From wave (weak) turbulence to strong turbulence.
rotating, stably-stratified and other flows

Nature of Turbulence KITP, UCSB

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Outline

- Wave-turbulence with three-wave resonance.
- Rotating turbulence, inertial wave-turbulence at very small Rossby number. Moderate rotation ? Open issues.
- Turbulence in a stably stratified fluid. Waves *and* turbulence.
- Generalized Lin equations and detailed conservation laws
- Renormalizing or not the *bare* dispersion frequency. The example of weakly compressible turbulence.
- A general EDQNM context, towards strong anisotropy, strictly incompressible flows.
- Towards superfluid ? some recent results (Tchoufag & Sagaut, PoF, 2011)

exact equations for “slow” amplitudes

- Velocity in 3D Fourier space

$$\hat{\boldsymbol{u}}(\boldsymbol{k},t) = a_+(\boldsymbol{k},t)e^{\imath\omega_k t}\boldsymbol{N}(\boldsymbol{K}) + a_-(\boldsymbol{k},t)e^{-\imath\omega_k t}\boldsymbol{N}(-\boldsymbol{k})$$

Linear limit: dispersion law ω_k, eigenmodes $\boldsymbol{N}(\pm\boldsymbol{k})$, $a_\pm$ are just constant.
Nonlinear ?

- From basic NS equations

$$\dot{a}_s(\boldsymbol{k},t) = \sum_{s',s''}\int G_{kpq}^{ss's''} e^{\imath\left(s\omega_k + s'\omega_p + s''\omega_q\right)t} a_{s'}(\boldsymbol{p},t)a_{s''}(\boldsymbol{q},t)d^3\boldsymbol{p},$$

with $s, s', s'' = \pm 1, \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$.

Preliminary discussion before any additional assumption or “closure”

- The “resonant” operator, be far or close to the resonant condition (below), deeply alters the cascade, *drastically reducing* it by phase-mixing as a first effect, and rendering it *anisotropic*.
- Nonlinearity concentrates on the manifold of resonant triads, e.g. for rotating 3D turbulence

$$\omega_k = 2\Omega \frac{k_{\|}}{k}, \rightarrow \qquad \frac{k_{\|}}{k} \pm \frac{p_{\|}}{p} \pm \frac{q_{\|}}{k} = 0$$

  at arbitrarily large rotation rate, or vanishing Rossby number.
- Caveat 1 : the “slow manifold” of inertial waves $k_{\|} = 0$ is the 2D manifold (strict Taylor/Proudman theorem says no more). “slow/rapid” separation from amplitude $a_{\pm 1}(\boldsymbol{k}, \epsilon t)$to phase ω_k incorrect *close to it*. The 2D mode is not at all an “attractor” or a “condensate” ! ... artefact from DNS ?
- Caveat 2 : The “slow” amplitudes are random, not smooth, variables, so that a statistical approach is needed (see Leslie Smith for a direct DNS approach, but with special care !!)

More complex or very different cases ?

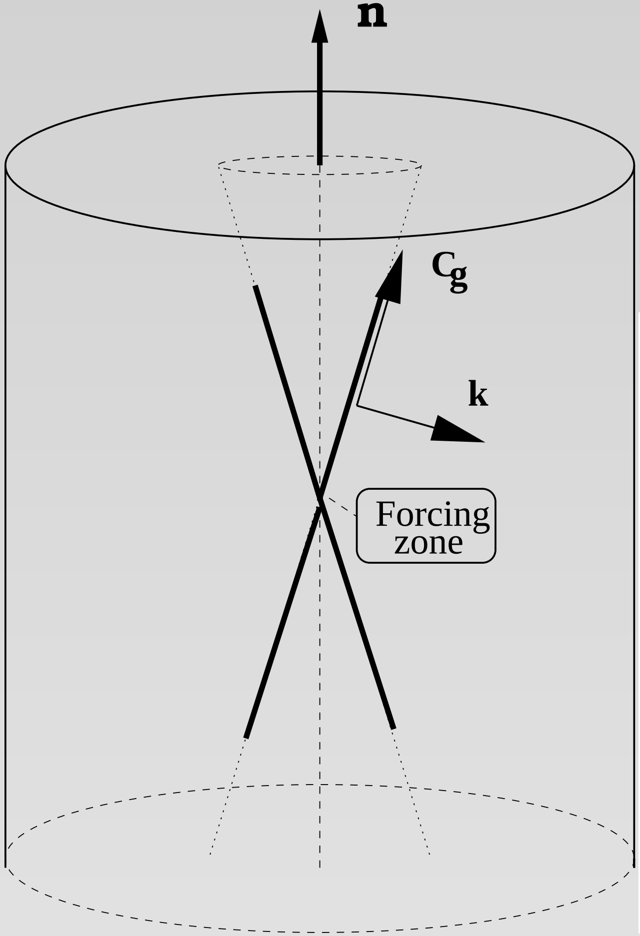
- Existence of a 3D "slow" mode, in addition to the low-dimension zero-frequency mode : solenoidal mode in weakly compressible turbulence (one slide further), toroidal (+ VSHF limit) mode in stably-stratified turbulence, QG (3D) mode in rotating and stably-stratified turbulence.

