Transitional shear flows: Computing exact coherent states in two dimensions

Cédric Beaume[1]

Edgar Knobloch[2] Greg Chini[3,4] Keith Julien[5]

[1]School of Mathematics, University of Leeds
[2]Department of Physics, University of California, Berkeley
[3]Program in Integrated Applied Mathematics, University of New Hampshire
[4]Department of Mechanical Engineering, University of New Hampshire
[5]Department of Applied Mathematics, University of Colorado, Boulder

January 10, 2017

Asymptotic reduction of nonlinear flows

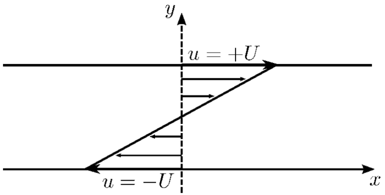
Strong restraint ⇒ reduce the flow in a particular direction, anisotropy

Small parameter ⇒ asymptotically consistent simplification of equations

- Boundary layers: P. Hall & W. D. Lakin, *Proc. R. Soc. London A* (1988)
- Langmuir circulation: G. P. Chini, K. Julien & E. Knobloch *Geophys. Astrophys. Fluid Dyn.* (2009)
- Rayleigh–Bénard convection: P. J. Blennerhassett & A. P. Bassom, *IMA J. Appl. Math.* (1994)
- Strongly constrained convection: K. Julien & E. Knobloch, *J. Math. Phys.* (2007)

Plane parallel shear flows

Plane Couette Flow



Wall BCs: $u = \pm 1$, $v = w = 0$
Forcing: $\mathbf{f}(y) = \mathbf{0}$

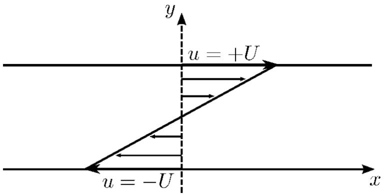
Navier–Stokes equation & incompressibility condition

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{v} + \mathbf{f}$$

$$\nabla \cdot \mathbf{v} = 0 \qquad Re = UH/\nu$$

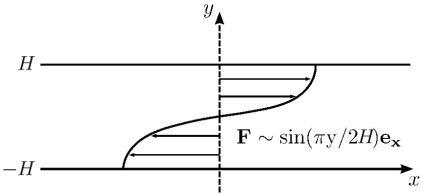
Plane parallel shear flows

Plane Couette Flow



Wall BCs: $u = \pm 1,\ v = w = 0$
Forcing: $\mathbf{f}(y) = \mathbf{0}$

Waleffe Flow



Wall BCs: $\partial_y u = 0,\ v = 0, \partial_y w = 0$
Forcing: $\mathbf{f}(y) = \frac{\sqrt{2}\pi^2}{4Re} \sin\left(\frac{\pi y}{2}\right)\hat{\mathbf{e}}_{\mathbf{x}}$

Waleffe, *Phys. Fluids* **9** 883–900 (1997)

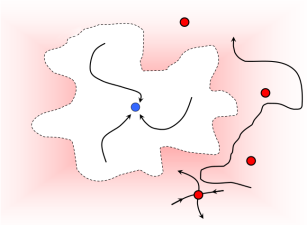
Navier–Stokes equation & incompressibility condition

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{v} + \mathbf{f}$$

$$\nabla\cdot\mathbf{v} = 0 \qquad Re = UH/\nu$$

Exact coherent structures (ECS)

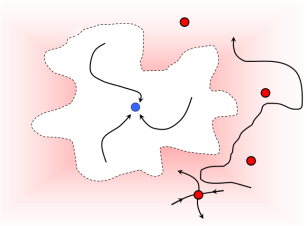
Turbulent and laminar states are both observable at $Re > Re_c$

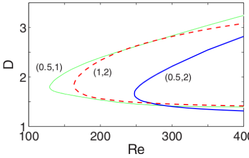


Exact coherent structures (ECS)

Turbulent and laminar states are both observable at $Re > Re_c$

- Marginal threshold: edge
  - Constrained dynamics $\Rightarrow$ edge states
  - Fixed points: lower branch states



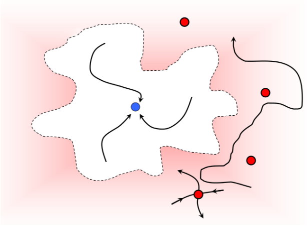


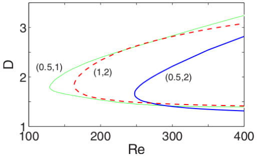
Schneider Gibson, Lagha, De Lillo & Eckhardt, Phys. Rev. E (2008)

Exact coherent structures (ECS)

Turbulent and laminar states are both observable at $Re > Re_c$

- Marginal threshold: edge
  - Constrained dynamics $\Rightarrow$ edge states
  - Fixed points: lower branch states
- Turbulence: pinball
  - Bounces from fixed point to fixed point
  - Typically upper branch states



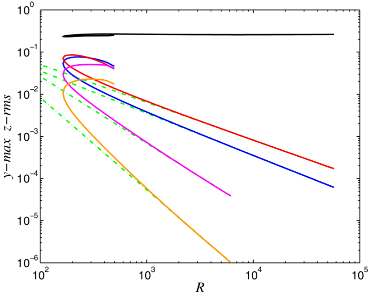


Schneider Gibson, Lagha, De Lillo & Eckhardt, Phys. Rev. E (2008)

