The Mean and the Invariant Measure

The Role of Noise

Existence Theory

Existence of the Invariant Measure

Hopf's Theory

Intermittency

Summary

The invariant measure of the stochastic Navier-Stokes equation

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Outline

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The Mean of the Solution to the Navier-Stokes Equations

- The flow satisfies the Navier-Stokes Equation

$$u_t + u \cdot \nabla u = \nu \Delta u + \nabla \{ \Delta^{-1} [trace(\nabla u)^2] \}$$

- However, in turbulence the fluid flow is not deterministic
- Instead we want a statistical theory
- In applications the flow satisfies the Reynolds Averaged Navier-Stokes Equation (RANS)

$$\overline{u}_t + \overline{u} \cdot \nabla \overline{u} = \nu \Delta \overline{u} - \nabla \overline{p} + (\overline{u} \cdot \nabla \overline{u} - \overline{u \cdot \nabla u})$$

- What does the bar $\overline{u}$ mean?

An Invariant Measure

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Summary

- In experiments and simulations it is an ensamble average
$$\bar{u} = \langle u \rangle$$
- Mathematically speaking the mean is an expectation, if ϕ is any bounded function on H
$$E(\phi(u)) = \int_H \phi(u) d\mu(u) \tag{1}$$
- The mean is defined by the invariant measure μ on the function space H where u lives
- The invariant measure determines the statistical theory. If we can we compute it, we know the whole story

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Noise-driven Instabilities

- Let $U(x_1)j_1$ denote the mean flow, taken to be in the x_1 direction
- If $U' < 0$ the flow is unstable
- The largest wavenumbers (k) can grow the fastest
- The initial value problem is ill-posed
- There is always white noise in the system that will initiate this growth
- If U is large the white noise will continue to grow
- This situation is typical for turbulent flow

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Saturation by the Nonlinearities

- This growth of the noise does not continue forever
- The exponential growth is saturated by the nonlinear terms in the Navier-Stokes equation
- The result is large noise that now drives the turbulent fluid
- Thus for fully developed turbulence we get the Stochastic Navier-Stokes driven by the large noise

$$du = (\nu\Delta u - u \cdot \nabla u + \nabla\{\Delta^{-1}[trace(\nabla u)^2]\})dt \quad (2)$$

$$+ \sum_{k \neq 0} h_k^{1/2} db_t^k e_k \quad (3)$$

- Determining how fast the coefficients $h_k^{1/2}$ decay as $k \to \infty$ is now a part of the problem

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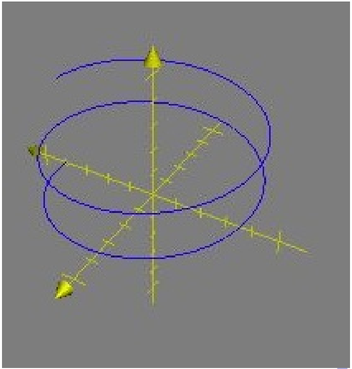
Uniform flow + Rotation

- To get existence of turbulent solution we need strong unidirectional flow $U\, j_1$
- To deal with the Fourier coefficients $\widehat{u}(0, k_2, k_3, t)$ that do not depend on k_1 we start with some rotation with amplitude A, angular velocity Ω and axis of rotation x_1



Onsager's Conjecture in Three Dimensions

Theorem (1)

Let the velocity U of the mean flow and the product of the amplitude A and the frequency Ω of the rotation be sufficiently large, with U and AΩ also satisfying a non-resonance conditions. Then the solution of the stochastic PDE (2) is uniformely bounded in the H^p Sobolev norm in $L^2(\Omega, P)$

$$\text{ess sup}_{t\in[0,\infty)} E(\|u\|^2_{\frac{11}{6}^+})(t) \leq \frac{\left(\sum_{k\neq 0} \frac{3(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2\nu|k|^2} h_k\right)}{\left(1 - C\left(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}\right)\right)} \quad (4)$$

where $B = min(|U|, A\Omega, CA\Omega)$ is large, δ small and C is a constant.

More general flows

- In numerical studies of isotropic turbulence:
- Y. Yang, D. I. Pullin and I. Bermejo-Moreno (2010), Multi-scale geometric analysis of Lagrangian structures in isotropic turbulence. J. Fluid Mech. 654 : 233-270
- Turbulent structure on difference scales in the inertial range alternate between tubular and sheet-like structures, with tubular structures typical on the largest scale
- The proof goes through for finitely many such largest scale tubular structure, assuming they are not in resonance
- Thus existence may not be a problem for most turbulent flows

What are the properties of these solutions?

