Unitary Invertible Field Theories and Short-Range Entangled Phases

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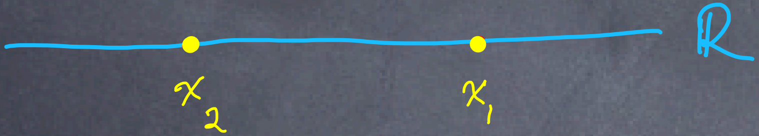
Joint work with Mike Hopkins
(arXiv:1604.06527)

Three "Mathematical" Objects

1. An ordered pair (x_1, x_2) of distinct real numbers



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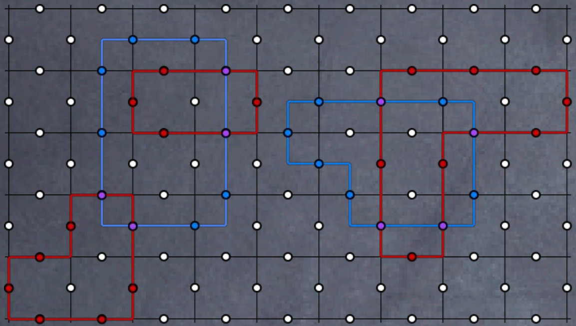

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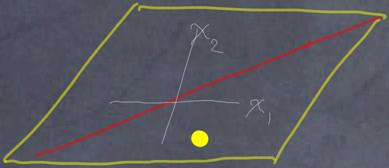

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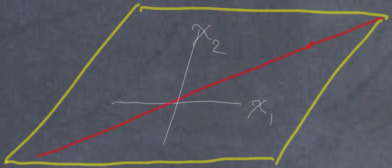

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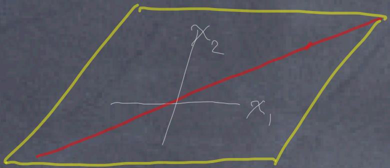

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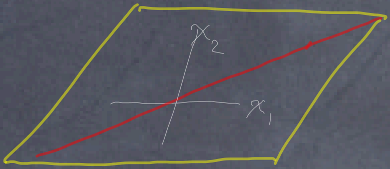

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Remark: $\pi_0\mathcal{M}$ is nontrivial because of *gap* condition $x_1 \neq x_2$

Moduli Spaces

2. A knot in Euclidean 3-space $\mathbb{E}^3$

Moduli space $\mathcal{M}$ is trickier to define, e.g. it is infinite-dimensional

Geometric Invariant: $\kappa\colon \mathcal{M} \longrightarrow \mathbb{R}^{>0}$ (total curvature)

Topological Invariant: $\pi_0\mathcal{M}$ is the subject of knot theory

$\pi_0\mathcal{M} \longrightarrow$ polynomials (Jones, ...)

Moduli Spaces

❸ A gapped short-range-entangled d-dimensional lattice system with fixed symmetry group

$\mathcal{M}$ not defined!

Geometric Invariant: Correlation functions

Topological Invariant: $\pi_0\mathcal{M}$ is the set of topological phases of matter

Transition to Field Theory

Two principles:

- The deformation class of a quantum system is determined by its low energy behavior
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- Allows to define and determine moduli space $\mathcal{M}'$ for TFT
- Axiom System applies to scale-dependent theories. . .

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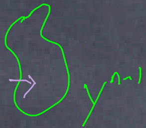
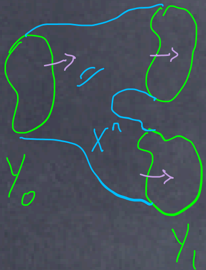
$$M^n \rightsquigarrow \mathcal{D} \rightsquigarrow \mathbb{E}^n \rightsquigarrow X^n$$

Definition: An n-dimensional *field theory* is a homomorphism

$$F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \longrightarrow \mathrm{Vect}_{\mathbb{C}}^{\mathrm{top}}$$



$\overset{F}{\longmapsto}$ $F(Y)$ topological vector space



$\overset{F}{\longmapsto}$ $F(X)\colon F(Y_0) \longrightarrow F(Y_1)$ contracting linear map