Asymptotic scaling

Basic characteristic: **streamwise rolls** are *weak* compared to **streamwise streaks**

Observation: Lower branch states in plane Couette flow



Fourier decomposition for steady-state ECS:

$$\mathbf{u}(\mathbf{x}) = \sum_{n=-\infty}^{n=+\infty} \hat{\mathbf{u}}_n(y,z)e^{in\alpha x}$$

Scalings:

- $\hat{u}_0 = O(1)$
- $(\hat{v}_0, \hat{w}_0) = O(Re^{-1})$
- $\hat{\mathbf{u}}_1 = O(Re^{-0.9})$
- $\hat{\mathbf{u}}_n = o(Re^{-1})$ for $n > 1$

Wang, Gibson & Waleffe, *Phys. Rev. Lett.* **98** 204501 (2007)

Methodology

Follow Wang *et al.*, *Phys. Rev. Lett.* **98** 204501 (2007)

- $\epsilon \equiv 1/Re \ll 1$
- $T = \epsilon t \Rightarrow \partial_t \to \partial_t + \epsilon \partial_T$
- Decompose: $(\mathbf{v}, p) = (\bar{\mathbf{v}}, \bar{p})(y, z, T) + (\mathbf{v}', p')(x, y, z, t, T)$
  $\overline{(\cdot)}$ = average over (x,t), and $(\cdot)'$ = fluctuation about mean
- Define $\mathbf{v} = u\hat{\mathbf{e}}_\mathbf{x} + \mathbf{v}_\perp$ and expand

$$\begin{aligned} u &\sim \bar{u}_0 + \epsilon\left(\bar{u}_1 + u_1'\right) + \ldots \\ \mathbf{v}_\perp &\sim \epsilon\left(\bar{\mathbf{v}}_{1\perp} + \mathbf{v}_{1\perp}'\right) + \ldots \end{aligned}$$

$\mathbf{v_1}'(x, y, z, t, T) = \mathbf{v_1}'(y, z, t, T)e^{i\alpha x} + c.c.$
Streamfunction-vorticity: $\bar{v}_1 = -\partial_z \phi_1, \quad \bar{w}_1 = \partial_y \phi_1, \quad \omega_1 = \nabla_\perp^2 \phi_1$

Reduced model

Mean equations

$$\begin{aligned}
\partial_T u_0 + J(\phi_1, u_0) &= \nabla_\perp^2 u_0 + f(y) \\
\partial_T \omega_1 + J(\phi_1, \omega_1) &+ 2\overline{(\partial_y^2 - \partial_z^2)(\mathcal{R}(v_1 w_1^*))} \\
&+ 2\overline{\partial_y \partial_z (w_1 w_1^* - v_1 v_1^*)} = \nabla_\perp^2 \omega_1
\end{aligned}$$

$J(a,b) = \partial_y a \partial_z b - \partial_z a \partial_y b$, $\mathcal{R}$ real part, * complex conjugate

Reduced model

Mean equations

$$\begin{aligned}
\partial_T u_0 + J(\phi_1, u_0) &= \nabla_\perp^2 u_0 + f(y) \\
\partial_T \omega_1 + J(\phi_1, \omega_1) &+ 2\overline{(\partial_y^2 - \partial_z^2)(\mathcal{R}(v_1 w_1^*))} \\
&+ 2\overline{\partial_y \partial_z (w_1 w_1^* - v_1 v_1^*)} = \nabla_\perp^2 \omega_1
\end{aligned}$$

$J(a,b) = \partial_y a \partial_z b - \partial_z a \partial_y b$, $\mathcal{R}$ real part, * complex conjugate

Fluctuation equations

$$\begin{aligned}
(\alpha^2 - \nabla_\perp^2) p_1 &= 2i\alpha(v_1 \partial_y u_0 + w_1 \partial_z u_0) \\
\partial_t \mathbf{v}_{1\perp} + u_0 i\alpha \mathbf{v}_{1\perp} &= -\nabla_\perp p_1
\end{aligned}$$

Why do we like it?

Reduced model

$$\begin{aligned}
\partial_T u_0 + J(\phi_1, u_0) &= \nabla_\perp^2 u_0 + f(y) \\
\partial_T \omega_1 + J(\phi_1, \omega_1) &= \nabla_\perp^2 \omega_1 - 2\overline{(\partial_y^2 - \partial_z^2)(\mathcal{R}(v_1 w_1^*))} - 2\overline{\partial_y \partial_z (w_1 w_1^* - v_1 v_1^*)} \\
(\alpha^2 - \nabla_\perp^2) p_1 &= 2i\alpha(v_1 \partial_y u_0 + w_1 \partial_z u_0) \\
\partial_t \mathbf{v}_{1\perp} + u_0 i\alpha \mathbf{v}_{1\perp} &= -\nabla_\perp p_1 + \epsilon \nabla_\perp^2 \mathbf{v}_{1\perp}
\end{aligned}$$

