Color of Turbulence:

stochastic dynamical modeling of turbulent flows

Mihailo Jovanović

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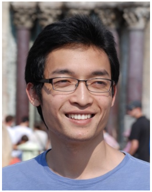
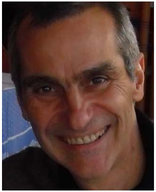
joint work with

Armin Zare



Yongxin Chen

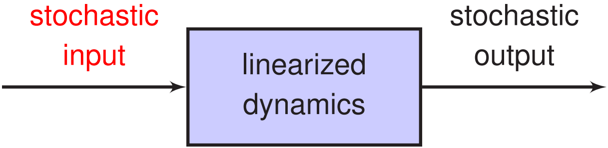


Tryphon Georgiou

Recurrence, self-organization, and the dynamics of turbulence

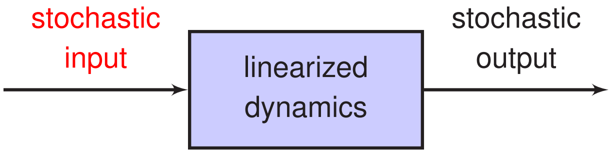
Turbulence modeling

$$\begin{aligned}\dot{x} &= A\,x + B\,d \\ y &= C\,x\end{aligned}$$



Turbulence modeling

$$
\begin{aligned}
\dot{x} &= A\,x + B\,d \\
y &= C\,x
\end{aligned}
$$



- OBJECTIVE
  - ⋆ combine physics-based with data-driven modeling
  - ⋆ account for statistical signatures of turbulent flows using stochastically-forced linearized models

- **Proposed approach**
  - ⋆ view second-order statistics as data for an inverse problem

- **Key questions**
  - ⋆ Can we identify forcing statistics to reproduce available statistics?
  - ⋆ Can this be done by white in-time stochastic process?

- PROPOSED APPROACH
  - ⋆ view second-order statistics as data for an inverse problem

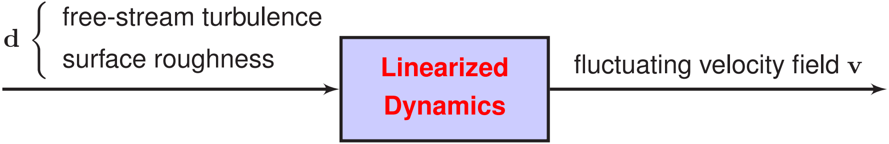
- KEY QUESTIONS
  - ⋆ Can we identify forcing statistics to reproduce available statistics?
  - ⋆ Can this be done by white in-time stochastic process?

OUR CONTRIBUTION

principled way of turbulence modeling as an inverse problem

Input-output analysis

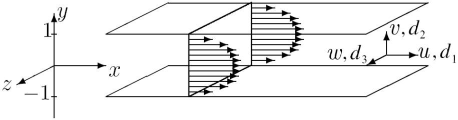
- TOOL FOR QUANTIFYING SENSITIVITY
  - ⋆ **spatio-temporal frequency responses**



Input-output analysis

- TOOL FOR QUANTIFYING SENSITIVITY
  - ⋆ **spatio-temporal frequency responses**

$\mathbf{d}$ $\begin{cases} \text{free-stream turbulence} \\ \text{surface roughness} \end{cases}$ → **Linearized Dynamics** → fluctuating velocity field $\mathbf{v}$



$$\underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\mathbf{d}} \xrightarrow{\text{amplification}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\mathbf{v}}$$

IMPLICATIONS FOR

physics:	**insight into mechanisms**
control:	**control-oriented modeling**

- STOCHASTIC FORCING

$$\left.\begin{array}{rl} \textbf{white} & \text{in } t \text{ and } y \\ \textbf{harmonic} & \text{in } x \text{ and } z \end{array}\right\} \quad \mathbf{d}(x,y,z,t) \; = \; \hat{\mathbf{d}}(y,t)\, \mathrm{e}^{\mathrm{i}k_x x}\, \mathrm{e}^{\mathrm{i}k_z z}$$