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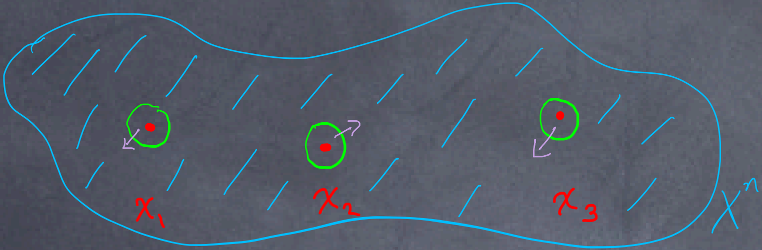

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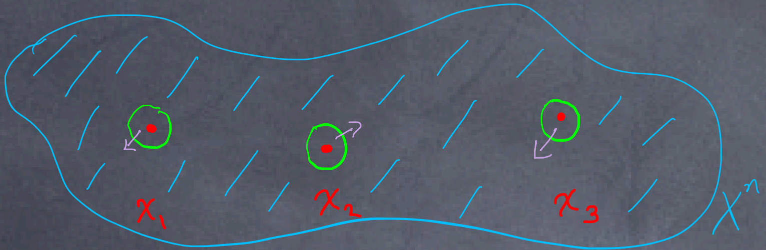

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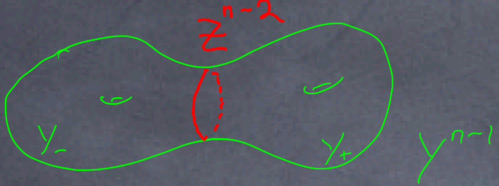
N.B.: The most far-reaching assertion is that the field theory can be encoded by *compact* Riemannian manifolds

Locality and Unitarity

The state space $F(Y^{n-1})$ depends *locally* on Y



$$F(Y) \text{ ``=''} \langle F(Y_-), F(Y_+) \rangle_{F(Z)}$$

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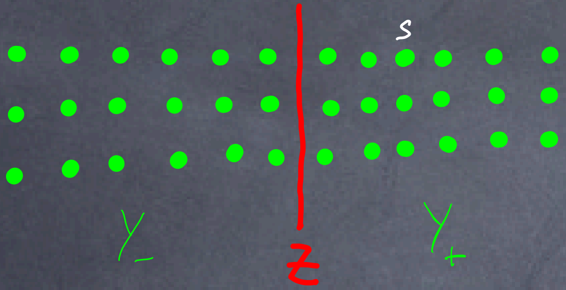

$$\mathcal{H} = \bigotimes_s \mathcal{H}_s$$

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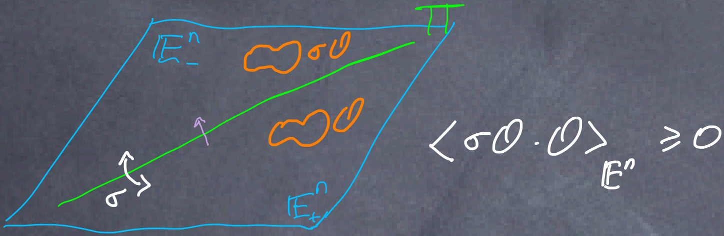
Wick-rotated unitarity: *reflection positivity*



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We propose a solution for invertible topological theories

Main Theorem

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spacetime dimension

symmetry group

Thom spectrum

stabilized symmetry group

Anderson dual to sphere spectrum

torsion subgroup

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Conjecture (F.-Hopkins): There is a 1:1 correspondence

$$\pi_0 \mathcal{M}'(n, H_n) \cong [MTH, \Sigma^{n+1} I\mathbb{Z}]$$

Comparison with Previous Results

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Will show some concrete results at end of lecture

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G_n	$\mathcal{G}_n$/translations
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$$\begin{array}{ccccccc}
1 & \longrightarrow & K & \longrightarrow & G_n & \xrightarrow{\rho_n} & O_{1,n-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K(\mathbb{C}) & \longrightarrow & G_n(\mathbb{C}) & \xrightarrow{\rho_n} & O_n(\mathbb{C}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
1 & \longrightarrow & K & \longrightarrow & H_n & \xrightarrow{\rho_n} & O_n
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compact Lie group