- They are stochastic processes that are continuous both in space and in time
- But they are not smooth in space, they are Hölder continuous with exponent $\frac{1}{3}$
- However, there is no blow-up in finite time
- Instead the solutions roughens
- The solutions start smooth, $u(x, 0) = 0$, but as the noise gets amplified they roughen, until they have reached the characteristic roughness $\chi = \frac{1}{3}$ in the statistically stationary state
- Neither ∇u nor $\nabla \times u$ are continuous in general

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A Unique Invariant Measure

- We must prove the existence of a unique invariant measure to prove the existence of Kolmogorov's statistically invariant stationary state
- A measure on an infinite-dimensional space H is invariant under the transition semigroup $R(t)$, if

$$R_t \int_H \varphi(x) d\mu = \int_H \varphi(x) d\mu$$

- Here $x = u(t)$ is the solution of N-S and R_t is induced by the Navier-Stokes flow
- We define the invariant measure by the limit

$$\lim_{t \to \infty} E(\phi(u(t))) = \int_H \phi(u) d\mu(u) \tag{5}$$

The Existence and Uniqueness of the Measure

- The limit exist if the sequence of probability measures is tight, the semi-group is strongly Feller and the flow is irreducible
- Tightness amounts to proving the existence of a bounded and compact invariant set for the N-S flow
- Strongly Feller is a generalization of a method developed by McKean (2002) in "Turbulence without pressure". The variation of Navier-Stokes equation is solved to get the estimate

$$|P_t(\phi(u)) - P_t(\phi(v))| \leq \|\phi\|_\infty |u - v|_2$$

- Irreducibility reduces to a problem in stochastic control theory

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Can we Compute the Invariant Measure?

- The characteristic equation of the invariant measure is

$$\varphi(h) = \int_H e^{i<h,x>} d\mu(x)$$

- In 1952 Hopf wrote down a functional differential equation for the characteristic function
- Can we write down a functional differential equation for the measure?
- Let $w = \int_H \phi(x) d\mu(x)$, then w satisfies the IVP

$$w_t = \frac{1}{2}\mathrm{Tr}[QD_x^2 w] + < A(u)x, D_x w >$$

$$w(x,0) = \phi(x)$$

Find a linear SPDE that has the same measure

- The Navier-Stokes equation is

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \Delta u - \nabla p + \nabla\{\Delta^{-1}[trace(\nabla u)^2]\} \quad (6)$$

- Let $w = D_{u^0} u$, where u^0 is the initial condition

$$\frac{\partial w}{\partial t} + u \cdot \nabla w + w \cdot \nabla u = \Delta w + 2\nabla\{\Delta^{-1}[trace(\nabla u \cdot \nabla w)]\} \quad (7)$$

- Adding "white" noise *CdB* we get the stochastic PDE

$$\begin{aligned} dw &= [\Delta w - u \cdot \nabla w - w \cdot \nabla u \\ &+ 2\nabla\{\Delta^{-1}[trace(\nabla u \cdot \nabla w)]\}]dt + CdB \end{aligned} \quad (8)$$

Ito Diffusion and Girsanov's Theorem

The Ito diffusion for the "Eulerian" fluid particle is

$$dX_t = -u(X_t, t)dt + \sqrt{2\nu}dB_t$$

We can write the solution of (8) without the pressure term as

$$w(x, t) = E[f(X_t)] + \int_0^t E[e_k(X_{t-s})]CdB_s$$

By Girsanov's theorem

$$w(x, t) = E[f(B_t)M_t] + \int_0^t E[e_k(B_{t-s})M_{t-s}]CdB_s \quad (9)$$

where M_t is the Martingale

$$M_t = exp\{-\int_0^t u(B_s, s) \cdot dB_s - \frac{1}{2}\int_0^t |u(B_s, s)|^2 ds\}$$

The Invariant Measure

Theorem (2)

The invariant measure of Equation (2) can be expressed as the convolution

$$\mu_\infty = \mathcal{N}_{2\nu} * \mathcal{N}_{V_\infty}$$

where

$$V_\infty = \int_0^\infty e^{Ds} C ds$$

where e^{Ds} is the semi-group generated by the bounded operator $Dw = 2\nabla\{\Delta^{-1}[trace(\nabla u \cdot \nabla w)]\}$