- 2D system (y, z) but 3 components (streamwise, wall-normal, spanwise)
- Mean system has unit effective Re
- Fluctuation equations are: (i) **inviscid**; (ii) **quasi-linear** and (iii) **singular** for equilibrium ECS on **critical layer** $u_0(y, z) = 0$

$$(\alpha^2 - \nabla_\perp^2) p_1 + \frac{2}{u_0}\left(\nabla_\perp u_0 \cdot \nabla_\perp p_1 - \epsilon \nabla_\perp u_0 \cdot \nabla_\perp^2 \mathbf{v}_{1\perp}\right) = 0$$

Generalized Rayleigh equation

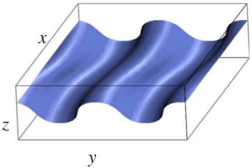
- **Critical regions!**

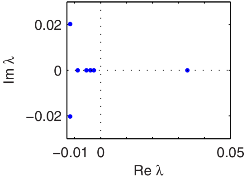
Problem statement

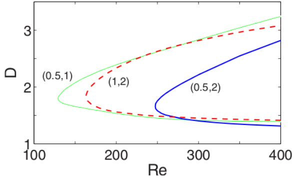
Calculating ECS is not easy!

They are:

- Fully nonlinear
- Unstable
- Not connected to the laminar state







Schneider *et al.*, *Phys. Rev. E* **78**, 037301 (2008)

Physical insight

Slow mean variables:

$$\begin{aligned}\partial_T u_0 + J(\phi_1, u_0) &= \nabla_\perp^2 u_0 + f(y) \\ \partial_T \omega_1 + J(\phi_1, \omega_1) &= \nabla_\perp^2 \omega_1 \\ &\quad -2\overline{(\partial_y^2 - \partial_z^2)(\mathcal{R}(v_1 w_1^*))} - 2\overline{\partial_y \partial_z (w_1 w_1^* - v_1 v_1^*)}\end{aligned}$$

Fast fluctuating variables:

$$\begin{aligned}(\alpha^2 - \nabla_\perp^2) p_1 &= 2i\alpha(v_1 \partial_y u_0 + w_1 \partial_z u_0) \\ \partial_t \mathbf{v}_{1\perp} + u_0 i\alpha \mathbf{v}_{1\perp} &= -\nabla_\perp p_1 + \epsilon \nabla_\perp^2 \mathbf{v}_{1\perp}\end{aligned}$$

Assume (u_0, ω_1) steady when solving for $(p_1, \mathbf{v}_{1\perp})$

Fluctuation system is linear $\Rightarrow$ eigenvalue problem

Iterative algorithm

Reduced model

$$\begin{aligned}
\partial_T u_0 + J(\phi_1, u_0) &= \nabla_\perp^2 u_0 + f(y) \\
\partial_T \omega_1 + J(\phi_1, \omega_1) &= \nabla_\perp^2 \omega_1 - 2(\partial_y^2 - \partial_z^2)(\mathcal{R}(v_1 w_1^*)) - 2\partial_y \partial_z (w_1 w_1^* - v_1 v_1^*) \\
(\alpha^2 - \nabla_\perp^2) p_1 &= 2i\alpha(v_1 \partial_y u_0 + w_1 \partial_z u_0) \\
\partial_t \mathbf{v}_{1\perp} + u_0 i\alpha \mathbf{v}_{1\perp} &= -\nabla_\perp p_1 + \epsilon \nabla_\perp^2 \mathbf{v}_{1\perp}
\end{aligned}$$

Step 1: choose a fluctuation amplitude A and a profile u_0

Step 2: compute the fastest non-oscillatory growing $\mathbf{v}_{1\perp}$ mode

Step 3: use A and the result of Step 2 to compute the Reynolds stresses

Step 4: time-advance u_0 and ω_1 to a steady state

Then: repeat Steps 2–4 until convergence

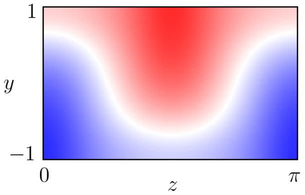
Repeat to find A_{opt} such that the converged solution has marginal fluctuations.

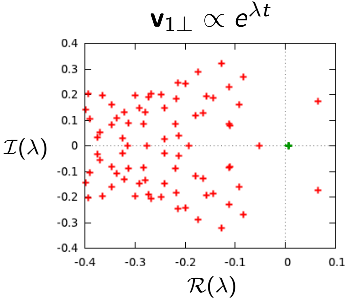
Hall & Sherwin, *J. Fluid Mech.* **661**, 178–205 (2010)
Beaume, *Proc. Geophys. Fluid Dyn. Program*, 389–412 (2012)
Mantič-Lugo, Arratia & Gallaire, *Phys. Fluids* **27**, 074103 (2015)

Initial iterate in Waleffe flow

$L_z = \pi$, $\alpha = 0.5$, $Re = 400$

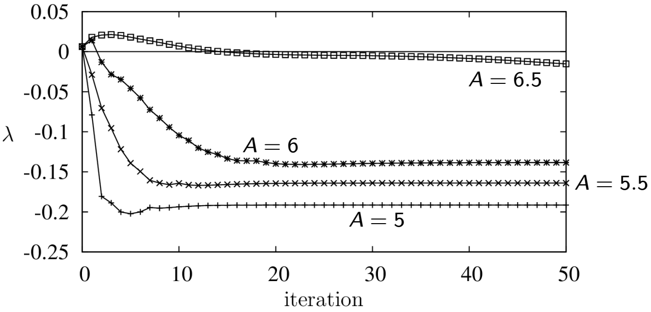
Fake streaks: set $\omega_1(y,z) = 20\sin(\pi y/2)\sin(2z)$ and converge the equation on u_0:





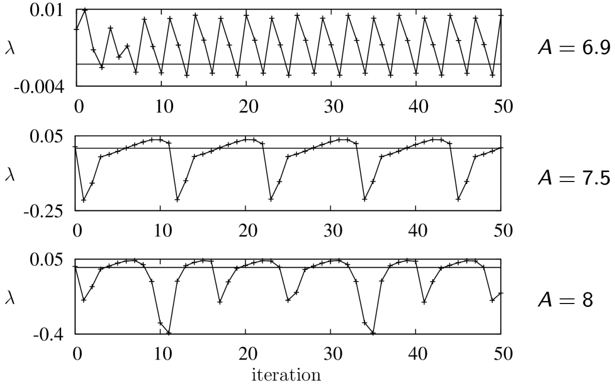
Toward a solution of Waleffe flow

$L_z = \pi$, $\alpha = 0.5$, $Re = 400$



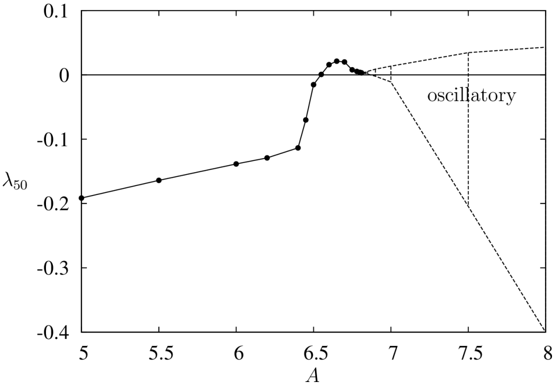
Toward a solution of Waleffe flow

$L_z = \pi$, $\alpha = 0.5$, $Re = 400$



Toward a solution of Waleffe flow

$L_z = \pi$, $\alpha = 0.5$, $Re = 400$



Candidates at $A \approx 6.5$ and $A \approx 6.8$

Need to converge them!

Stokes preconditioning

$$\gamma_t \partial_t U = N(U) + \gamma_D L U \quad (= 0)$$

Tuckerman's Stokes preconditioner (1989)

Semi-implicit Euler scheme:

$$U(t + \triangle t) = \left(I - \frac{\triangle t \gamma_D}{\gamma_t} L \right)^{-1} \left(\frac{\triangle t}{\gamma_t} N[U(t)] + U(t) \right)$$

Substract $U(t)$:

$$U(t + \triangle t) - U(t) = \frac{\triangle t}{\gamma_t} \left(I - \frac{\triangle t \gamma_D}{\gamma_t} L \right)^{-1} (N[U(t)] + \gamma_D L U(t))$$

Usually, take $\triangle t \gg 1$:

$$U(t + \triangle t) - U(t) \approx -(\gamma_D L)^{-1} (N[U(t)] + \gamma_D L U(t))$$

⇒ Asymptotic Laplacian preconditioner

Adaptive Stokes preconditioning

Remember

In the general case (forget $\triangle t \gg 1$):

$$U(t+\triangle t) - U(t) = \frac{\triangle t}{\gamma_t} P^{-1} \left(N[U(t)] + \gamma_D L U(t) \right)$$

Stokes preconditioner: $P = I - \frac{\triangle t \gamma_D}{\gamma_t} L$

For steady flows, we can use different preconditioners for the mean and fluctuation equations while solving simultaneously.

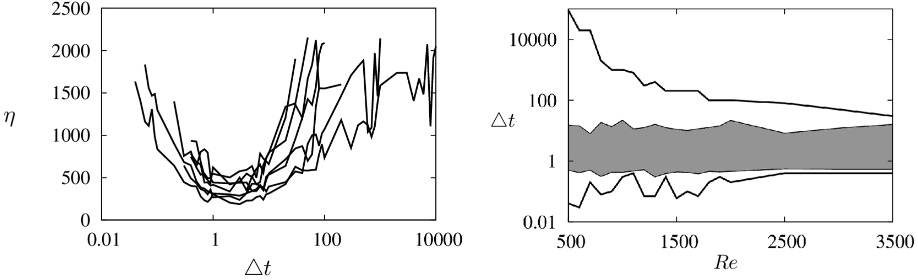
To precondition the slow, mean equations ($\gamma_t = \epsilon^{-1}$, $\gamma_D = 1$):

$$\triangle t = \epsilon^{-1} = Re \Rightarrow P = I - L$$

Beaume, Adaptive Stokes preconditioning for steady incompressible flows, *to appear in Commun. Comput. Phys.* (2017)

Adaptive Stokes preconditioning

For the fast, fluctuation equations ($\gamma_t = 1$, $\gamma_D = \epsilon$):



$$\Rightarrow \quad \triangle t \approx 1$$

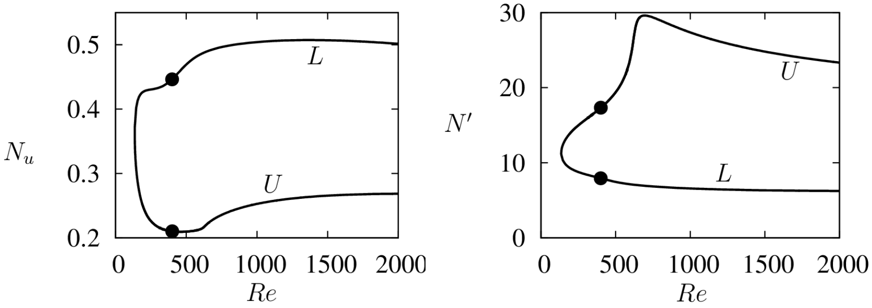
$$\Rightarrow \quad P = I - \epsilon L = I - Re^{-1} L$$

