appr.**

**Twist**

\[
t_x \rightarrow t_x e^{-i\theta/L}
\]

**Temporal gauge:**

\[
A_{t}^{(t)} = 0
\]

\[
B = \partial_\theta A_t - \partial_t A_\theta
\]

**Thouless ’83**

_With/without edges_

With/without edges
**Temporal gauge**

Temporal gauge: \( A_t^{(t)} = 0 \)

\[
B = \partial_\theta A_t - \partial_t A_\theta
\]

Gauge transformation \( \langle g' | \partial_\mu g' \rangle = \langle g | \partial_\mu g \rangle + i \partial_\mu \chi, \quad |g'\rangle = |g\rangle e^{i\chi} \)

**temporal** \[ A_\mu^{(t)}(t, \theta) = A_\mu(t, \theta) + i \partial_\mu \chi(t, \theta) \]

**general** \[
\chi(t, \theta) = i \int_0^t d\tau A_t(\tau, \theta) + i \int_0^\theta dv A_{\theta}(0, v)
\]

\[
A_t^{(t)}(t, \theta) = A_t(t, \theta) + i \partial_t \chi(t, \theta) = 0
\]

\[
A_\theta^{(t)}(t, \theta) = A_\theta(t, \theta) + i \partial_\theta \chi(t, \theta)
\]

\[
= A_\theta(t, \theta) - A_\theta(0, \theta) - \partial_\theta \int_0^t d\tau A_t(\tau, \theta)
\]

\[
A_\theta^{(t)}(t, \theta) \neq A_\theta^{(t)}(t + T, \theta)
\]

non periodic gauge fixing

**C :** \( (0,0) \rightarrow (0,\theta) \rightarrow (t,\theta) \)
**Pumped charge & Center of mass (CM)**

\[ H(\theta, t) = \sum_{j}^{L} \left[ -t_{x}e^{-i \theta \frac{j}{L}} c_{j+1}^\dagger c_{j} + h.c. + v_{j}(t)c_{j}^\dagger c_{j} \right] \]

**Twist**: gauged out for an open system (with edges)

\[ H(\theta, t) = \mathcal{U} H(0, t) \mathcal{U}^\dagger \quad \mathcal{U}(\theta) = \prod_{j=1}^{j_{0}} e^{-i \theta n_{j}(j-j_{0})/L_{x}} \]

\[ j_{0} = L/2 \]

\[ \theta \text{ independent} \]

\[ |g(\theta)\rangle = \mathcal{U}(\theta)|g(0)\rangle \]

\[ \mathcal{U}_{c_{j}U_{\dagger}} = e^{i \theta j/L_{x}} c_{j} \]

\[ \mathcal{U}_{c_{j}U_{\dagger}} = c_{j}^\dagger e^{-i \theta j/L_{x}} \]

\[ A_{\theta} = \langle g(\theta) | \partial_{\theta} g(\theta) \rangle = \langle g(0) | \mathcal{U}^\dagger \partial_{\theta} \mathcal{U} | g(0) \rangle = -iP(t) \]

Center of mass (CM)

\[ P(t) = \sum_{x_{j}} x_{j} \rho_{j} \]

\[ \sum_{j} \rho_{j} = N \]

\[ \rho_{j} = \rho(x_{j}) = \langle g(0) | n_{j} | g(0) \rangle \]

\[ x_{j} = (j-j_{0})/L \in [-1/2, 1/2] \]

\[ P(t) = \mathcal{O}(N^{0}) \]

**Insulator**

\[ \Delta Q = P(T) - P(0) \]

\[ \text{ill-defined for periodic system} \]
\( \phi = 1/7, \rho = 3/7, C = 3 \) **With/without edges**

**periodic**

**with edges**

*Not so much difference*
Center of mass with edges

\[ \phi = \frac{1}{7}, \quad N \sim L \gg 1 \]

\[ E_F = -1.5, \rho \sim \frac{2}{7} \]

\[ P(t) = \sum_j x_j \rho_j \sim O(N^0) \]

\[ x_j = (j - j_0)/L \in [-1/2, 1/2] \]

\[ \rho_j = \rho(x_j) = \langle g(0)|n_j|g(0) \rangle \]

\[ L = 139 \]

Many jump, discontinuities
Singular motion of CM due to edge states

\[ E(t) = P(t) \]

\[ P(t_i) = \sum_{j} x_j \rho_j \]

\[ \rho_j = \rho(x_j) = \langle g(0) | n_j | g(0) \rangle \]

\[ x_j = (j - j_0)/L \in [-1/2, 1/2] \]

\[ \Delta P(t_i) = P(t_i^+) - P(t_i^-) \]

-1/2: becomes unoccupied at R
+1/2: becomes occupied at R
+1/2: becomes unoccupied at L
-1/2: becomes occupied at L

\[ j \sim L \quad (L \to \infty) \]

\[ j_0 = L/2 \]

\[ \phi = 2/7 \]
Singular motion of CM

due to edge states

Many jump!?

\[
P(t) = \sum_{j} x_j \rho_j
\]

\[
x_j = (j - j_0)/L \in [-1/2, 1/2]
\]

\[
\rho_j = \rho(x_j) = \langle g(0) | n_j | g(0) \rangle
\]

Contribution of the edge state for \( P \) is

\[
\frac{j - j_0}{L} \rightarrow \pm \frac{1}{2}
\]

\[
j_0 = L/2 \quad (L \to \infty)
\]

\[
\Delta P(t_i) = P(t_i^+) - P(t_i^-)
\]

\[
\begin{cases}
-1/2: \text{becomes unoccupied at } R \\
+1/2: \text{becomes occupied at } R \\
+1/2: \text{becomes unoccupied at } L \\
-1/2: \text{becomes occupied at } L
\end{cases}
\]
How much pumped?