- It is clear that this cannot be the whole story
- This measure give the Kolomogorov scaling of the even structure functions
- However, it is an infinite-dimensional Gaussian and all the odd structure functions vanish

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The central limit theorem

- We have to modify the noise
- Split the torus $\mathbb{T}^3$ into little boxes and consider the dissipation to be a stochastic process in each box
- By the central limit theorem the average

$$S_n = \frac{1}{n}\sum_{j=1}^{n} p_j$$

converges to a Gaussian random variable as $n \to \infty$

- This holds for any Fourier component (e_k) and the result is the infinite dimensional Brownian motion

$$\sum_{k \neq 0} c_k^{\frac{1}{2}} b_t^k e_k(x)$$

The large deviation principle

- In addition we get intermittency of the dissipation
- This intermittency is capture by the large deviation principle
- If these excursions are completely random then they are modeled by a Poisson process by the rate λ
- An application of the large deviation principle shows that the large deviations of S_n are bounded above by a Gamma process with rate λ
- This also holds in the direction of each Fourier component and gives,

$$\sum_{k \neq 0} d_k \gamma_t^k e_k(x)$$

Velocity fluctuation

- So far our noise is additive. There is also multiplicative noise due to velocity fluctuation
- The multiplicative noise, models the excursion (jumps) in the velocity gradient
- If these jumps are completely random they should be modeled by a Poisson process
- This gives the multiplicative noise term

$$u \sum_{k \neq 0} h_k p_{m_k}$$

where the p_{m_k} are independent Poisson processes and m_k are the numbers of jumps

Computation of the measure

Now the linearized equation

$$dz = (\nu\Delta z - u\cdot\nabla z - z\cdot\nabla u + 2\nabla\Delta^{-1}\mathrm{tr}(\nabla u\nabla z)dt + \sum_{k\neq 0}(c_k^{\frac{1}{2}}db_t^k + d_k\gamma_t^k)e_k(x) + z\sum_{k\neq 0}h_k dp_{m_k} \qquad (10)$$

$z(0) = z_0$, has the same invariant measure as the stochastic Navier-Stokes equation. We solve (10) using the additional help of the Feynmann-Kac formula, and Cameron-Martin

$$\begin{aligned} z &= e^{Kt}e^{\int_0^t dq}M_t z^0 \\ &+ \sum_{k\neq 0}\int_0^t e^{K(t-s)}e^{\int_s^t dq}M_{t-s}(c_k^{1/2}d\beta_s^k + d_k\gamma_s^k)e_k(x) \end{aligned}$$

where $K = \nu\Delta + 2\nabla\Delta^{-1}tr(\nabla u\nabla)$

The log-Poisson processes

- The Feynmann-Kac formula gives

$$\exp \int_s^t dq = \exp \sum_{k \neq 0} h_k (p_{m_k(t)} - p_{m_k(s)})$$

where $dp_{m(t)} = p_{m(t+dt)} - p_{m(t)}$ denotes the excursions in the time interval $(t, t + dt]$

- Integrating s from 0 to t gives a telescoping sum and we end up with the process $\exp \sum_{k \neq 0} h_k p_{m_k(t)}$
- These are log-Poisson processes as pointed out by She and Leveque (94), She and Waymire (95) and Dubrulle (94)

$$P_k = e^{h_k p_{m_k}} = e^{\gamma \ln |k|} e^{p_k \ln \beta} = |k|^\gamma \beta^{p_k}$$

where p_k is a Poisson process with mean $\mu_k = -\frac{\gamma \ln |k|}{\beta - 1}$

Computation of the structure functions

Lemma

The scaling of the structure functions is

$$S_p \sim C|x-y|^{\zeta_p},$$

where

$$\zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3})$$

$\frac{p}{3}$ *being the Kolmogorov scaling and* τ_p *the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's* $4/5$ *law,*

$$S_3 = -\frac{4}{5}\epsilon|x-y|$$

to leading order, were $\epsilon = \frac{d\mathcal{E}}{dt}$ *is the energy dissipation.*

