A dynamical model of the large-scale circulation in turbulent Rayleigh-Benard convection

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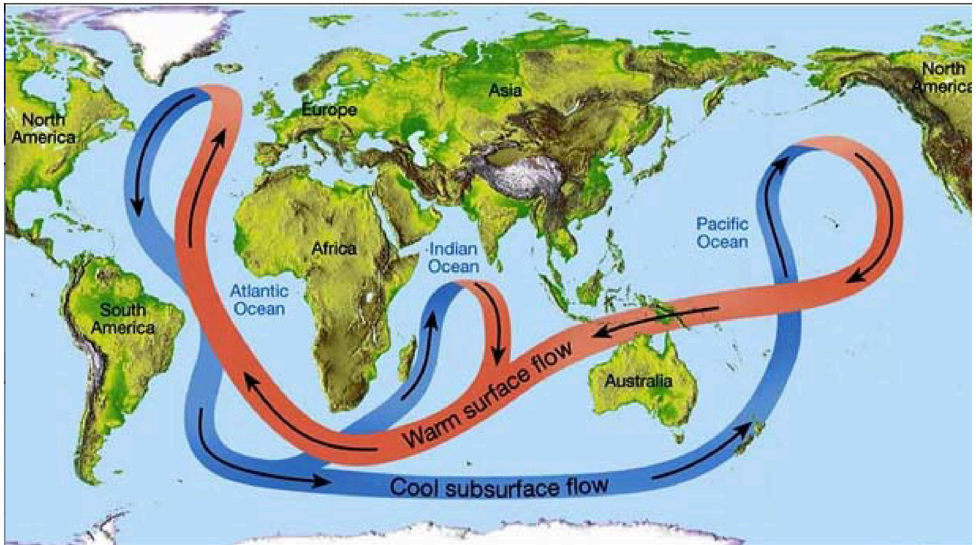
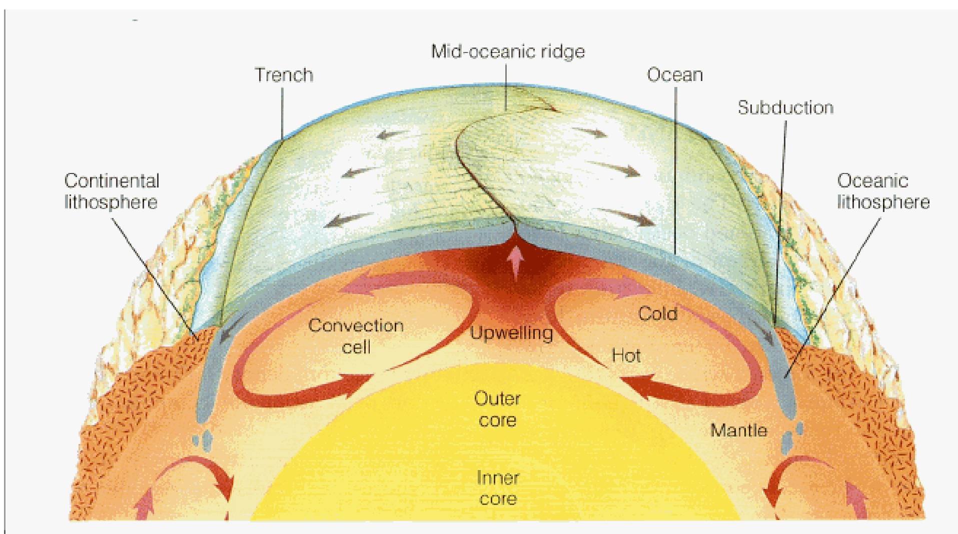
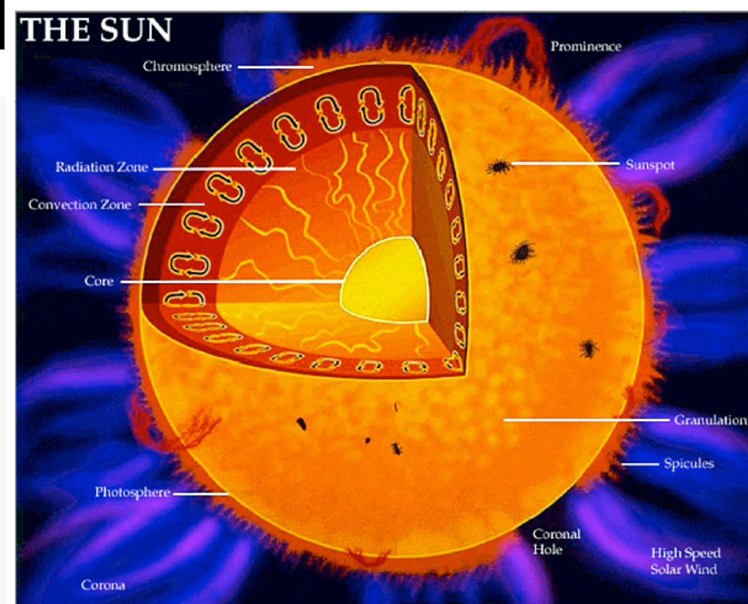
Large-Scale Circulations (LSCs) in nature



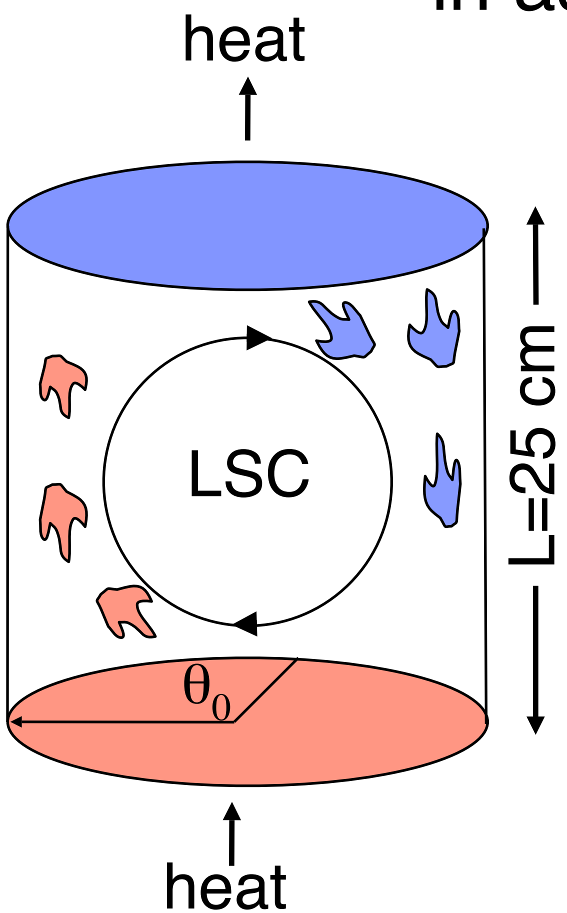
THERMOHALINE CIRCULATION - GREAT OCEAN CURRENT

NOAA





Turbulent Rayleigh-Benard convection in aspect ratio 1 cylinders



Observed LSC dynamics:

- meandering
- cessations
- preferred orientation, net rotation
- twisting and sloshing oscillations

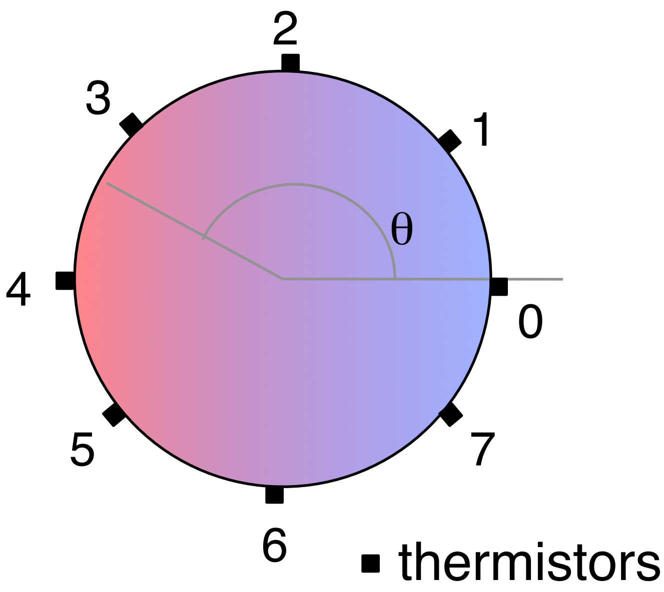
Low-dimensional model:

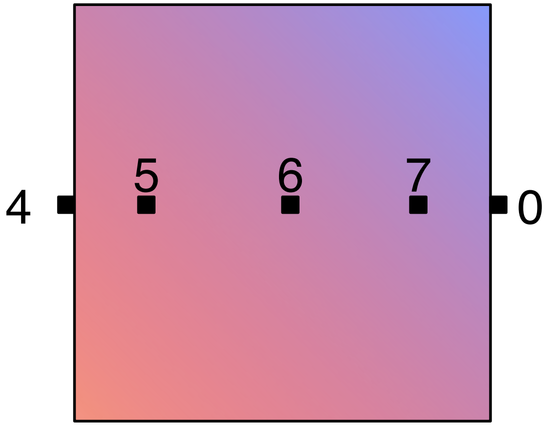
- dynamics described by a pair of physically motivated, stochastic ordinary differential equations.