Remember: slow, mean equations preconditioned by $P = I - L$

Beaume, Adaptive Stokes preconditioning for steady incompressible flows, *to appear in Commun. Comput. Phys.* (2017)

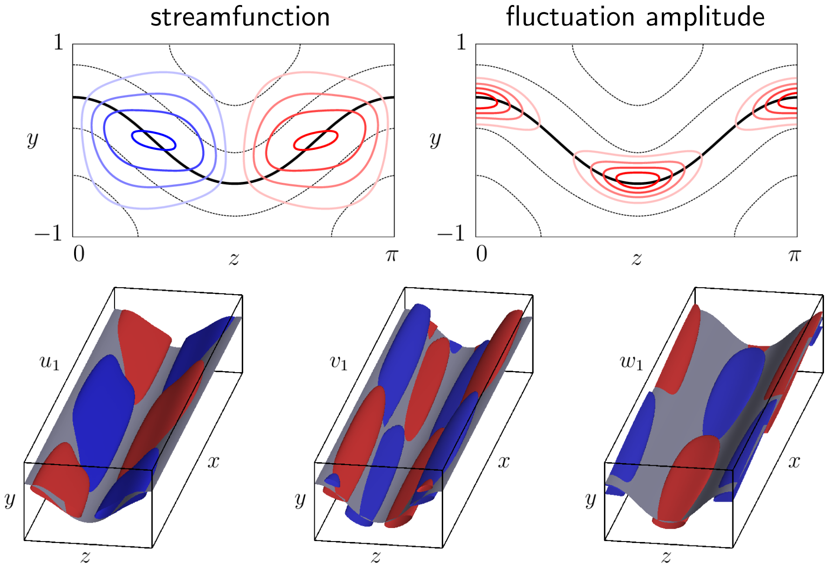
Results for Waleffe flow: $\alpha = 0.5$, $L_z = \pi$

$N_u \equiv \mathcal{D}^{-1} \int_{\mathcal{D}} u_0^2 \, dy \, dz$ $\qquad$ $N' \equiv \mathcal{D}^{-1} \int_{\mathcal{D}} (v_1^2 + w_1^2) \, dy \, dz$

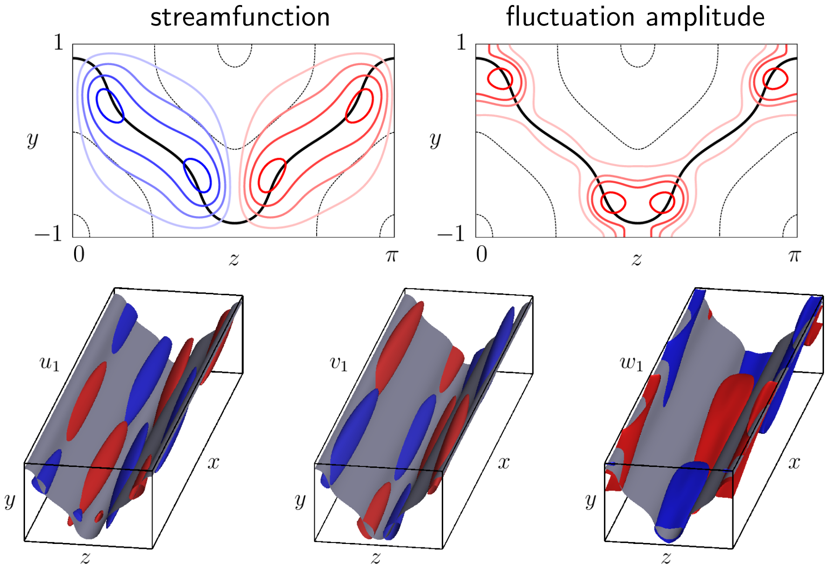


Note that trivial solution has $N_u = 1$ and $N' = 0$.

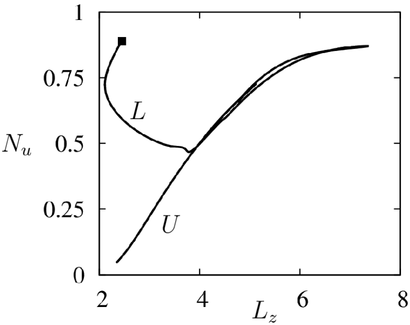
Lower branch states: $Re = 1500$, $\alpha = 0.5$, $L_z = \pi$

