Periodic solutions representing the origin of turbulent bands in channel flow

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Plane channel flow

Reynolds number: \( Re = \frac{U_c h}{v} \)

- \( U_c \) : centerline velocity

- Laminar flow becomes unstable at \( Re = Re_c^{[1]} \)
- Turbulence is observed at \( Re < Re_c \)

\[ [1] \text{ } Re_c = 5772, \text{ Orszag, JFM (1971)} \]

\( (L_x, L_z) = (32\pi, 16\pi) \)

(Streamwise velocity near the wall)
Isolated turbulent band

Isolated extending turbulent band in large computational domain

\[ (L_x, L_z) = (200h, 160h) \]

Xiong, Tao, Chen and Brandt, Phys. Fluids (2015)
**System & Parameters**

- **Governing equation**
  Incompressible Navier-Stokes equation
  
  \[
  \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} \\
  \nabla \cdot \mathbf{u} = 0
  \]

- **Dimensionless parameter**
  Reynolds number : \( Re_m = \frac{Uh}{v} \)
  
  \( (U : \text{constant bulk mean velocity}) \)

- **Boundary conditions**
  streamwise & spanwise \( \Rightarrow \) periodic
  wall-normal \( \Rightarrow \) no-slip impermeable

\[ x : \text{streamwise} \]
\[ y : \text{wall-normal} \]
\[ z : \text{spanwise} \]

\[ x \in [0, L_x] \]
\[ y \in [-h, h] \]
\[ z \in [0, L_z] \]
System & Parameters

• Numerical domain & Grid numbers

<table>
<thead>
<tr>
<th>$(L_x, L_z)$</th>
<th>$(N_x, N_y, N_z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(500, 250)</td>
<td>(3840, 49, 1920)</td>
</tr>
<tr>
<td>(200, 200)</td>
<td>(768, 49, 768)</td>
</tr>
<tr>
<td>(100, 100)</td>
<td>(384, 49, 384)</td>
</tr>
</tbody>
</table>

$x$ : streamwise  
$y$ : wall-normal  
$z$ : spanwise

$x \in [0, L_x]$
$y \in [-1, 1]$
$z \in [0, L_z]$
Example

\((L_x, L_z) = (500, 250), \ Re_m = 440\)
Example

\[(L_x, L_z) = (500, 250), \quad Re_m = 440\]

(\(y = 0\))

- **Length & Angle**
- **Downstream edge**

\(t = 1410\)

\(u_x - U_{LF}\)

\(U_{LF}\) : Laminar flow velocity
Length & Angle

Measurement method

\[(L_x, L_z) = (500, 250), \quad Re_m = 470\]

\[c_x \approx 1.23, \quad c_z \approx -0.14\]

\[u^2 = (u_x - U_{LF})^2\]

\[\langle \cdot \rangle_x = \frac{1}{L_x} \int_0^{L_x} \cdot dx\]

\[\langle \cdot \rangle_z = \frac{1}{L_z} \int_0^{L_z} \cdot dz\]
Length & Angle

\[(L_x, L_z) = (500, 250), \quad Re_m = 440 \quad (Re = 3Re_m/2 = 660)\]
Length & Angle

\[(L_x, L_z) = (500, 250), \quad Re_m = 440 \quad (Re = 3Re_m/2 = 660)\]

\[x/h = 1292, \quad x/h = 1880, \quad x/h = 2096\]

\[(2+1)-d \text{ DP} \quad \beta = 0.583\]

Effects of spatial periodicity

\[(L_x, L_z) = (500, 250)\]
Example

$$(L_x, L_z) = (500, 250), \ Re_m = 440$$

① Length & Angle  
② Downstream edge

$U_{LF}$ : Laminar flow velocity
Downstream edge

$$(L_x, L_z) = (500, 250), \quad Re_m = 450$$

$c_x \approx 1.29$
$c_z \approx -0.14$
Downstream edge

\((L_x, L_z) = (500, 250), \; Re_m = 450\)

\[ u_x > U_{LF} \]

- Coherent
- Turbulent

\[ c_x \approx 1.29 \]
\[ c_z \approx -0.14 \]
Downstream edge

$$(L_x, L_z) = (500, 250), \; Re_m = 450$$

$$(y = 0)$$

$c_x \approx 1.29$

$c_z \approx -0.14$

$u_x - U_{LF} = +0.1$

$Q = 0.075, \omega_x > 0$

$Q = 0.075, \omega_x < 0$

$Q$ : second invariant of velocity gradient tensor
Isolation of downstream edge

⇒ Add the spatially localized damping force

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} - F(x, z, t)(\mathbf{u} - U_{LF} \mathbf{e}_x) \]

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ F(x, z, t) = \alpha f(x, z, t) \]

\( \alpha \) : damping intensity

\( f(x, z, t) \) : spatial distribution function

\( f = 0 \)

\( 0 < f < 1 \) : hyperbolic-tangent type

\( f = 1 \)

\( C \) : local maximum point of \( u_x \) (\( y=0 \))

\( (r_x, r_z) \)

\( C' \) : \( (r_x-d, r_z+d) \)

\( d \approx 19 \)
Isolation of downstream edge

Damping mask in minimal turbulent channel

\[
\frac{\partial \mathbf{u}}{\partial t} = F \times (\text{RHS of Navier-Stokes})
\]

\[
F(y) = 1 \quad \text{if } y \leq \delta_1, \quad F(y) = F_0 < 1 \quad \text{if } y \geq \delta_2
\]

\[
\omega_x^+ = \pm 0.18, \quad u^+ - \bar{u}^+ = -3.5
\]