Prandtl number $\sigma = 4.4$ (water)

Rayleigh number $R = 1.1 \times 10^{10}$

Finding the orientation θ_0 of the LSC





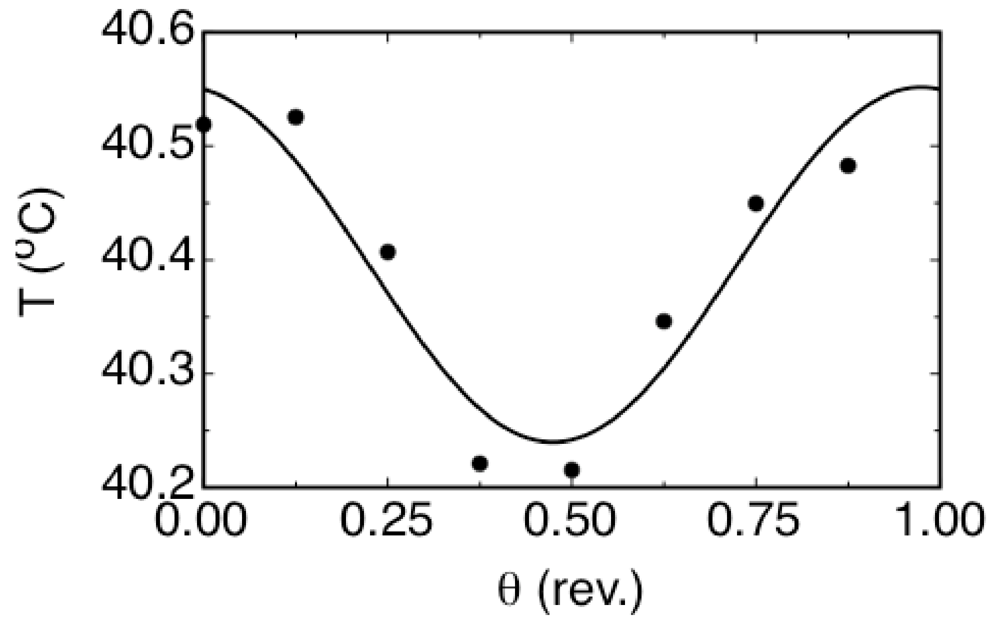
fit to measured temperatures:

$$T = T_0 + \delta \cos(\theta_0 - \theta)$$

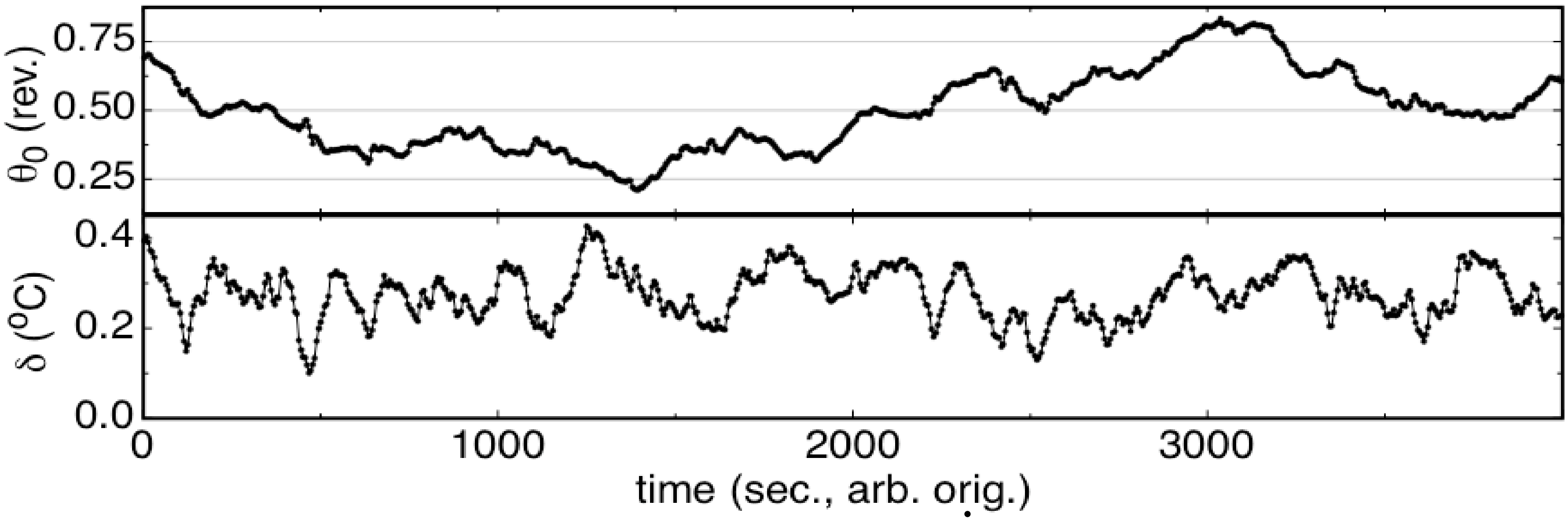
T_0 = average temperature

δ = amplitude

θ_0 = orientation

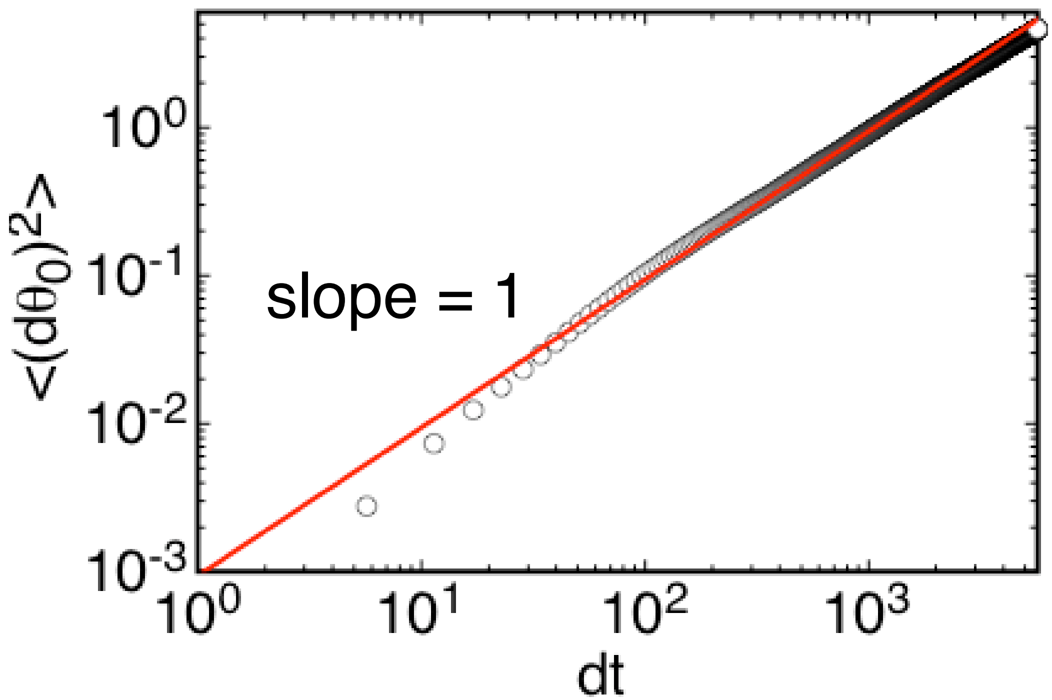


Meandering of the LSC orientation



Orientation fluctuations are diffusive

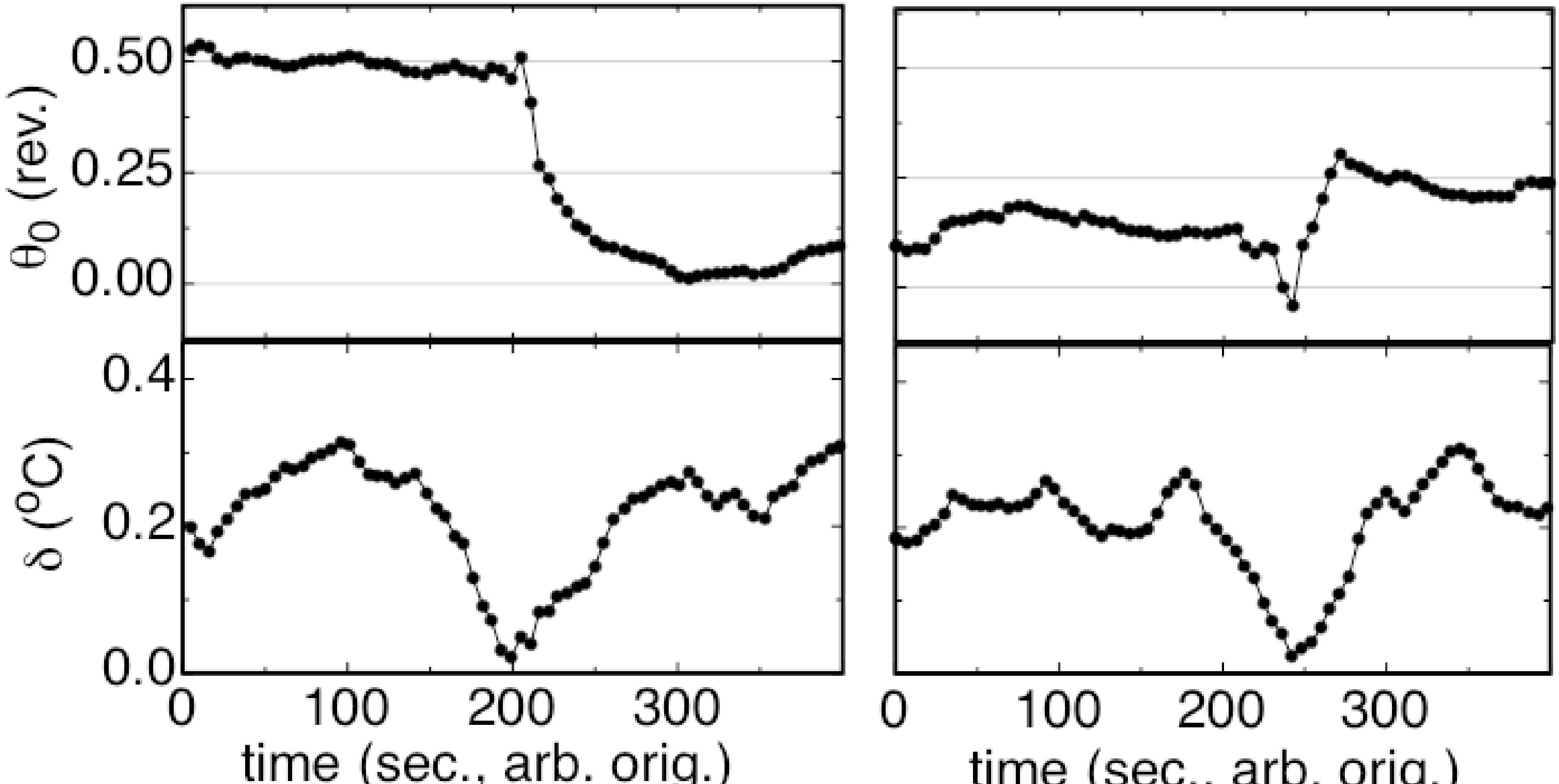
$d\theta_0$ = orientation change over a time interval dt.



$$<(d\theta_0)^2>=D_\theta dt$$

D_θ = diffusivity

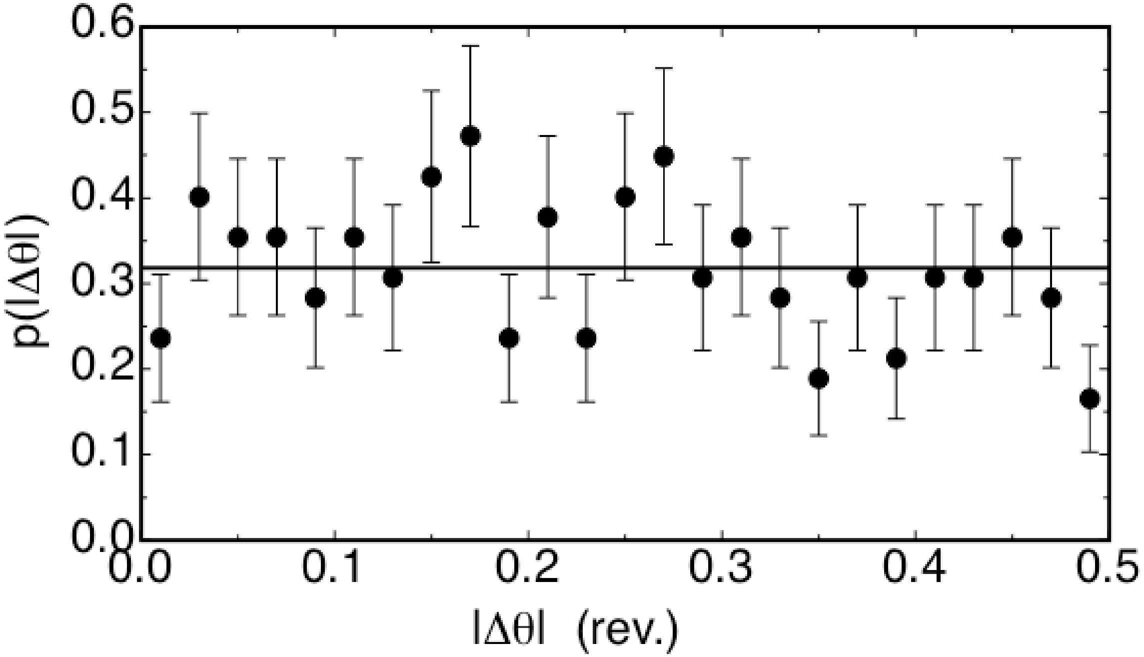
Cessations



average time between cessations: 1.5 days

circulation period: 49 s

LSC forgets orientation during cessations



No preference for reversals ($|\Delta\theta| = 0.5$)

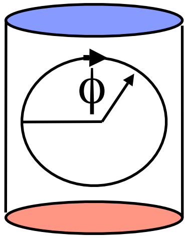
Brown & Ahlers, J. Fluid Mech. 568 (2006)

Model of the LSC dynamics

- Order-of-magnitude stochastic model
- Based on Navier-Stokes equations
- Empirical observation of LSC structure allows volume averages to turn PDEs into ODEs
- only 2 variables required -- θ_0 and δ.

Brown & Ahlers, Phys. Fluids 20 (2008)

Langevin equation for LSC strength



Navier-Stokes: (buoyancy) (drag)

$$\dot{u}_\phi = g\alpha(T - T_0) + \nu\nabla^2 u_\phi$$

volume average:

$$\frac{2\dot{U}}{3} = \frac{2g\alpha\delta}{3\pi} - \frac{12\nu^{1/2}U^{3/2}}{L^{3/2}} \longleftarrow \langle\nabla^2 u_\phi\rangle = \frac{U}{\lambda^2}\frac{6\lambda}{L} = \frac{12UR_e^{1/2}}{L^2}$$

U = max. velocity

Boundary layer width:

$$\lambda = \frac{L}{2}R_e^{-1/2}$$

Assume $\delta \propto$ U: (both characterize LSC strength)

$$\dot{\delta} = \frac{\delta}{\tau_\delta} - \frac{\delta^{3/2}}{\tau_\delta\sqrt{\delta_0}} + f_\delta(t)$$

(buoyancy) (drag) (turbulent fluctuations)

$\text{diffusivity} = D_\delta$

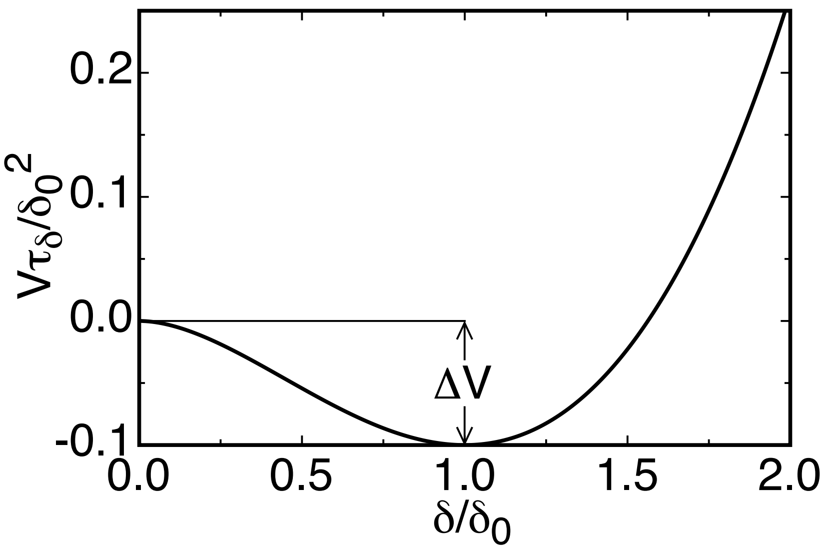
stable amplitude: $\delta_0 = \frac{18\pi\Delta T\sigma R_e^{3/2}}{R}$

damping timescale: $\tau_\delta = \frac{L^2}{18\nu R_e^{1/2}} \sim 47\ \text{s}$

Diffusion in a potential well

$$\dot{\delta} = \underbrace{\frac{\delta}{\tau_\delta} - \frac{\delta^{3/2}}{\tau_\delta\sqrt{\delta_0}}}_{\dot{\delta}_d} + f_\delta(t)$$

potential: $V \equiv -\int \dot{\delta}_d d\delta$

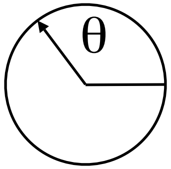


fixed points ($\dot{\delta} = 0$)