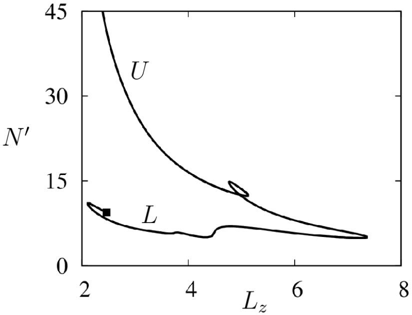


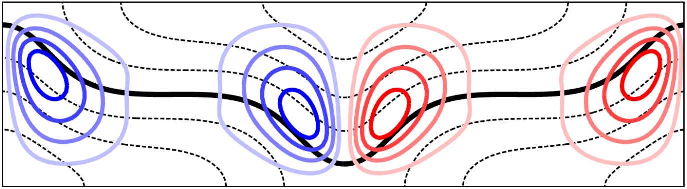
Upper branch states: $Re = 1500$, $\alpha = 0.5$, $L_z = \pi$



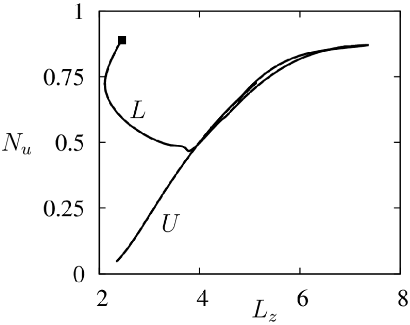
Dependence on L_z: $Re = 1500$, $\alpha = 0.5$

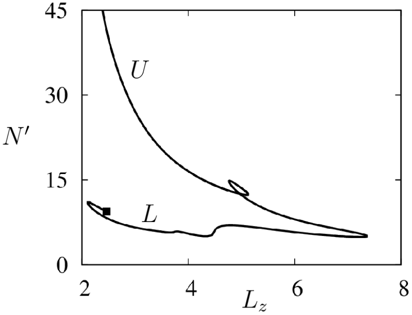


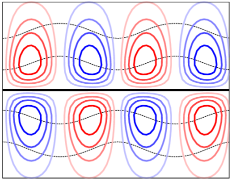


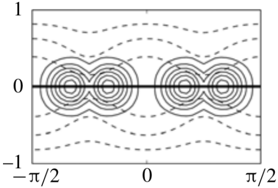


Dependence on L_z: $Re = 1500$, $\alpha = 0.5$





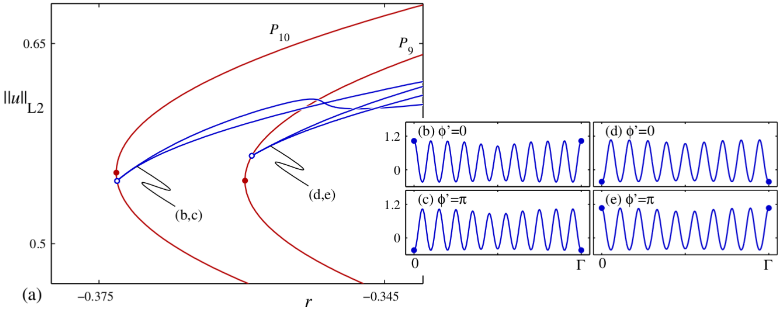




Gibson & Brand, *J. Fluid Mech.* **745**, 25–61 (2014)

Modulated patterns: The postulate

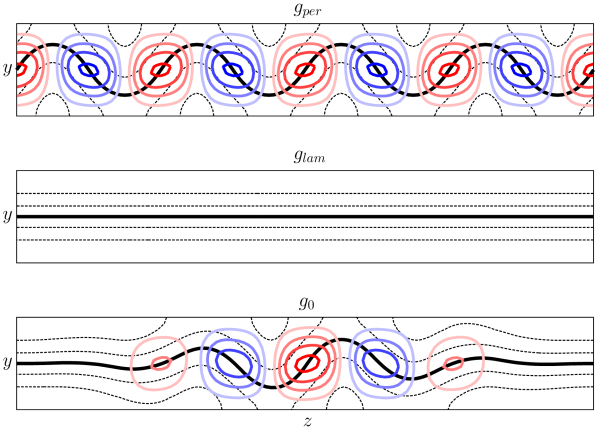
Saddle-nodes of subcritical branches in large domains yield modulational instabilities



Bergeon, Burke, Knobloch & Mercader, *Phys. Rev. E* (2008)

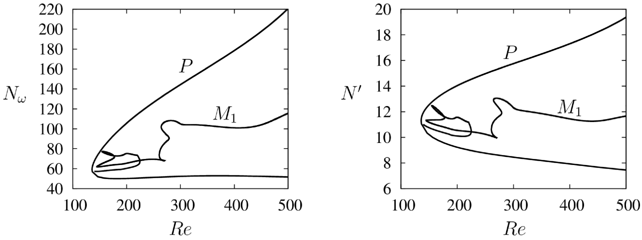
Modulated patterns: Artificial modulation

Extend solutions to a $L_z = 4\pi$ domain

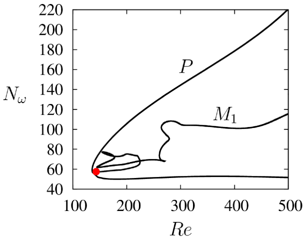


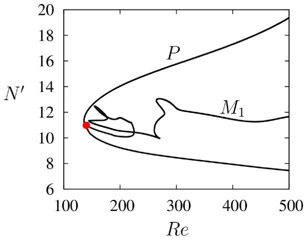
$$g_0 = \left[1 - \frac{\chi}{2}\left(1 + \cos\left(\frac{z}{2}\right)\right)\right] g_{per} + \left[\frac{\chi}{2}\left(1 + \cos\left(\frac{z}{2}\right)\right)\right] g_{lam}$$