Farrell & Ioannou, Phys. Fluids A '93

Bamieh & Dahleh, Phys. Fluids '01

Jovanović & Bamieh, J. Fluid Mech. '05

- **Stochastic forcing**

$$\left.\begin{array}{rl} \textbf{white} & \text{in } t \text{ and } y \\ \textbf{harmonic} & \text{in } x \text{ and } z \end{array}\right\} \quad \mathbf{d}(x,y,z,t) \;=\; \hat{\mathbf{d}}(y,t)\, \mathrm{e}^{\mathrm{i}k_x x}\, \mathrm{e}^{\mathrm{i}k_z z}$$

Farrell & Ioannou, Phys. Fluids A '93

Bamieh & Dahleh, Phys. Fluids '01

Jovanović & Bamieh, J. Fluid Mech. '05

- **Deterministic forcing**

$$\left.\begin{array}{rl} \textbf{deterministic} & \text{in } y \\ \textbf{harmonic} & \text{in } x,\, z,\, t \end{array}\right\} \quad \mathbf{d}(x,y,z,t) \;=\; \hat{\mathbf{d}}(y)\, \mathrm{e}^{\mathrm{i}k_x x}\, \mathrm{e}^{\mathrm{i}k_z z}\, \mathrm{e}^{\mathrm{i}\omega t}$$

Trefethen et al., Science '93

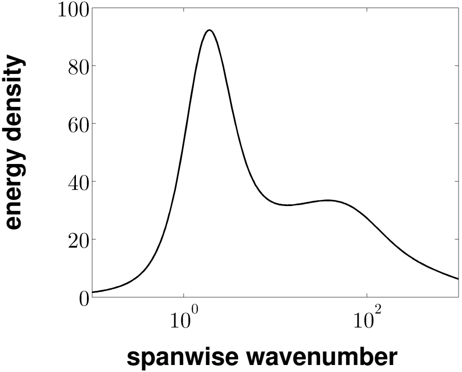
Jovanović, PhD Thesis '04

McKeon & Sharma, J. Fluid Mech. '10

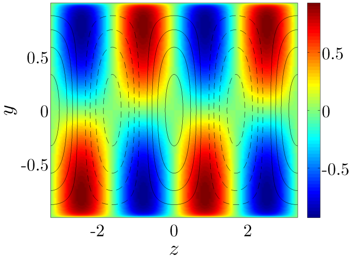
Input-output analysis of turbulent flows

- STREAMWISE CONSTANT FLUCTUATIONS

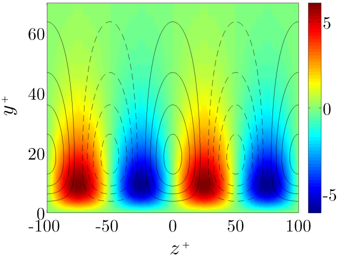
channel flow with $R_\tau = 547$



channel-wide streaks



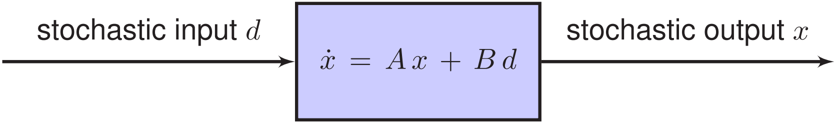
near-wall streaks



del Álamo & Jiménez, J. Fluid Mech. '06

Hwang & Cossu, *McKeon & coworkers*

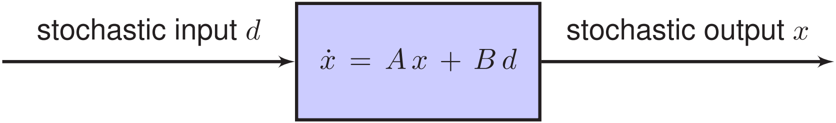
Response to stochastic inputs



- LYAPUNOV EQUATION
  - ⋆ propagates white correlation of d into colored statistics of x

$$A X + X A^* = - B W B^*$$

Response to stochastic inputs



- LYAPUNOV EQUATION
  - ⋆ propagates white correlation of d into colored statistics of x