$\delta = 0$: unstable

$\delta = \delta_0$: stable

cessations:
diffusion over potential barrier
(Arrhenius-Kramers)
frequency $\omega_c = (2\pi\tau_\delta)^{-1}\exp(-2\Delta V/D_\delta)$
predicted: 0.5 per day
measured: 1.5 per day



Langevin equation for azimuthal motion

(rotational inertia)

from Navier-Stokes: $\dot{u}_\theta + (\vec{u} \cdot \vec{\nabla}) u_\theta = 0$

volume average: $\frac{L\ddot{\theta}_0}{3} = -\frac{2U\dot{\theta}_0}{3}$

Convert to δ and θ_0 :

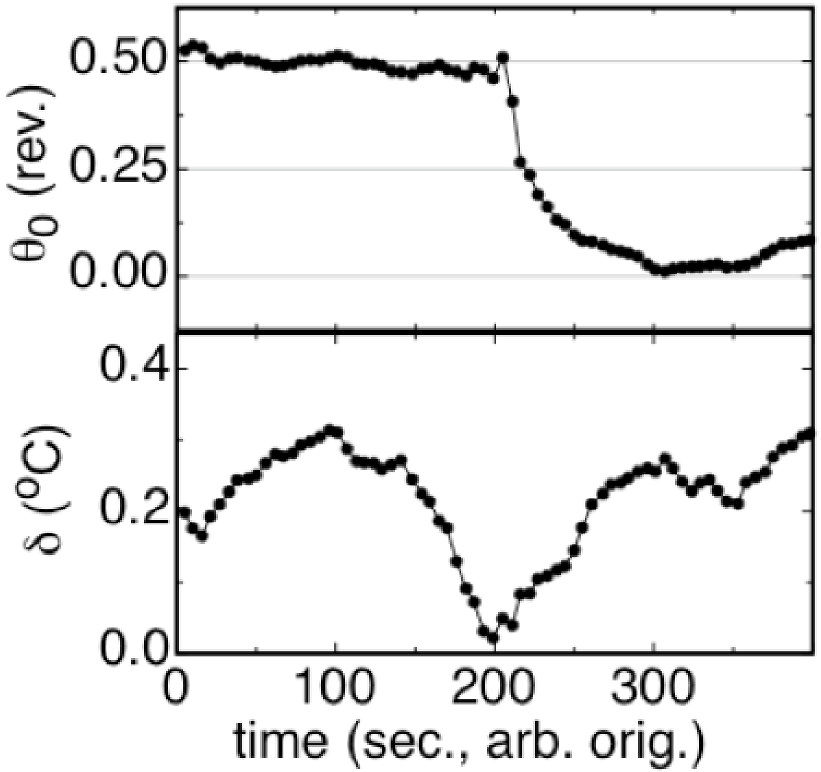
$$\boxed{\ddot{\theta}_0 = -\frac{\dot{\theta}_0 \delta}{\tau_{\dot{\theta}} \delta_0} + f_{\dot{\theta}}(t)}$$

(rotational inertia) (turbulent fluctuations)

diffusivity $D_{\dot{\theta}}$

damping timescale: $\tau_{\dot{\theta}} = \frac{L^2}{2\nu R_e} \sim 7 \text{ s}$

Orientational memory loss due to lack of rotational inertia during cessations



$$\ddot{\theta}_0 = -\frac{\dot{\theta}_0 \delta}{\tau_{\dot{\theta}} \delta_0} + f_{\dot{\theta}}(t)$$

(rotational inertia)

During cessations: small δ

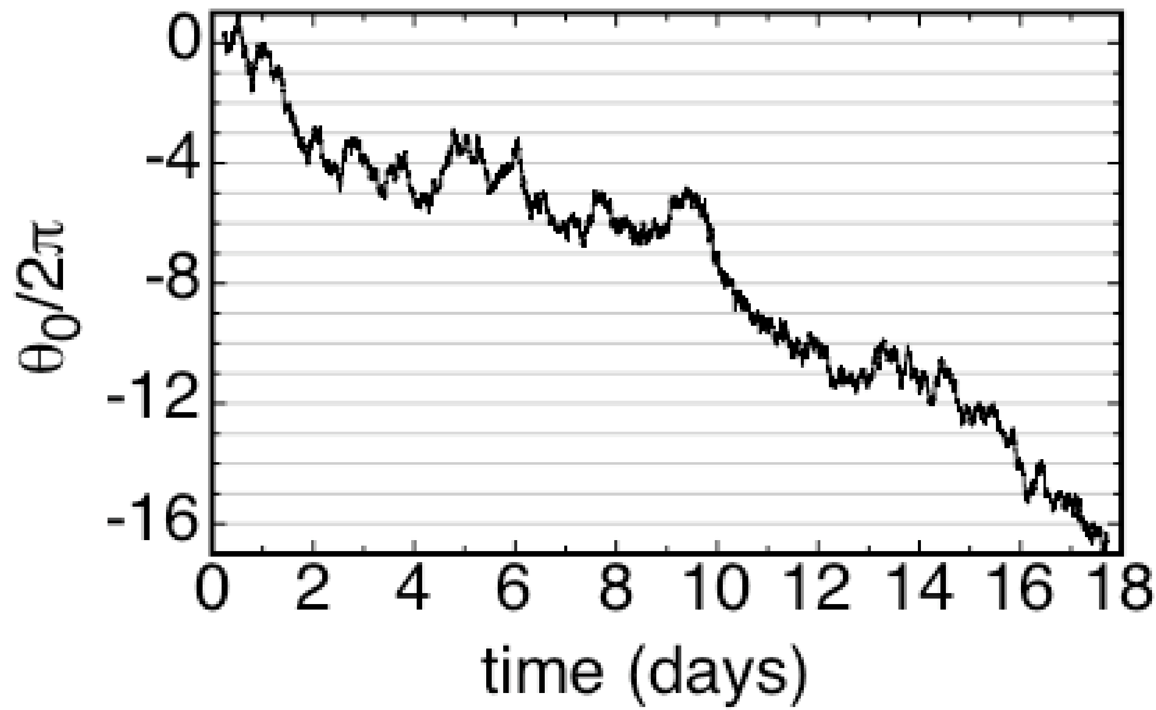
⇒ no rotational inertia

⇒ no damping of diffusion

⇒ orientational memory loss

Asymmetric forcing described with additional terms
Example: Earth’s Coriolis force