Modulated patterns: M_1 states, $L_z = 4\pi$

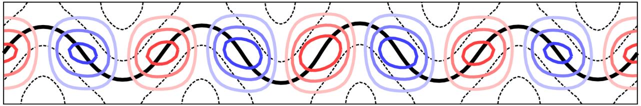


Modulated patterns: M_1 states, $L_z = 4\pi$

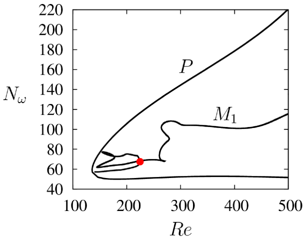


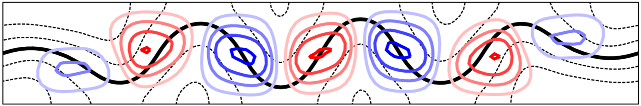


$Re \approx 140$

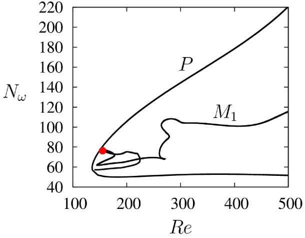


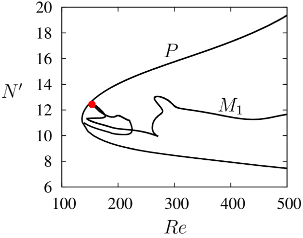
Modulated patterns: M_1 states, $L_z = 4\pi$



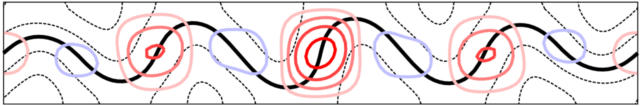


Modulated patterns: M_1 states, $L_z = 4\pi$

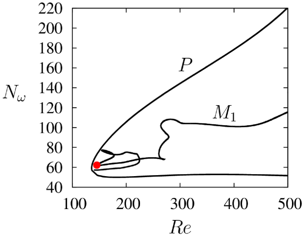


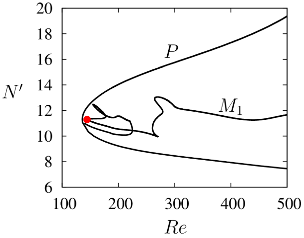


$Re \approx 155$

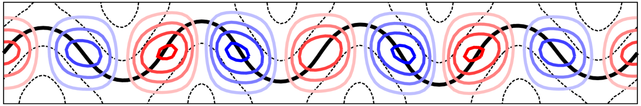


Modulated patterns: M_1 states, $L_z = 4\pi$

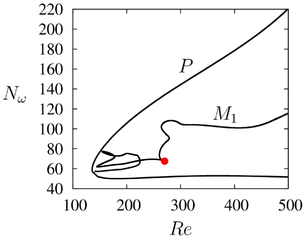


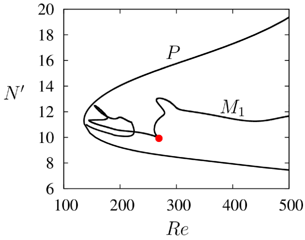


$Re \approx 145$

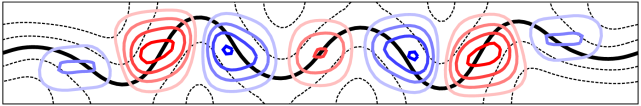


Modulated patterns: M_1 states, $L_z = 4\pi$

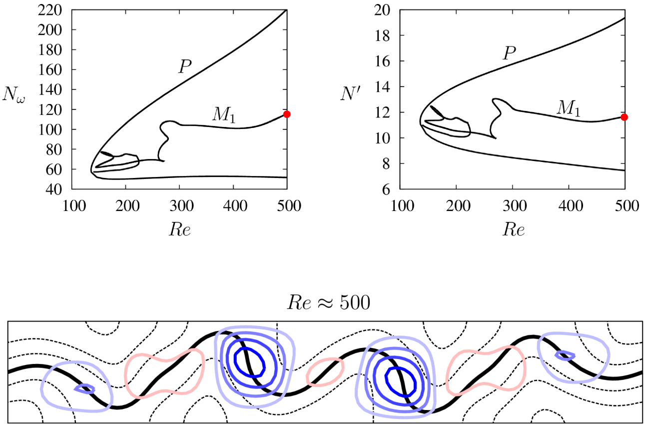




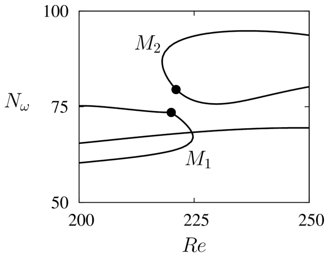
$Re \approx 271$



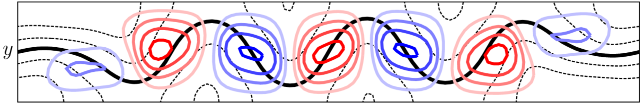
Modulated patterns: M_1 states, $L_z = 4\pi$