$$A\,X + X\,A^* = -\,B\,W\,B^*$$

  - ⋆ colored-in-time d

$$A\,X + X\,A^* = -\,\overbrace{(B\,H^* + H\,B^*)}^{Z}$$

$$H := \lim_{t \to \infty} \mathbf{E}\,(x(t)\,d^*(t)) + \frac{1}{2}\,B\,W$$

Georgiou, IEEE TAC '02

- THEOREM

$X = X^* \succeq 0$ is the steady-state covariance of (A, B)

$$\Updownarrow$$

there is a solution H to

$$B\,H^* + H\,B^* = -(A\,X + X\,A^*)$$

$$\Updownarrow$$

$$\operatorname{rank}\begin{bmatrix} AX + XA^* & B \\ B^* & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

Georgiou, IEEE TAC '02

Lyapunov equation

discrete-time dynamics: $x_{t+1} \;=\; A\,x_t \;+\; B\,d_t$

white-in-time input: $\mathbf{E}\,(d_t\,d_\tau^*) \;=\; W\,\delta_{t\,-\,\tau}$

- LYAPUNOV EQUATION

$$
\begin{aligned}
{\color{red}X_{t\,+\,1}} \;&:=\; \mathbf{E}\left(x_{t\,+\,1}\,x_{t\,+\,1}^*\right) \\
&=\; \mathbf{E}\Big(\,(A\,x_t \;+\; B\,d_t)\,(x_t^*\,A^* \;+\; d_t^*\,B^*)\,\Big) \\
&=\; A\,\mathbf{E}\,(x_t\,x_t^*)\,A^* \;+\; B\,\mathbf{E}\,(d_t\,d_t^*)\,B^* \\
&=\; A\,{\color{red}X_t}\,A^* \;+\; B\,{\color{blue}W}\,B^*
\end{aligned}
$$

⋆ continuous-time version

$$
\frac{\mathrm{d}\,{\color{red}X_t}}{\mathrm{d}\,t} \;=\; A\,{\color{red}X_t} \;+\; {\color{red}X_t}\,A^* \;+\; B\,{\color{blue}W}\,B^*
$$

Outline

- STRUCTURED COVARIANCE COMPLETION PROBLEM
  - ⋆ embed available statistical features in turbulence models
  - ⋆ complete unavailable data (via convex optimization)
- TURBULENCE MODELING
  - ⋆ case study: turbulent channel flow
  - ⋆ verification in linear stochastic simulations
- ALGORITHM
  - ⋆ Alternating Minimization Algorithm (AMA)
  - ⋆ works as proximal gradient on the dual problem
- SUMMARY AND OUTLOOK

Details

- THEORY AND ALGORITHMS

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 00, NO. 00, 2016 1

Low-Complexity Modeling of Partially Available Second-Order Statistics: Theory and an Efficient Matrix Completion Algorithm

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- TURBULENCE MODELING

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Colour of turbulence

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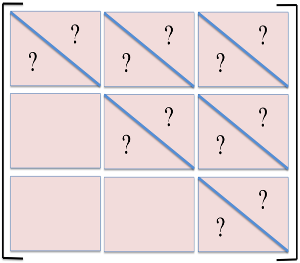
STRUCTURED COVARIANCE COMPLETION

Problem setup

known entries of X



$$A\,X + X\,A^* \;=\; -\underbrace{(B\,H^* + H\,B^*)}_{Z}$$

- PROBLEM DATA
  - ⋆ system matrix A; output matrix C
  - ⋆ partially available entries of X
- UNKNOWNS
  - ⋆ missing entries of X
  - ⋆ disturbance dynamics Z $\left\{\begin{array}{l}\text{input matrix } B \\ \text{input power spectrum } H\end{array}\right.$

Inverse problem

- CONVEX OPTIMIZATION PROBLEM

$$
\begin{array}{lll}
\underset{X,\,Z}{\text{minimize}} & -\log\det(X) \;+\; \gamma\,\|Z\|_* & \\
\text{subject to} & A\,X \;+\; X\,A^* \;+\; Z \;=\; 0 & \textbf{physics} \\
 & (C\,X\,C^*)_{ij} \;=\; G_{ij} \quad \text{for given} \;\; i,j & \textbf{available data}
\end{array}
$$