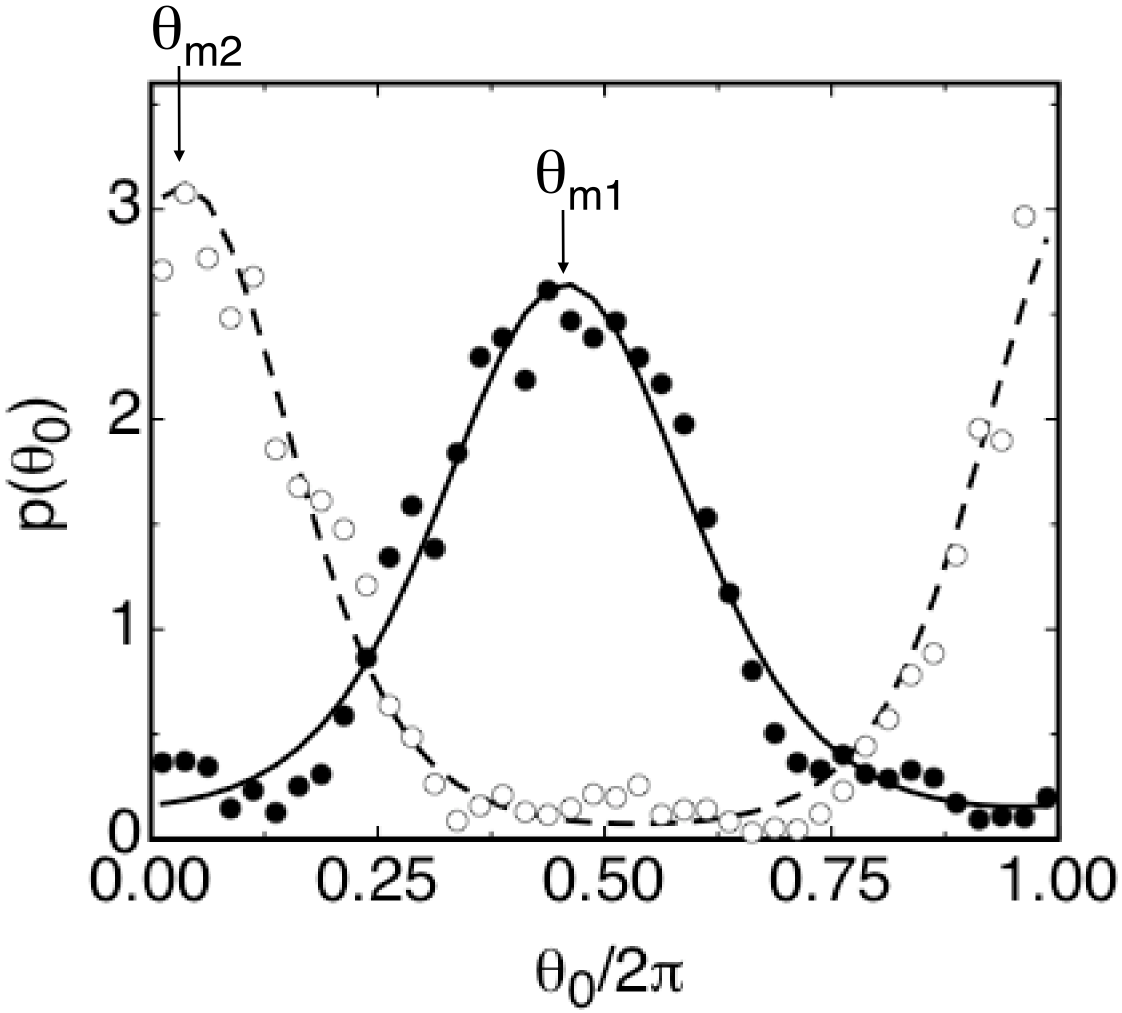
Net Rotation

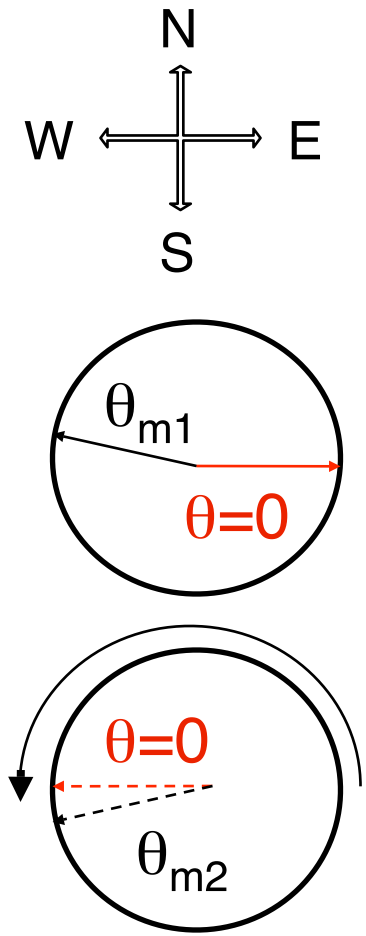


total running time: 258 days

net rate: 0.30 +/- 0.08 rev./day

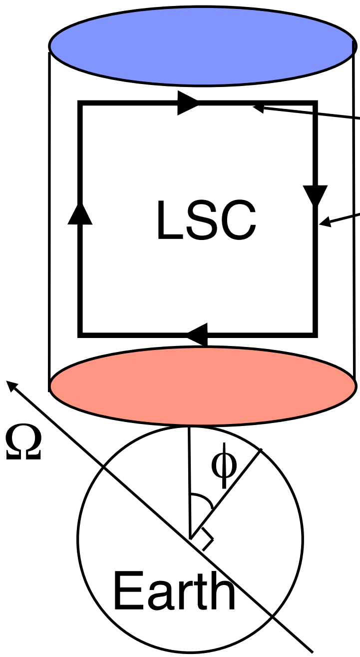
Preferred Orientation





Brown & Ahlers, Phys. Fluids 18 (2006)

Asymmetric forcing described with additional terms: Azimuthal potential due to Coriolis force



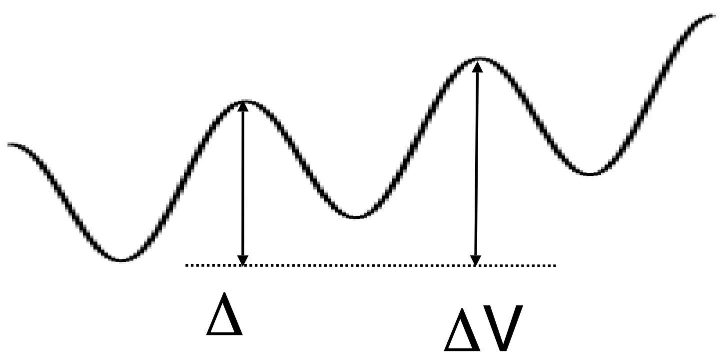
deflection of horizontal legs causes net rotation

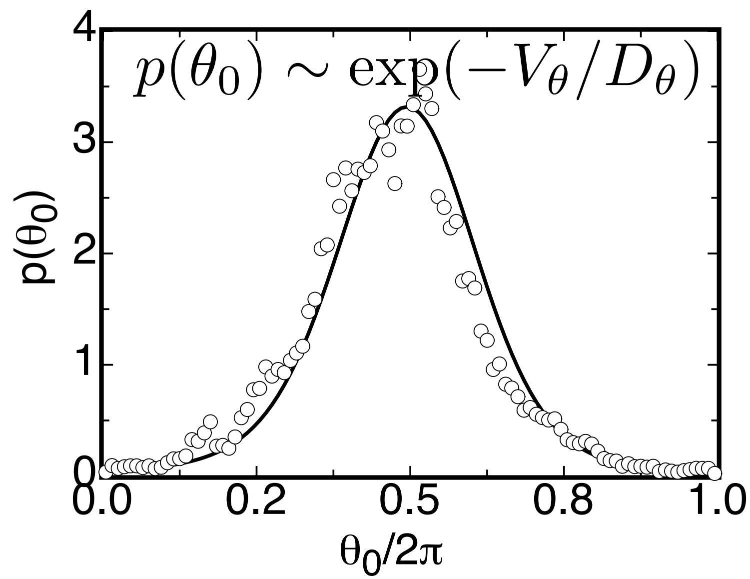
deflection of vertical legs causes preferred θ_0

overdamped: $(\ddot{\theta}_0 = 0)$

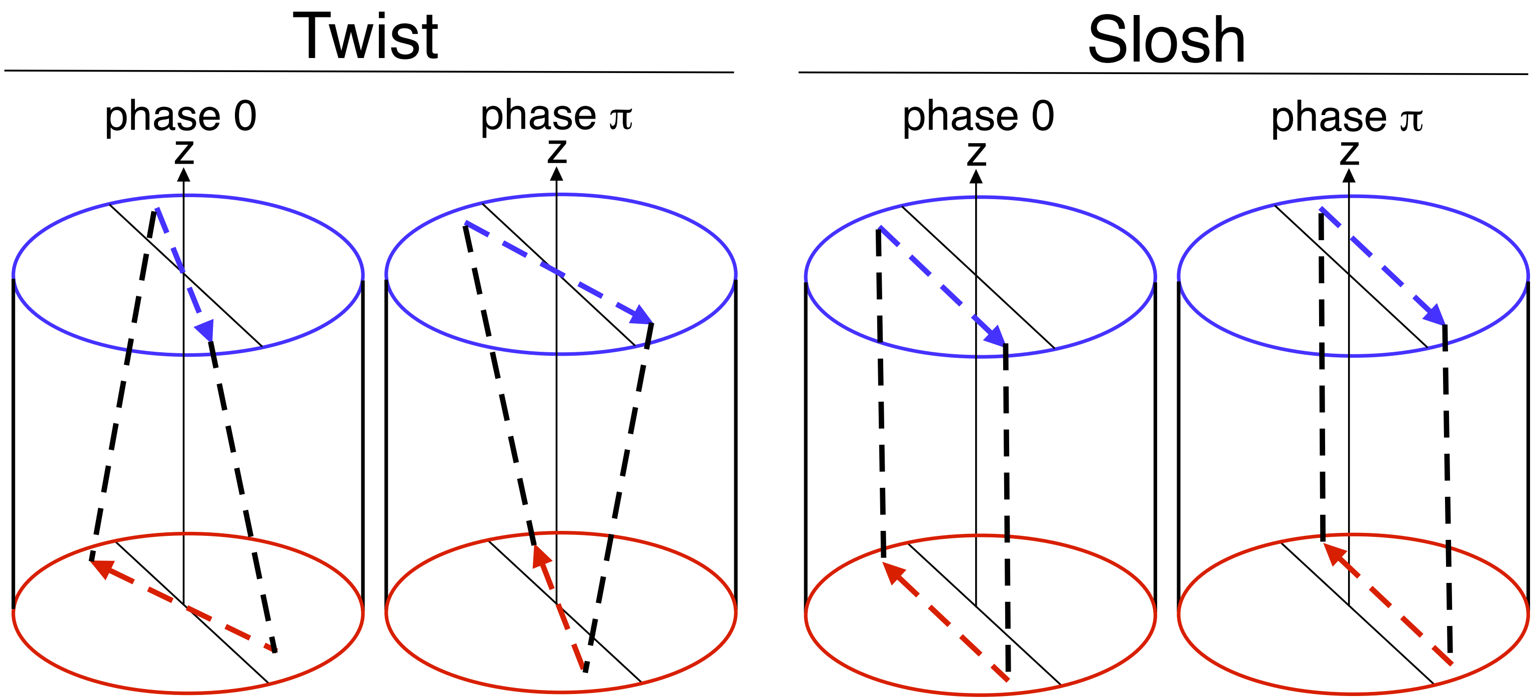
$$\frac{\dot{\theta}_0}{\tau_{\dot{\theta}}} = \omega_\phi \left(2\Omega \cos\phi \sin\theta_0 - \Omega \sin\phi\right) + f_{\dot{\theta}}(t)$$