$P(t)$ is periodic function!

\[
\Delta P(t_i) = P(t_i^+) - P(t_i^-)
\]

- $1/2$: become unoccupied at $R$
- $+1/2$: become occupied at $R$
- $+1/2$: become unoccupied at $L$
- $-1/2$: become occupied at $L$

\[
\Delta Q = \sum_i \int_{t_i^-}^{t_i^+} dt \partial_t P(t) = \sum_i [P(t_{i+1}^-) - P(t_i^+)]
\]

\[
= - \sum_i [P(t_i^+) - P(t_i^-)] = - \sum_i \Delta P(t_i)
\]

Bulk-edge correspondence in time domain due to edge states

Pump by bulk

Patch work in time domain

Integrate along the red curves

Periodicity in time
Quantization & conservation law

\[ \Delta Q = - \sum_i \Delta P(t_i) \]

Number of the discontinuities (SUM) are EVEN!
Conservation of charge & periodicity in time

become occupied ↔ become unoccupied

paired
Quantization & conservation law

\[ \Delta P(t_i) = P(t_i^+) - P(t_i^-) \]

\[ \begin{align*}
-1/2: & \text{ become unoccupied at } R \\
+1/2: & \text{ become occupied at } R \\
+1/2: & \text{ become unoccupied at } L \\
-1/2: & \text{ become occupied at } L 
\end{align*} \]

\[ \Delta Q = - \sum_i \Delta P(t_i) = - \sum_i \left( \pm \frac{1}{2} \right) = \text{integer } I \]

Number of the discontinuities (SUM) are EVEN!

Conservation of charge & periodicity in time

become occupied \(\leftrightarrow\) become unoccupied

paired
Quantization & conservation law

Identify the discontinuities as massive Dirac fermions

\[ \Delta P(t_i) = P(t_i^+) - P(t_i^-) \]

\[ = \begin{cases} 
-1/2: & \text{become unoccupied at } R \\
+1/2: & \text{become occupied at } R \\
+1/2: & \text{become unoccupied at } L \\
-1/2: & \text{become occupied at } L 
\end{cases} \]

\[ \Delta Q = - \sum_i \Delta P(t_i) = - \sum_i \left( \pm \frac{1}{2} \right) = \text{integer } I \]

Edge states correspond to massive Dirac fermions

\[ \frac{1}{2} \text{sgn } m \]
Pumped charge as a Chern number

Skipping the discontinuities

$P(t)$
Pumped charge as a Chern number

Skipping the discontinuities

$P(t)$
Pumped charge as a Chern number

\[ iA^{(t)}(t) = P(t) \]

CM is not well defined for the bulk (Bloch state)

\[ iA^{(t)}(t) \text{ is still well defined (non periodic in time)} \]

\[ \Delta Q = i \int_0^T dt \partial_t A^{(t)}(t) \]

(Thouless ‘83)
Pumped charge as a Chern number

\[ \Delta Q = i \int_0^T dt \partial_t A^{(t)} = \frac{1}{2\pi i} \int_0^T dt \int_0^{\Delta k} dk_x b(k_x, t) \equiv C \]

\[ b = \partial_{k_x} a_t - \partial_t a_{k_x} \]

\[ a^{(t)}_{k_x} = \text{Tr}_M A^{(t)}_{k_x} \]

\[ A^{(t)}_{k_x} = u^\dagger \partial_{k_x} u \]

\[ u = (u_1, \ldots, u_M), \]

\[ u_\ell(k_x, t) \text{ Block state of the } \ell\text{-th band} \]

\[ I(\text{edge}) = C(\text{bulk}) \]

**Bulk-edge correspondence between the topological numbers**

Then the Chern number is integer as well! (non trivial)
Note!

CM is only well-defined with edges
no way to define CM with periodic boundary condition

Pumped charge is carried by bulk
but is described
by the discontinuity due to edge states

This is the bulk-edge correspondence

Discontinuity: breakdown of the adiabaticity
due to gapless edge states, then it is never observed in
real experiments of finite speed pump!
Edge states: Do **Not** contribute the experiments **BUT still**

**protect** quantization of the pumped charge

**Note!**

Discontinuity: breakdown of the adiabaticity due to gapless edge states, then it cannot be observed in real experiments of finite speed pump!

*(if the system is large enough)*
Thank you