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The Proof

$$
\begin{aligned}
S_1(x,y,t) &= E(|u(x,t)-u(y,t)|) \\
&= 2\sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{d_k}{\alpha_k}\frac{(1-e^{-\lambda_k t})}{\lambda_k} E(P_k)\sin(\pi k\cdot(x-y)) \\
&= 2\sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{d_k}{\alpha_k}\frac{(1-e^{-\lambda_k t})}{\lambda_k|k|^{\tau_1}} \sin(\pi k\cdot(x-y)).
\end{aligned}
$$

Now, if $\alpha_k = 1/|k|^{1/3}$, and for $|x-y|$ small,

$$
S_1(x,y,\infty) \sim \frac{2\pi^{\zeta_1}}{C}\sum_{k\in\mathbb{Z}^3\setminus\{0\}} d_k|x-y|^{\zeta_1},
$$

where $\zeta_1 = 1/3+\tau_1 \approx 0.37$.

The higher order structure functions

$$S_2(x,y,\infty) \sim \frac{16\pi^{\zeta_2}}{C^2} \sum_{k\in\mathbb{Z}^3} d_k^2 |x-y|^{\zeta_2} + \frac{2\pi^{2/3}}{C} \sum_{k\in\mathbb{Z}^3} c_k |x-y|^{2/3},$$

where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$

$$S_n = 2^n(\#) \sum_{k\in\mathbb{Z}^3} \frac{d_k^n}{\alpha^n} \frac{(1-e^{-2\lambda_k t})^n}{\lambda_k^n |k|^{\tau_n}} \sin^n(\pi k \cdot (x-y))$$

Thus

$$S_n(x,y,\infty) \sim \frac{2^n \pi^{\zeta_n}}{C^n}(\#) \sum_{k\in\mathbb{Z}^3\setminus\{0\}} d_k^n |x-y|^{\zeta_n},$$

to leading order for $|x-y|$ small

Conclusions

- Starting with sufficiently fast constant flow in some direction there exist turbulent solutions
- These solutions are Hölder continuous with exponent $\frac{1}{3}$ (Onsager's conjecture)
- There exist a unique invariant measure corresponding to these solutions
- The invariant measure can be computed and it gives the modified Kolmogorov scaling of the structure function with the intermittency corrections
- Knowing the structure function (cumulants k_n) one can then compute the PDF from the Gram Charlier series

$$f(x) = \exp\left[\sum_{3}^{\infty} k_n \frac{(-D_x)^n}{n!}\right] \frac{\exp^{-\frac{(x-\mu)}{2\sigma}}}{\sqrt{2\pi\sigma}}$$

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Applications

- Better closure approximations for RANS
- There is no universal way now of approximating the eddy viscosity (closure problem)

$$\bar{u}\cdot\nabla\bar{u}-\overline{u\cdot\nabla u}\sim\nu_{\text{eddy}}|\overline{\nabla u}|(\overline{\nabla u}+\overline{\nabla^T u})$$

- Knowing μ we will be able to solve the closure problem to any desired degree of accuracy
- The same applies to LES (Large Eddy Simulations)
- Better subgrid models for LES

The Artist by the Water's Edge
Leonardo da Vinci Observing Turbulence

Existence and Uniqueness of the Invariant Measure

- This limit exist if we can show that the sequence of associated probability measures are tight
- If the N-S semigroup maps bounded function on H onto continuous functions on H, then it is called Strongly Feller
- Irreducibility says that for an arbitrary b in a bounded set

$$P(sup_{\{t<T\}}\|u(t) - b\| < \epsilon) > 0$$

The spectrum of K

Lemma

The spectrum of K on $W_2^{11/6}(\mathbb{T}^3)$ is discrete and

$$-\nu\, 4\pi^2|k|^2 - C\|u\|^2_{\frac{11}{6}+}|k|^{2/3} \le \lambda_k \le -\nu\, 4\pi^2|k|^2 + C\|u\|^2_{\frac{11}{6}+}|k|^{2/3}$$

Proof.

We consider the operator H were $K = H\partial^{2/3}$ acting on the functions $\partial^{2/3}f$, $f \in W_2^{11/6}(\mathbb{T}^3)$, the Sobolev space with index $11/6$ based on L^2. Formally Hf can be written as

$$Hf = \partial^{4/3}f + 2\nabla\Delta^{-1}\mathrm{tr}(\partial^{4/3}uf)$$

where the operator $\partial^{4/3}$ has discrete spectrum and $2\nabla\Delta^{-1}\mathrm{tr}(\partial^{4/3}u\partial^{2/3}\cdot)$ is a compact perturbation. □