Inverse problem

- CONVEX OPTIMIZATION PROBLEM

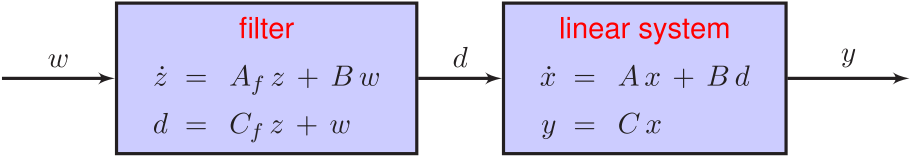
$$\begin{array}{lll} \underset{X,Z}{\text{minimize}} & -\log\det(X) \, + \, \gamma\,\|Z\|_* & \\ \text{subject to} & A\,X \, + \, X\,A^* \, + \, Z \, = \, 0 & \textbf{physics} \\ & (C\,X\,C^*)_{ij} \, = \, G_{ij} \quad \text{for given } \; i,j & \textbf{available data} \end{array}$$

⋆ nuclear norm: proxy for rank minimization

$$\|Z\|_* \; := \; \sum \sigma_i(Z)$$

Fazel, Boyd, Hindi, Recht, Parrilo, Candès, Chandrasekaran, . . .

Filter design



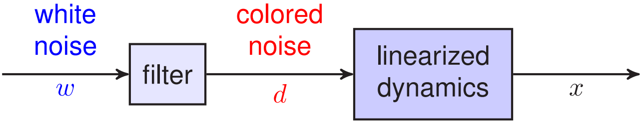
⋆ **white-in-time input**

$$\mathbf{E}\left(w(t_1)\, w^*(t_2)\right) \;=\; \Omega\, \delta(t_1 \,-\, t_2)$$

⋆ **filter dynamics**

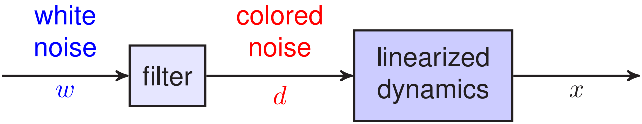
$$\begin{aligned} A_f &= A \,+\, B\, C_f \\ C_f &= \left(H^* \,-\, \frac{1}{2}\, \Omega\, B^* \right) X^{-1} \end{aligned}$$

Low-rank modification

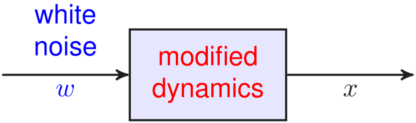


colored input: $\dot{x} \;=\; A\,x \;+\; B\,d$

Low-rank modification



colored input: $\dot{x} \;=\; A\,x \;+\; B\,d$



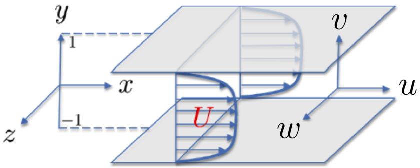
low-rank modification: $\dot{x} \;=\; (A \;+\; B\,C_f)\,x \;+\; B\,w$

TURBULENCE MODELING

Turbulent channel flow

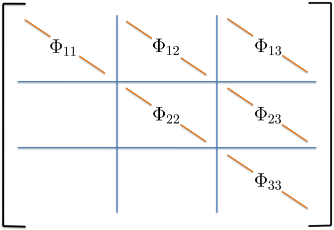
output covariance:

$$
\begin{aligned}
\Phi(\mathbf{k}) \quad &:= \quad \lim_{t \to \infty} \mathbf{E}\left(\mathbf{v}(t, \mathbf{k})\, \mathbf{v}^*(t, \mathbf{k})\right) \\
\mathbf{v} \quad &= \quad [\, u \;\; v \;\; w \,]^T \\
\mathbf{k} \quad &- \quad \text{horizontal wavenumbers}
\end{aligned}
$$