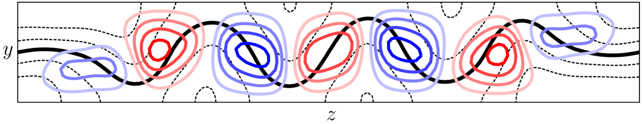
Modulated patterns: Imperfect bifurcations



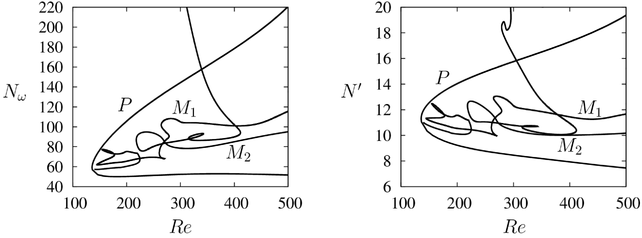
M_1 at $Re \approx 220.0320$



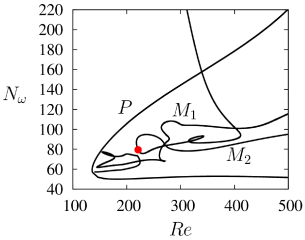
M_2 at $Re \approx 221.0741$

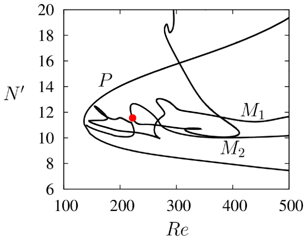


Modulated patterns: M_2 states, $L_z = 4\pi$

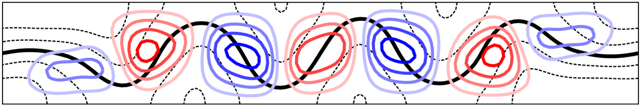


Modulated patterns: M_2 states, $L_z = 4\pi$

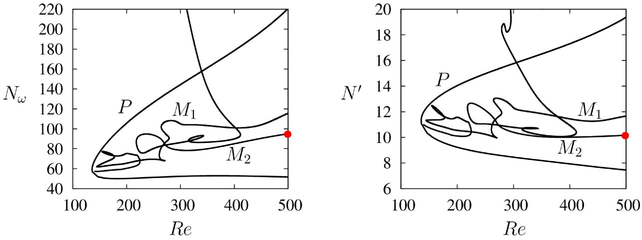




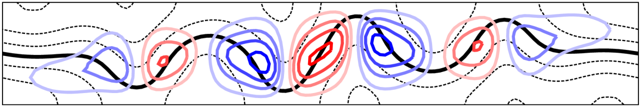
$Re \approx 221$



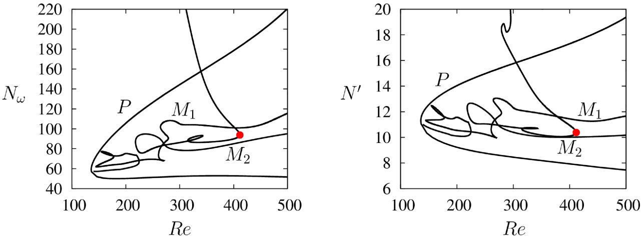
Modulated patterns: M_2 states, $L_z = 4\pi$



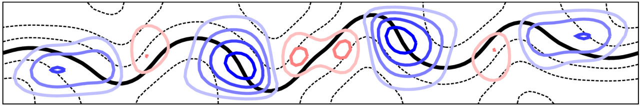
$Re \approx 500$



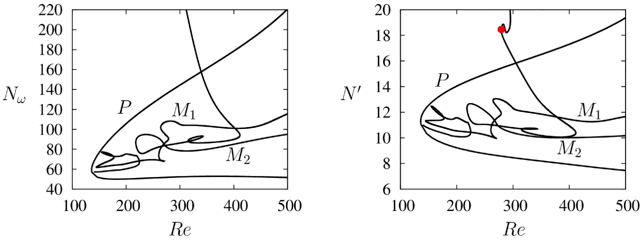
Modulated patterns: M_2 states, $L_z = 4\pi$



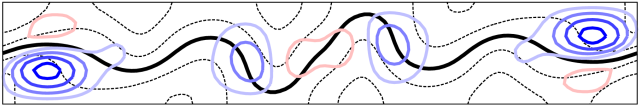
$Re \approx 411$



Modulated patterns: M_2 states, $L_z = 4\pi$



$Re \approx 278$



Conclusions

✓ Closed reduced description of ECS in parallel shear flows

✓ Efficient numerical technique

✓ Lower, upper and modulated state branches obtained

⇒ Localized pattern formation? (see Gibson, Kerswell, Schneider...)

⇒ What level of accuracy do we achieve?

⇒ Can we model temporal dynamics? (see Farrell, Gayme, Ioannou, Thomas, Marston, Tobias...)

References:

C. Beaume, *Proc. Geophys. Fluid Dyn. Program*, 389–412 (2012)

C. Beaume, E. Knobloch, G. P. Chini & K. Julien, *Fluid Dyn. Res.* **47**, 015504 (2015)

C. Beaume, G. P. Chini, K. Julien & E. Knobloch, *Phys. Rev. E* **91**, 043010 (2015)

C. Beaume, E. Knobloch, G. P. Chini & K. Julien, *Phys. Scr.* **91**, 024003 (2016)

C. Beaume, *to appear in Commun. Comput. Phys.* (2017)