$$V_\theta = -\int \dot{\theta}_{0,d}\, d\theta_0$$





Oscillation modes



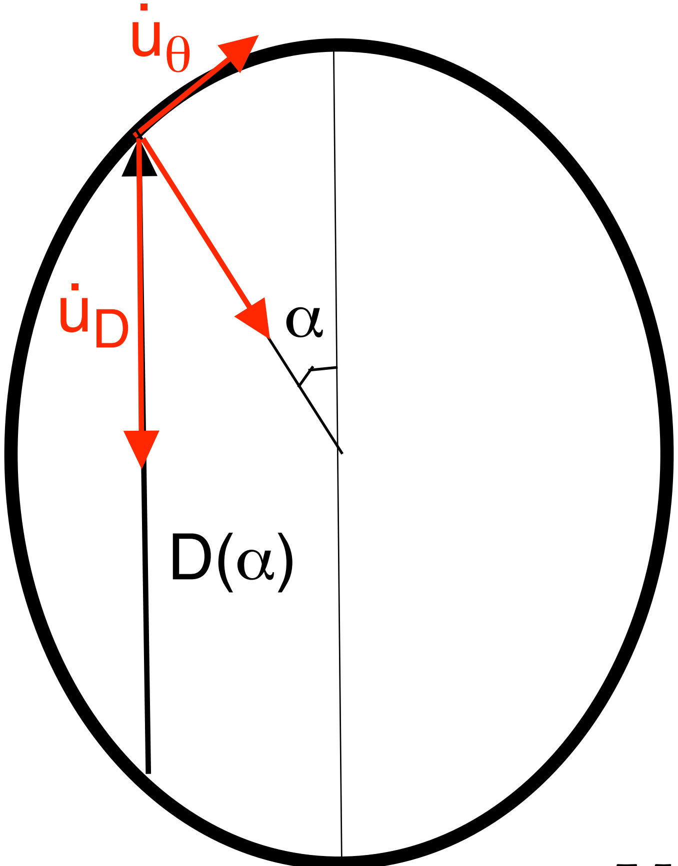
Funfschilling & Ahlers,
Phys. Rev. Lett. 92 (2004)

Xi, Zhou, Zhou, Chan & Xia,
J. Fluid Mech. 630 (2009)

equivalently: traveling waves of hot(upward) and cold(downward) fluid on opposite sides of the container

What is responsible for the restoring force?

Slosh displacement: Pressure forcing from side wall



centripetal acceleration:

$$\dot{u}_D = \omega_\phi^2 D(\alpha)/2$$

$$\dot{u}_\theta = \dot{u}_D L^2 D'(\alpha)/D(\alpha)^3$$

restoring acceleration:

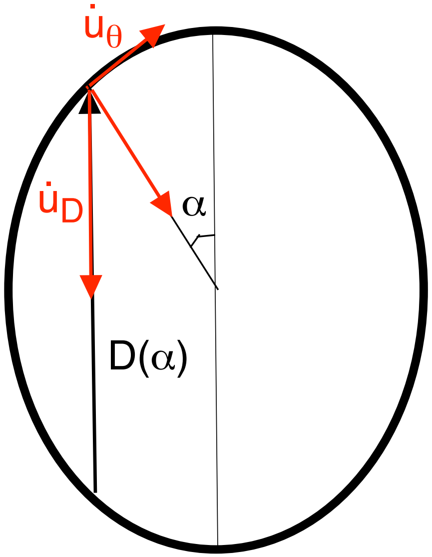
$$\ddot{\alpha}_g = \frac{2\dot{u}_\theta}{D(\alpha)} = -\omega_\phi^2 L^2 \frac{D'(\alpha)}{D(\alpha)^3}$$

$$\omega_\phi = \text{circulation rate}$$

$$V_g \equiv -\int \ddot{\alpha}_g d\alpha = \frac{\omega_\phi^2 L^2}{2D(\alpha)^2}$$

Potential V_g a function of LSC diameter

Oscillations around potential minimum



$$V_g = \frac{\omega_\phi^2 L^2}{2D(\alpha)^2}$$

$$\ddot{\alpha} = -\frac{\dot{\alpha}}{\tau_{\dot{\theta}}} - \omega_\phi^2 \alpha + f_{\dot{\theta}}(t) \;\; (\text{for small } \alpha)$$

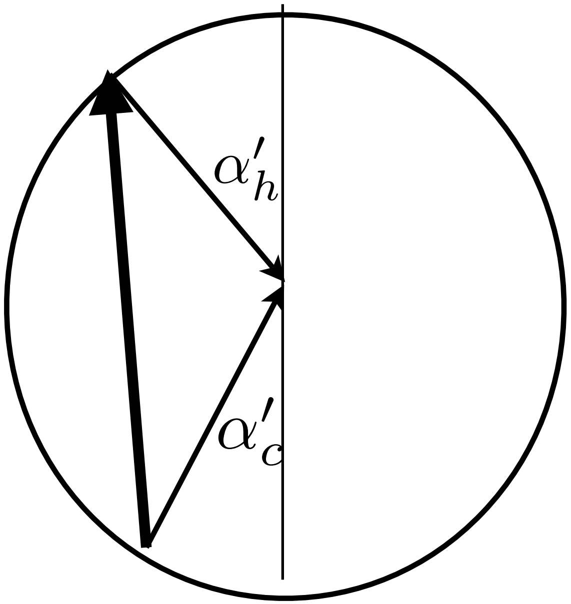
(damping): rotational inertia

(restoring): side wall pressure

(driving): turbulent fluctuations

Potential V_g a function of LSC diameter

Advected oscillations become traveling waves



damping: (rotational inertia)

turbulent fluctuations

restoring force: (wall pressure)

advection

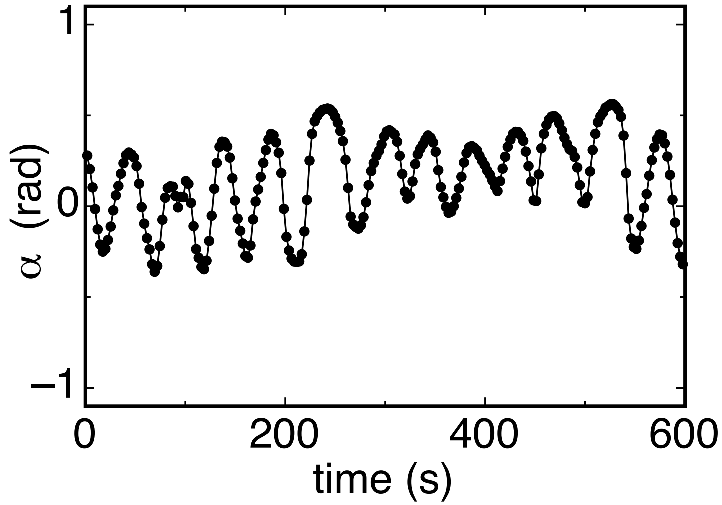
$$\ddot{\alpha}'_h(z,t) = -\frac{\omega_\phi L}{2}\frac{d\dot{\alpha}'_h(z,t)}{dz} - \omega_\phi^2\frac{\alpha'_h(z,t)-\alpha'_c(z,t)}{2} - \frac{\dot{\alpha}'_h}{\tau_{\dot{\theta}}} + f_h(z,t)$$

$$\ddot{\alpha}'_c(z,t) = \frac{\omega_\phi L}{2}\frac{d\dot{\alpha}'_c(z,t)}{dz} + \omega_\phi^2\frac{\alpha'_h(z,t)-\alpha'_c(z,t)}{2} - \frac{\dot{\alpha}'_c}{\tau_{\dot{\theta}}} + f_c(z,t)$$

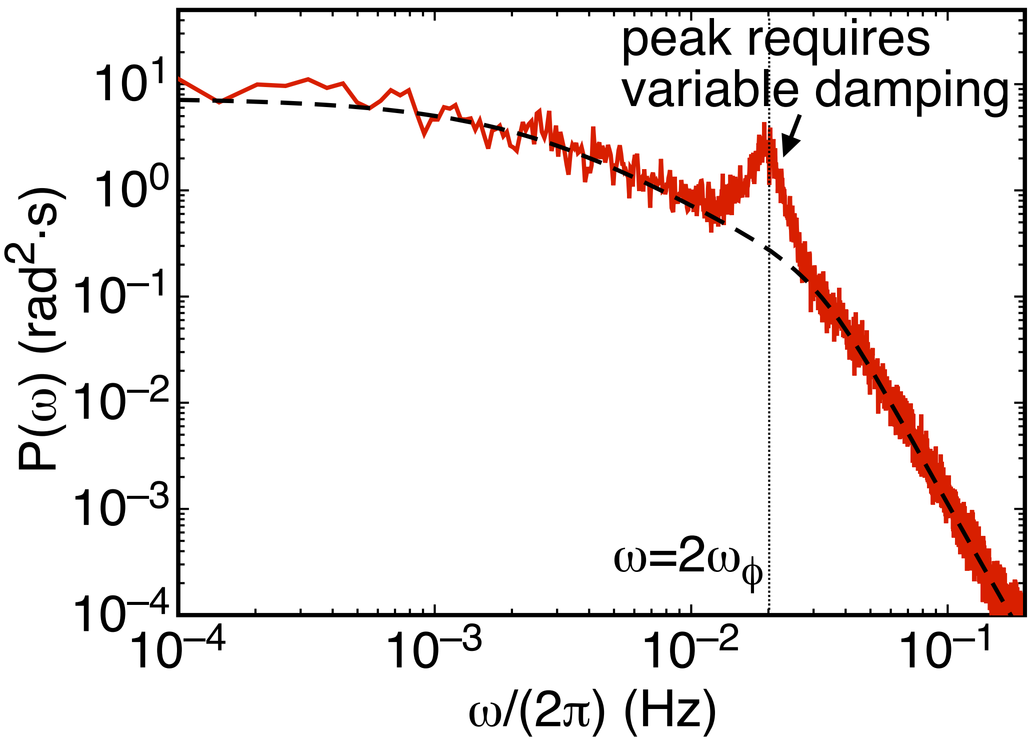
$$\text{traveling wave solutions: } \alpha_h = A\exp(ink_0 z - in\omega_\phi t)$$