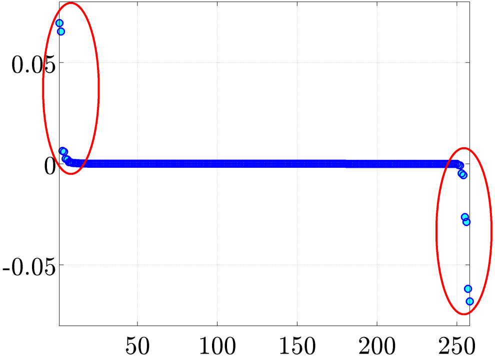
$$
A \; = \; \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}
$$

known elements of $\Phi(\mathbf{k})$



- KEY OBSERVATION
  - ⋆ white-in-time forcing: **too restrictive**

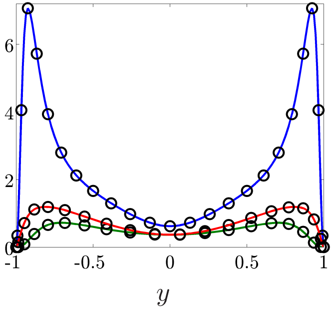
$$\lambda_i \left(A X_{\mathrm{dns}} \; + \; X_{\mathrm{dns}} A^* \right)$$



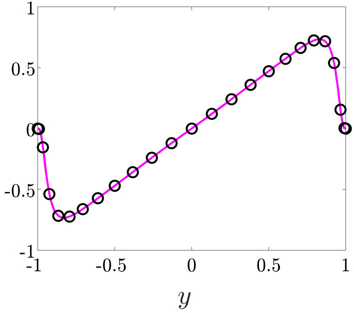
Jovanović & Georgiou, APS DFD '10

One-point correlations

normal stresses

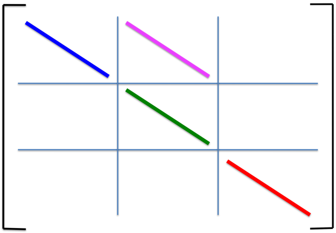


shear stress



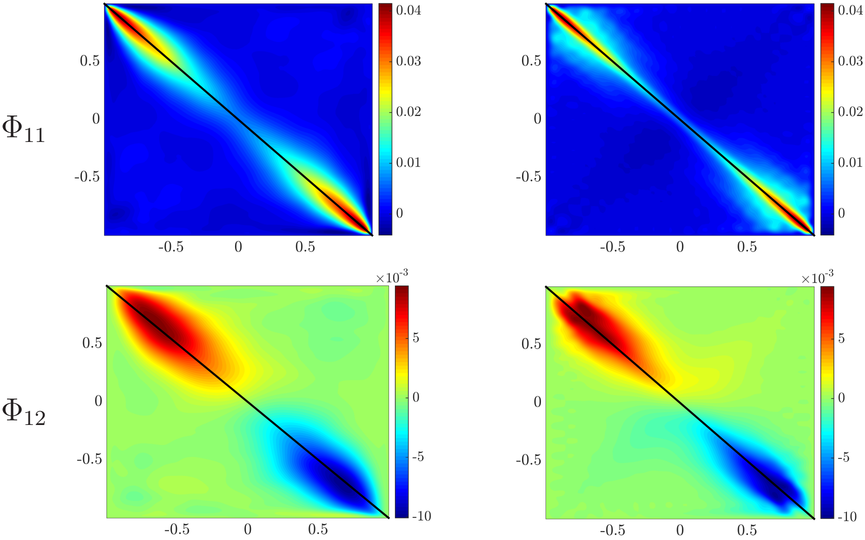
Nonlinear simulations —

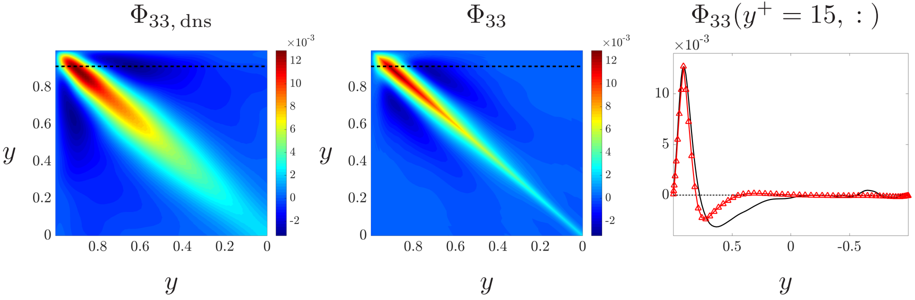
Solution to inverse problem ○



Two-point correlations