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Relativistic invariance: $\mathrm{Image}(\rho_n) = SO_n$ or O_n

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Minimal internal symmetry group K:

states/symmetry	H_n	K	k_0
bosons only	SO_n	$\{1\}$	1
bosons, time-reversal (T)	O_n	$\{1\}$	1
fermions allowed	Spin_n	$\{\pm 1\}$	-1
fermions, $T^2 = (-1)^F$	Pin_n^+	$\{\pm 1\}$	-1
fermions, $T^2 = \mathrm{id}$	Pin_n^-	$\{\pm 1\}$	-1

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Theorem: There are 10 stable symmetry groups H of this type:

$K = \mathbb{T}$	Spin^c, Pin^c,
	$\mathrm{Pin}^{\tilde{c}+} := \mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathbb{T}$
	$\mathrm{Pin}^{\tilde{c}-} := \mathrm{Pin}^- \ltimes_{\{\pm 1\}} \mathbb{T}$
$K = \{\pm 1\}$	Spin, Pin^+, Pin^-
$K = SU_2$	$\mathrm{Spin} \times_{\{\pm 1\}} SU_2$
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(topological insulators)

$K = \{\pm 1\}$ Spin, Pin^+, Pin^-

$K = SU_2$ $\mathrm{Spin} \times_{\{\pm 1\}} SU_2$

$\mathrm{Pin}^+ \times_{\{\pm 1\}} SU_2$

$\mathrm{Pin}^- \times_{\{\pm 1\}} SU_2$

(topological superconductors)

Uniform treatment via Clifford algebras

Theorem: There are embeddings $H_n(s) \hookrightarrow \mathrm{Cliff}_{+n} \otimes D(s)$ compatible with Clifford multiplication.

s	H^c	K	Cartan	D
0	Spin^c	$\mathbb{T}$	A	$\mathbb{C}$
1	Pin^c	$\mathbb{T}$	AIII	$\mathrm{Cliff}^{\mathbb{C}}_{-1}$

s	H	K	Cartan	D
0	Spin	$\{\pm 1\}$	D	$\mathbb{R}$
-1	Pin^+	$\{\pm 1\}$	DIII	Cliff_{-1}
-2	$\mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{T}$	AII	Cliff_{-2}
-3	$\mathrm{Pin}^- \times_{\{\pm 1\}} SU_2$	SU_2	CII	Cliff_{-3}
4	$\mathrm{Spin} \times_{\{\pm 1\}} SU_2$	SU_2	C	$\mathbb{H}$
3	$\mathrm{Pin}^+ \times_{\{\pm 1\}} SU_2$	SU_2	CI	Cliff_{+3}
2	$\mathrm{Pin}^- \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{T}$	AI	Cliff_{+2}
1	Pin^-	$\{\pm 1\}$	BDI	Cliff_{+1}

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time-reversal?

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Computations

$\mathcal{M}'_{\text{top}}(n, H_n) :=$ moduli space of reflection positive invertible n-dimensional extended topological field theories with symmetry group H_n