Observed oscillations: dominant n=2 mode

traveling wave solutions: $\alpha_h' = A\exp(ink_0 z - in\omega_\phi t)$

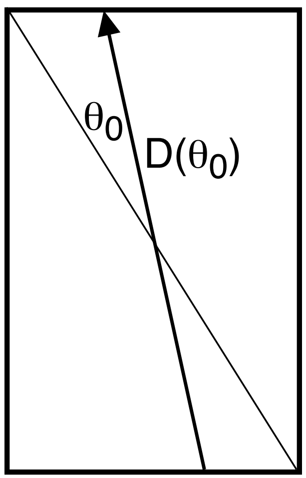


Power spectrum of slosh α



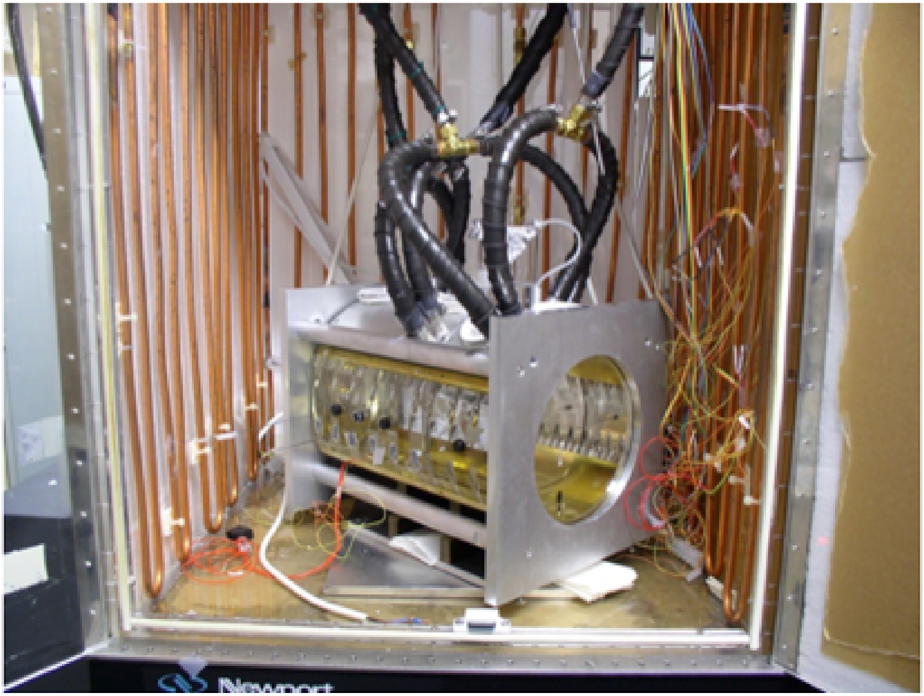
Brown & Ahlers, J. Fluid Mech. 638 (2009)

What about different container geometries? Same diameter function $D(\theta_0)$ describes pressure forcing from side wall



$$V_g \equiv -\int \ddot{\theta}_g d\theta = \frac{\omega_\phi^2 L^2}{2D(\theta_0)^2}$$

Experiment with adjustable aspect ratio cross section

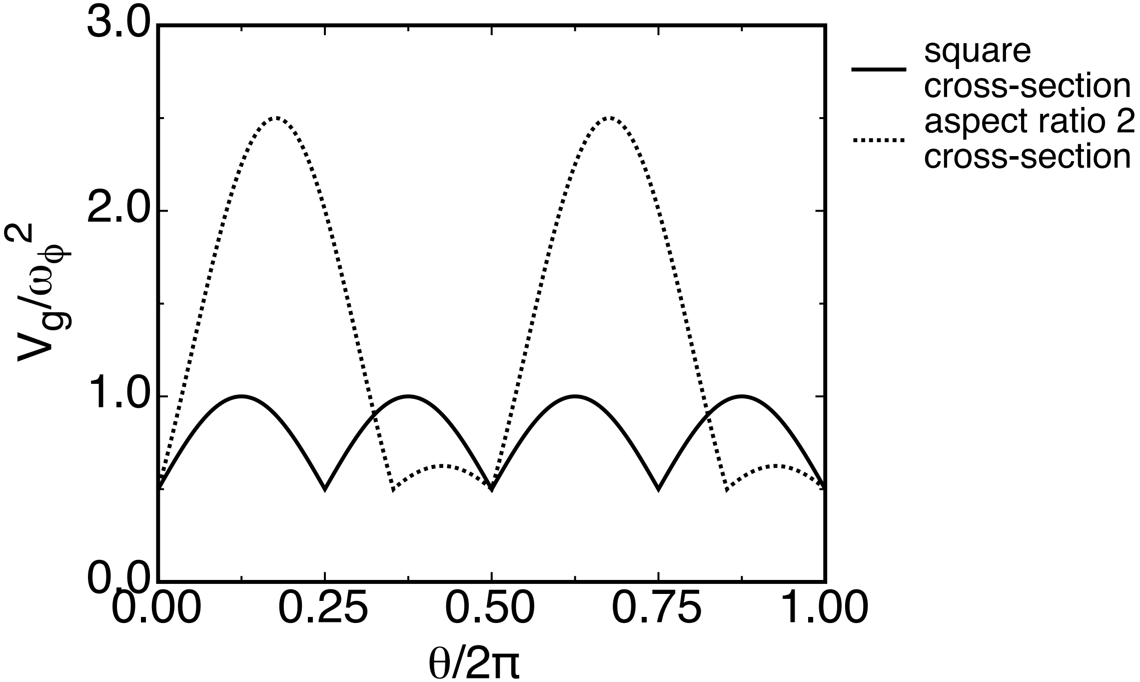


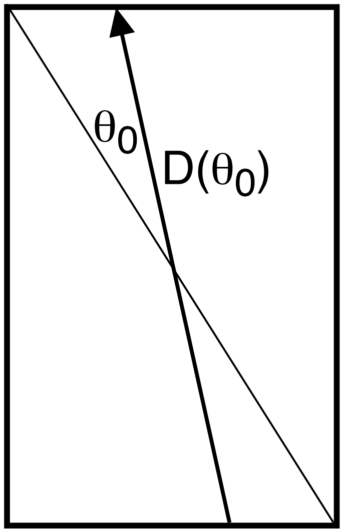
H. Song, P.Tong, Hong Kong
University of Science & Technology

dynamics:
preferred orientation at corners
oscillations around corners
switching between corners

Potential for rectangular cross sections

$$V_g = \frac{\omega_\phi^2 L^2}{2D(\theta_0)^2}$$



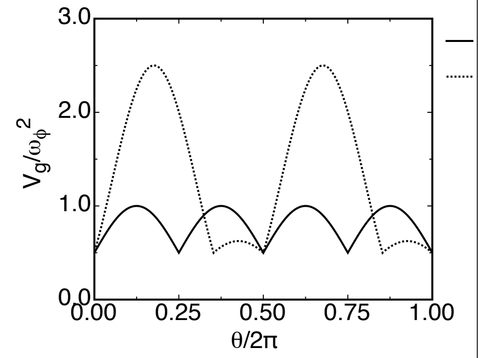


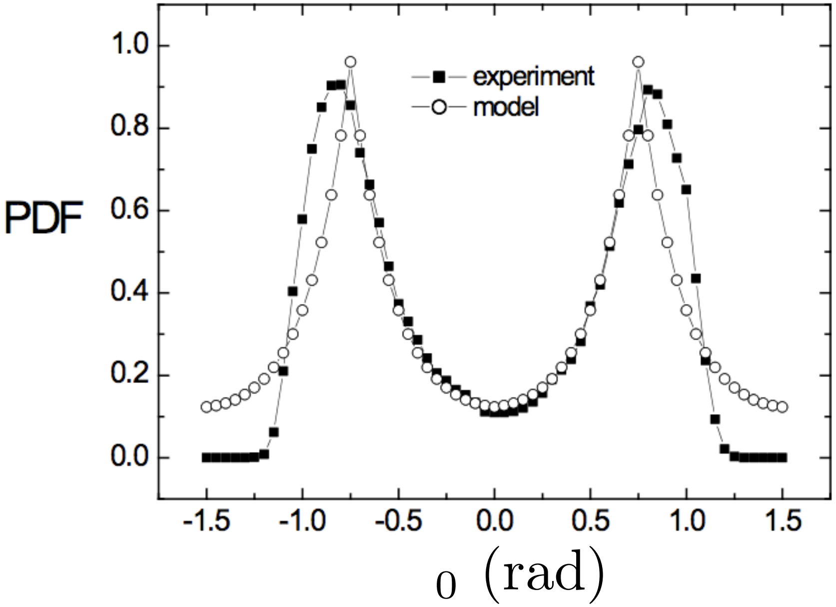
oscillations around potential minima at corners

p(θ_0) for aspect ratio 1 (sqaure)