nonlinear simulations **covariance completion**




$\Phi_{33,\,\mathrm{dns}}$
Φ_{33}
$\Phi_{33}(y^+ = 15,\ :\ )$
$\times 10^{-3}$
12
10
8
6
4
2
0
-2
0.8
0.6
0.4
0.2
0
0.5
-0.5
5
y

Importance of physics

- COVARIANCE COMPLETION PROBLEM

$$
\begin{array}{ll}
\underset{X,\,Z}{\text{minimize}} & -\log\det(X) \;+\; \gamma\,\|Z\|_* \\
\text{subject to} & A\,X \;+\; X\,A^* \;+\; Z \;=\; 0 \\
 & (C\,X\,C^*)_{ij} \;=\; G_{ij} \quad \text{for given } \; i,j
\end{array}
$$

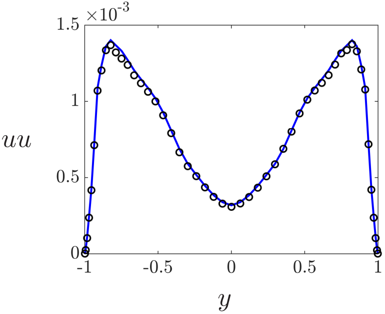
physics

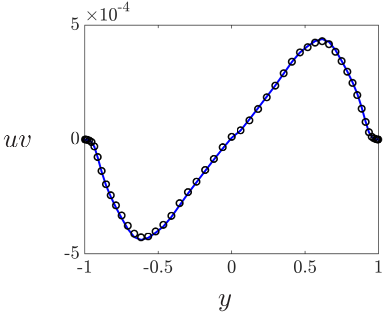
available data

physics helps!

Verification in stochastic simulations

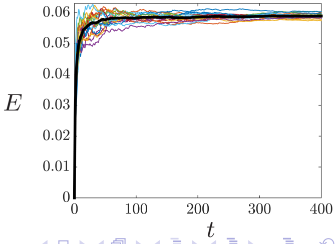
- $R_\tau = 180$; $k_x = 2.5$, $k_z = 7$