Isolation of downstream edge

Damping force in high-Re 2-d turbulent channel

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \frac{1}{Re} \nabla^2 u - H_{\sigma^2, \Omega}(x)(u - U_L), \]

\[ H_{\sigma^2, \Omega}(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{\Omega} dx' \exp \left( \frac{(x - x')^2}{2\sigma^2} \right) \]

**turbulence**  \( Re = 8000 \)

**ejection-jet cycle**

Isolated downstream edge

\[(L_x, L_z) = (200, 200), \quad Re_m = 470, \alpha = 0.1\]

\[
E_{3d} = \frac{1}{2V} \int_V (u - U_{LF} e_x)^2 \, dV
\]

sustain without turbulent ‘tail’
Periodic solution extracted using damping force

\((L_x, L_z) = (100, 100), \ Re_m = 450, \ \alpha = 0.1\)

\[
\begin{align*}
Z &\equiv 0.075, \\
\tau &\equiv 0
\end{align*}
\]

\(\tau \equiv 0.1\)

\(\mathcal{M} = 100, 100\)

\(\mathcal{R} \equiv 450\)

\(\alpha \equiv 0.1\)

\(\frac{\mathcal{E}_3d(10^{-4})}{t} \Rightarrow \text{'shear-layer' sinuous instability} \Rightarrow \text{staggered vortices} \Rightarrow \text{invariant solution} \Rightarrow \text{chaos}\)

\[\begin{align*}
\text{\textbullet} & \quad u_x - U_{LF} = +0.1 \\
\text{\textbullet} & \quad Q = 0.075, \ \omega_x > 0 \\
\text{\textbullet} & \quad Q = 0.075, \ \omega_x < 0
\end{align*}\]
Sustaining mechanism of downstream edge

\[(L_x, L_z) = (100, 100), \ Re_m = 460, \ \alpha = 0.1\]

Mean flow field for periodic solution

\[\langle u - U_{LF} e_x \rangle_t = \frac{1}{T} \int_0^T (u - U_{LF} e_x) \, dt\]

Wall-normal roll

High-speed region

\[\bar{u}_x - U_{LF} = 0.1\]
\[\bar{u}_x - U_{LF} = -0.1\]

velocity vectors on \(y=0\)
Sustaining mechanism of downstream edge

Mean flow field for periodic solution

\[
\langle \mathbf{u} - U_L e_x \rangle_t = \frac{1}{T} \int_0^T (\mathbf{u} - U_L e_x) \, dt
\]
Sustaining mechanism of downstream edge

Mean flow field for periodic solution

\[
\langle \mathbf{u} - U_{LF} \mathbf{e}_x \rangle_t = \frac{1}{T} \int_0^T (\mathbf{u} - U_{LF} \mathbf{e}_x) \, dt
\]
Sustaining mechanism of downstream edge

Mean flow field for periodic solution

\[
\langle u - U_{LF}e_x \rangle_t = \frac{1}{T} \int_0^T (u - U_{LF}e_x) \, dt
\]
Sustaining mechanism of downstream edge

If the roll were purely wall-normal

\[ y = 1 \]
\[ y = -1 \]

would decay

Real inclined rolls

\[ y = 1 \]
\[ y = -1 \]

stretching

\[ \omega_x > 0 \]
\[ \omega_x < 0 \]

can sustain!
Extension to turbulent band

\[(L_x, L_z) = (100, 100), Re_m = 460, \alpha = 0\]
Extension to turbulent band

$$(L_x, L_z) = (100, 100), \text{Re}_m = 460, \alpha = 0$$
Extension to turbulent band

\[(L_x, L_z) = (100, 100), Re_m = 460, \alpha = 0\]

Turbulent ‘tail’

\[c_x \approx 1.25\]
\[c_z \approx -0.16\]

- \(u_x - U_{LF} = +0.1\)
- \(Q = 0.1, \omega_x > 0\)
- \(Q = 0.1, \omega_x < 0\)

\[t = 250\]
Bifurcation diagram of periodic solution

\((L_x, L_z) = (100, 100), \alpha = 0.1\)
Snapshot

\((L_x, L_z) = (100, 100), Re_m = 460, \alpha = 0.1\)
Bifurcation diagram of periodic solution

\[(L_x, L_z) = (100, 100), \; Re_m = 460\]
Bifurcation diagram of periodic solution

\((L_x, L_z) = (100, 100), \ Re_m = 460\)
Relevance to full Navier-Stokes system

\[(L_x, L_z) = (100, 100), \, Re_m = 460\]
Relevance to full Navier-Stokes system

$(L_x, L_z) = (100, 100), \quad Re_m = 550$

\[\alpha = 0.4\]
\[\alpha = 0.1\]
\[\alpha = 0.03\]
\[\alpha = 0.01\]

\[10^{-2}\]
\[10^{-3}\]

Crisis?

$E_{3d}$ (Local maximum in time)

(Transient) turbulent bands

(Navier-Stokes)
Concluding remarks

- Turbulent bands of equilibrium length have been observed in large numerical domain.
- Turbulent bands can be sustained up to around $Re_m = 440$.
- Relative periodic orbits have been discovered in spatially-localized damping-forced Navier-Stokes system.
- Periodic solutions mathematically provide self-sustaining mechanism of downstream edge (physically, inclined and thus stretched wall-normal rolls).
- If damping force is reduced, upper-branch solution loses its stability and eventually chaotic solution appears to represent turbulent bands of longer array of complex vortices.
- Periodic solutions representing turbulent bands might be connected to full Navier-Stokes system (cf. Hof et al.’s invariant solutions).