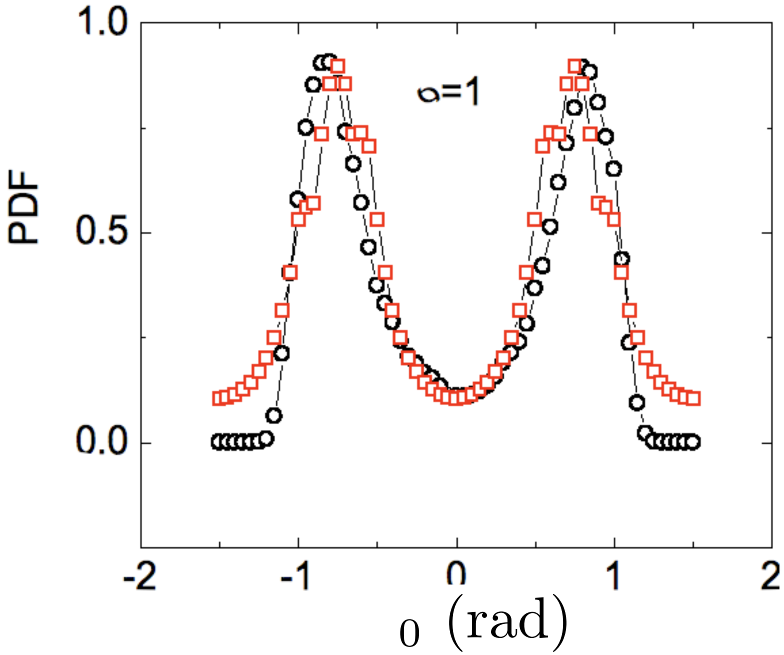
$$V_g \equiv -\int \ddot{\theta}_g d\theta = \frac{\omega_\phi^2 L}{2D(\theta_0)^2}$$

$$p(\theta_0) \propto \exp\left(-\frac{V_g(\theta_0)}{\tau_{\dot\theta} D_{\dot\theta}}\right)$$





correction for non-zero width of LSC

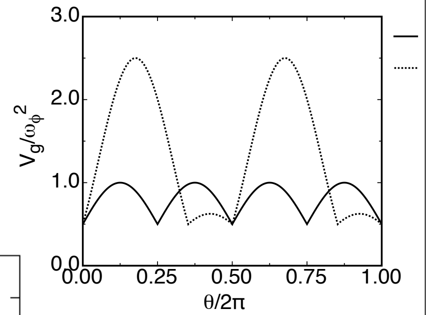


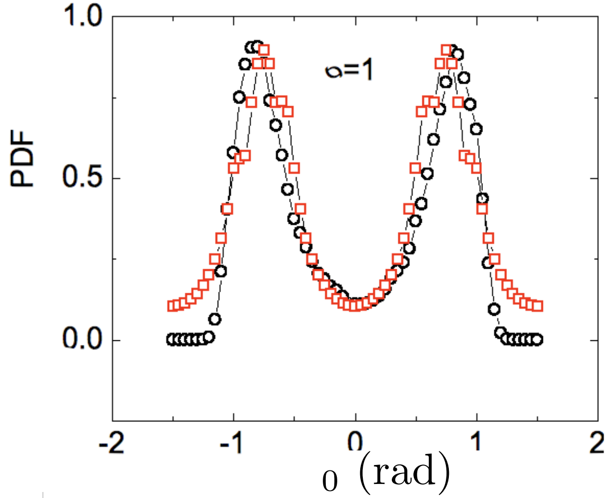
correction due to width of LSC: average potential over finite range (smooths potential near corners)

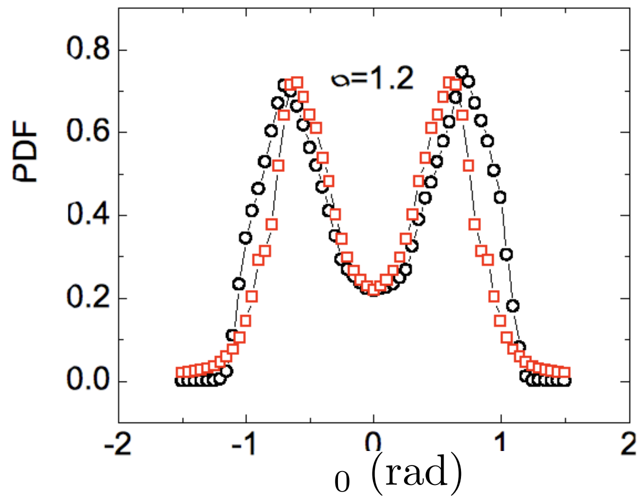
data from H. Song, P. Tong, HKUST

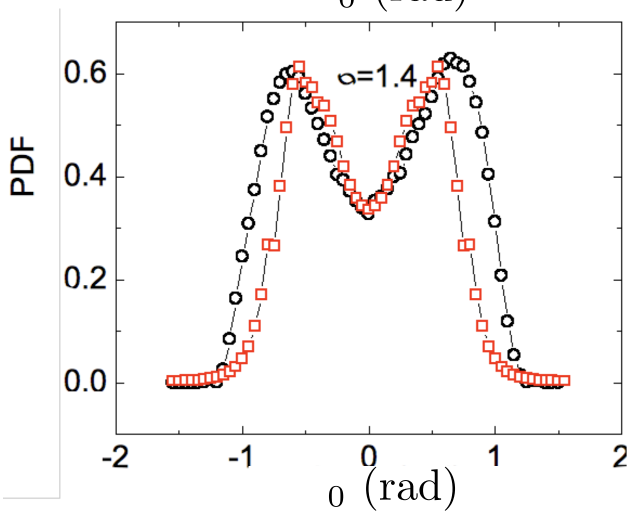
Aspect ratio dependence of p(θ0)

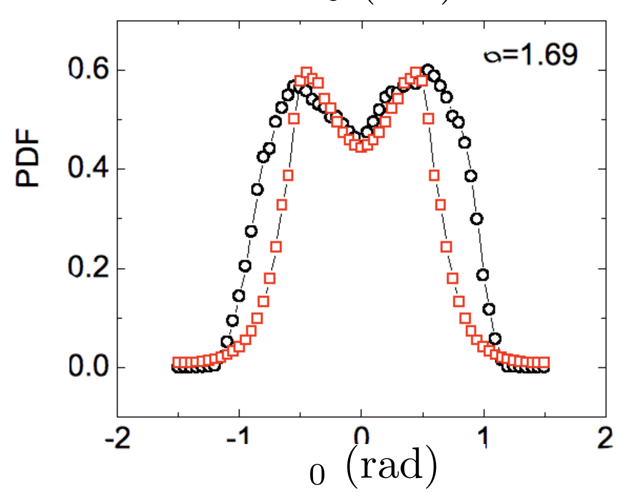
$$V_g \equiv -\int \ddot{\theta}_g d\theta = \frac{\omega_\phi^2 L}{2D(\theta_0)^2} \qquad p(\theta_0) \propto \exp\left(-\frac{V_g(\theta_0)}{\tau_{\dot{\theta}} D_{\dot{\theta}}}\right)$$











with correction for width of LSC (smooths potential near corners)

data from H. Song, P.Tong, HKUST

Conclusions

A pair of stochastic ordinary differential equations can describe the dynamics of the LSC for aspect ratio 1 cylinders:

- diffusive meandering
- cessations:
  - correspond to driving of system from stable to unstable fixed point by fluctuations
  - orientational memory loss is due to loss of rotational inertia
- Asymmetric forcings can be accounted for by adding terms to potentials:
  - Coriolis force (Brown & Ahlers, Phys. Fluids 18, 2006)
  - non-uniform heating (Brown & Ahlers, Phys. Fluids 18, 2006)
  - tilt (Brown & Ahlers, Phys. Fluids 20, 2008)
  - container geometry
- oscillations correspond to advected traveling waves with restoring force from the side wall (Brown & Ahlers, J. Fluid Mech. 638, 2009)
  - same potential as for container geometry
- Rayleigh number dependence (Brown & Ahlers, Phys. Fluids 18, 2008)