Direct Numerical Simulations —

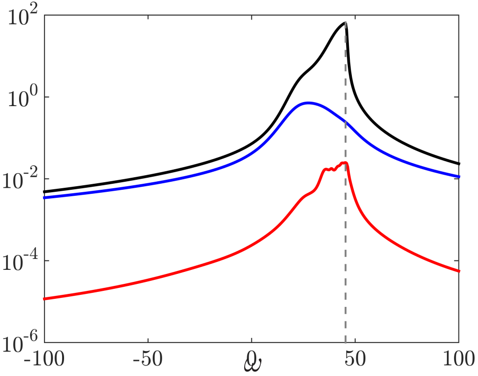
Linear Stochastic Simulations ○



Power spectral density

- $R_\tau = 186;\, k_x = 2.5,\, k_z = 7$

$$\mathrm{trace}\left(\mathcal{T}(\omega)\,\mathcal{T}^*(\omega)\right)$$



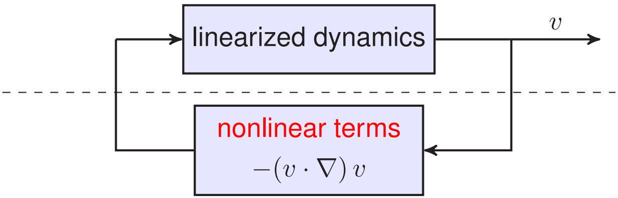
original linearized NS model

eddy-viscosity enhanced model

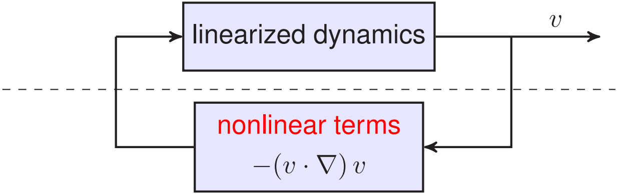
dynamics w/ low-rank modification

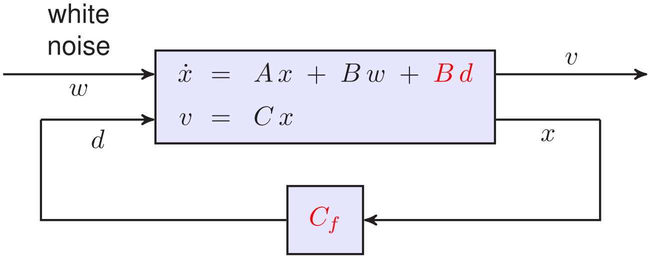
$$\mathcal{T}(\omega) \;=\; -C\,(\mathrm{i}\omega I \,+\, A)^{-1} B$$

Modeling nonlinear terms



Modeling nonlinear terms





equivalence at the level of 2nd order statistics

ALGORITHM

Primal and dual problems

- PRIMAL

$$\begin{array}{ll} \underset{X,\,Z}{\text{minimize}} & -\log\det(X) \;+\; \gamma\,\|Z\|_* \\ \text{subject to} & \mathcal{A}X \;+\; \mathcal{B}Z \;-\; \mathcal{C} \;=\; 0 \end{array}$$

Primal and dual problems

- Primal

$$\begin{array}{ll} \underset{X,\,Z}{\text{minimize}} & -\log\det(X) + \gamma\,\|Z\|_* \\ \text{subject to} & \mathcal{A}X + \mathcal{B}Z - \mathcal{C} = 0 \end{array}$$

- Dual

$$\begin{array}{ll} \underset{Y_1,\,Y_2}{\text{maximize}} & \log\det(\mathcal{A}^\dagger Y) - \langle G, Y_2 \rangle \\ \text{subject to} & \|Y_1\|_2 \leq \gamma \end{array}$$

$$\mathcal{A}^\dagger \;-\; \text{adjoint of } \mathcal{A}; \qquad Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

SDP characterization

$$Z = Z_+ - Z_-, \quad Z_+ \succeq 0, \quad Z_- \succeq 0$$

$$\boxed{\begin{array}{ll} \underset{X, Z_+, Z_-}{\text{minimize}} & -\log\det(X) + \gamma \,\mathrm{trace}(Z_+ + Z_-) \\ \text{subject to} & \mathcal{A}\, X + \mathcal{B}\, Z - \mathcal{C} = 0 \\ & Z_+ \succeq 0, \quad Z_- \succeq 0 \end{array}}$$

Customized algorithms

- Alternating Direction Method of Multipliers (ADMM)

  Boyd et al., Found. Trends Mach. Learn. '11

- Alternating Minimization Algorithm (AMA)

  Tseng, SIAM J. Control Optim. '91

Augmented Lagrangian

$$\mathcal{L}_\rho(X, Z; Y) = -\log\det(X) + \gamma \|Z\|_* + \langle Y, \mathcal{A}X + \mathcal{B}Z - \mathcal{C}\rangle + \frac{\rho}{2}\|\mathcal{A}X + \mathcal{B}Z - \mathcal{C}\|_F^2$$

Augmented Lagrangian

$$\mathcal{L}_\rho(X, Z; Y) \;=\; -\log\det(X) \;+\; \gamma\,\|Z\|_* \;+\; \langle Y, \mathcal{A}\,X \;+\; \mathcal{B}\,Z \;-\; \mathcal{C}\rangle$$
$$+\;\frac{\rho}{2}\,\|\mathcal{A}\,X \;+\; \mathcal{B}\,Z \;-\; \mathcal{C}\|_F^2$$