Theorem/Conjecture (F.-Hopkins): There is a 1:1 correspondence

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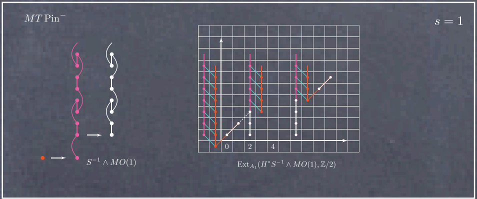
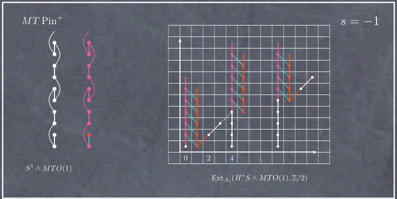
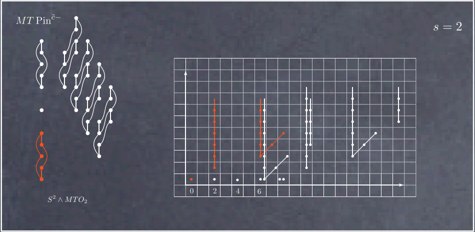
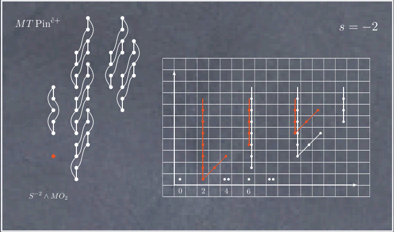
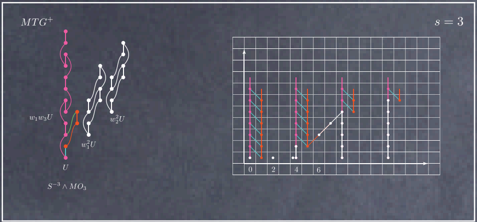
Compute RHS of (1) using Adams spectral sequence. Forthcoming paper with Jonathan Campbell to explain technique and more examples

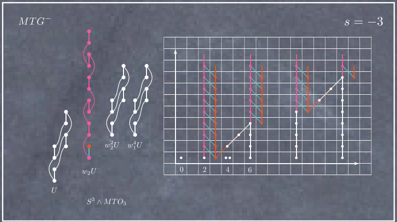

$MT\,\mathrm{Pin}^-$
$s = 1$
$S^{-1} \wedge MO(1)$
0
2
4
$\mathrm{Ext}_{A_1}(H^*S^{-1} \wedge MO(1), \mathbb{Z}/2)$


$MT\,\mathrm{Pin}^+$
$s = -1$
$S^1 \wedge MTO(1)$
0
2
4
$\mathrm{Ext}_{A_1}(H^*S \wedge MTO(1), \mathbb{Z}/2)$


$MT\,\mathrm{Pin}^{\tilde{c}-}$
$s = 2$
$S^2 \wedge MTO_2$
0
2
4
6


$MT\,\mathrm{Pin}^{\tilde{c}+}$
$s = -2$
$S^{-2} \wedge MO_2$
0
2
4
6


MTG^+
$s = 3$
w_1w_3U
w_2^2U
w_1^2U
U
$S^{-3} \wedge MO_3$
0
2
4
6


MTG^-
$s = -3$
w_2^2U
w_1^4U
w_2U
U
$S^3 \wedge MTO_3$
0
2
4
6

Computations

Class DIII (Pin^+):

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^+) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^+) \longrightarrow$	$\mathrm{coker}\,\Phi$
4	$16\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z}$	0
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

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3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
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3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
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- $TP_n(H) = \pi_0\mathcal{M}'(n, H_n)$ is group of topological phases (Main Thm)
- Φ is the map described above (essentially ABS)
- The FF_n groups are well-known. Many TP_n appear in the condensed matter literature (together with Φ) via other methods

Class AII ($\mathrm{Pin}^{\tilde{c}+}$):

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^{\tilde{c}+}) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^{\tilde{c}+}) \longrightarrow$	$\mathrm{coker}\,\Phi$
4	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2$
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	0	0	0
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
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- Metlitski asked about $TP_4(\mathrm{Pin}^{\tilde{c}+})$ vs. bordism computation
- The results in 3 dimensions are also known via non-bordism means

Class CI ($G^+ = \mathrm{Pin}^+ \times_{\{\pm 1\}} SU_2$):

n	$\ker\Phi \longrightarrow$	$FF_n(G^+) \xrightarrow{\Phi}$	$TP_n(G^+) \longrightarrow$	$\mathrm{coker}\,\Phi$
4	$4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	0	0	0	0
2	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

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3	0	0	0	0
2	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
1	0	0	0	0
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- Unsure if $TP_{2,3}(G^+)$ are in the CM literature (predictions)

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n	$\ker \Phi \longrightarrow$	$FF_n(G^-) \xrightarrow{\Phi}$	$TP_n(G^-) \longrightarrow$	$\operatorname{coker} \Phi$
4	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2$
3	0	0	0	0
2	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
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- Open Question: Extended reflection positivity in general