- METHOD OF MULTIPLIERS
  - ⋆ minimizes $\mathcal{L}_\rho$ jointly over X and Z

$$\left(X^{k+1}, Z^{k+1}\right) \;:=\; \underset{X,\,Z}{\operatorname{argmin}}\; \mathcal{L}_\rho(X, Z; Y^k)$$

$$Y^{k+1} \;:=\; Y^k \;+\; \rho\left(\mathcal{A}\,X^{k+1} \;+\; \mathcal{B}\,Z^{k+1} \;-\; \mathcal{C}\right)$$

ADMM vs AMA

- ADMM

$$
\begin{aligned}
X^{k+1} &:= \underset{X}{\operatorname{argmin}} \ \mathcal{L}_{\rho}(X, Z^k; Y^k) \\
Z^{k+1} &:= \underset{Z}{\operatorname{argmin}} \ \mathcal{L}_{\rho}(X^{k+1}, Z; Y^k) \\
Y^{k+1} &:= Y^k + \rho\left(\mathcal{A}\, X^{k+1} + \mathcal{B}\, Z^{k+1} - \mathcal{C}\right)
\end{aligned}
$$

ADMM vs AMA

- ADMM

$$
\begin{aligned}
X^{k+1} &:= \underset{X}{\operatorname{argmin}}\ \mathcal{L}_{\rho}(X, Z^k; Y^k) \\
Z^{k+1} &:= \underset{Z}{\operatorname{argmin}}\ \mathcal{L}_{\rho}(X^{k+1}, Z; Y^k) \\
Y^{k+1} &:= Y^k + \rho\left(\mathcal{A}\,X^{k+1} + \mathcal{B}\,Z^{k+1} - \mathcal{C}\right)
\end{aligned}
$$

- AMA

$$
\begin{aligned}
X^{k+1} &:= \underset{X}{\operatorname{argmin}}\ \mathcal{L}_{0}(X, Z^k; Y^k) && \textbf{matrix inverse} \\
Z^{k+1} &:= \underset{Z}{\operatorname{argmin}}\ \mathcal{L}_{\rho}(X^{k+1}, Z; Y^k) && \textbf{sv-thresholding} \\
Y^{k+1} &:= Y^k + \rho_k\left(\mathcal{A}\,X^{k+1} + \mathcal{B}\,Z^{k+1} - \mathcal{C}\right)
\end{aligned}
$$

Properties of AMA

- COVARIANCE COMPLETION VIA AMA
  - ⋆ proximal gradient on the dual problem
  - ⋆ sub-linear convergence with constant step-size

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STEP-SIZE SELECTION

- ⋆ Barzilla-Borwein initialization followed by backtracking
- ⋆ positive definiteness of X^{k+1}
- ⋆ sufficient dual ascent

Zare, Chen, Jovanović, Georgiou, IEEE TAC '16

Challenges

- TURBULENCE MODELING
  - ⋆ development of turbulence closure models
  - ⋆ modeling higher-order moments
  - ⋆ design of flow estimators/controllers

- ALGORITHMIC
  - ⋆ alternative rank approximations
    (e.g., iterative re-weighting, matrix factorization)
  - ⋆ improving scalability

- THEORETICAL
  - ⋆ conditions for exact recovery
  - ⋆ convergence rate of AMA with BB step-size initialization

Summary

- THEORETICAL AND ALGORITHMIC DEVELOPMENTS
  - ⋆ *Zare, Chen, Jovanović, Georgiou, IEEE TAC '16* (in press)
  - ⋆ *Zare, Jovanović, Georgiou, IEEE CDC '16*
- APPLICATION TO TURBULENT FLOWS
  - ⋆ *Zare, Jovanović, Georgiou, J. Fluid Mech. '16* (in press)

arXiv:1602.05105

- CUSTOMIZED ALGORITHMS FOR COVARIANCE COMPLETION
  - ⋆ ADMM vs AMA
  - ⋆ AMA works as a proximal gradient on the dual problem

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- COMPUTING RESOURCES
  - ⋆ University of Minnesota Supercomputing Institute
- DISCUSSIONS
  - ⋆ B. Bamieh, S. Chernyshenko, B. Farrell, Y. Hwang, P. Ioannou, J. Jimenez, B. McKeon, R. Moarref, P. Moin, J. Nichols, P. Schmid, J. Sillero