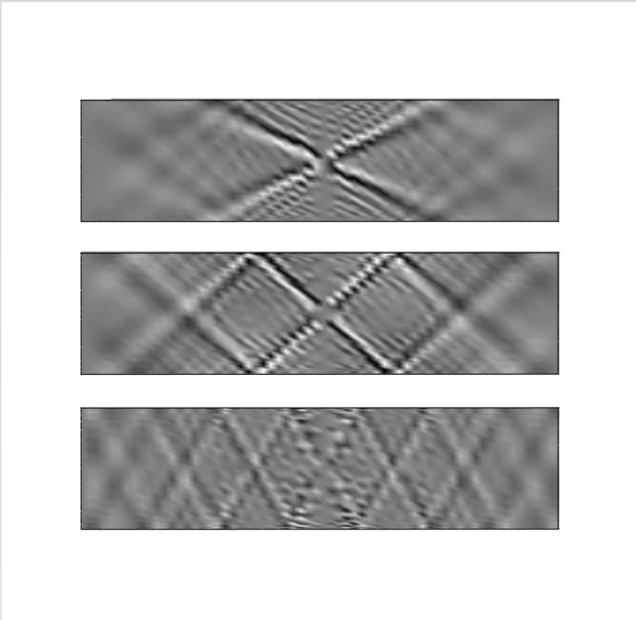
- Non-applicability of three-wave resonance: e.g. rotating stably-stratified turbulence $\omega_k = \sqrt{(2\Omega)^2 \frac{k_\parallel^2}{k^2} + N^2 \frac{\kappa_\perp^2}{k^2}}$, if $1/2 \leq |2\Omega/N| \leq 3/2 \longrightarrow$ *fourth-wave resonance* ? not to mention shallow water cases.

- Rossby waves. Are they really waves (or westward convected oscillations) ? not $\pm$ in the signed dispersion law, only first order in time dynamical equation !

- On the other hand, *non-dispersive* waves can be treated in wave turbulence theory, especially because of the $\pm$ sign of phase: e.g. Alfvén waves in Alfvénic MHD turbulence, with $\omega_k = \mathbf{V}_a \cdot \boldsymbol{k} = V_a k_\parallel$ (e.g. Galtier *et al.*, PRE,2001)

Wave aspects





Mc Ewan (1967), Mowbray & Rarity (1967), Godeferd & Lollini, JFM (1999)

From basic —isotropic —- EDQNM to anisotropic theory

for rotating turbulence

- Basic EDQNM $e(\boldsymbol{k}) = \frac{E(k)}{4\pi k^2}$,

  Lin equation $\frac{\partial E}{\partial t} + 2\nu k^2 E = T$

$$T(k) = \int\int\int \theta_{kpq} A_{kpq} e(q) \left(e(p) - e(k)\right) d^3\boldsymbol{p}$$

- EDQNM(2-3) $\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e(\boldsymbol{k}) = T^{(e)}(\boldsymbol{k})$

$$T^{(e)}(\boldsymbol{k}) = \sum_{s', s''=\pm 1} \int\int\int \frac{A(k, s'p, s''q)}{(\theta_{kpq})^{-1} + \imath(\omega_k + s'\omega_p + s''\omega_q)}$$

$$\times e(\boldsymbol{q}) \left(e(\boldsymbol{p}) - e(\boldsymbol{k})\right) d^3\boldsymbol{p}.$$

(Rediscussed further, also $e, Z, \mathcal{H}$, ED and Markovian)

From anisotropic EDQNM to Wave-Turbulence theory

- EDQNM(2-3) $T^{(e)}(\boldsymbol{k}) = \sum_{s',s''=\pm 1} \int\int\int \frac{A(k,s'p,s''q)}{(\theta_{kpq})^{-1}+\imath(\sigma_k+s'\sigma_p+s''\sigma_q)}$

$$\times e(\boldsymbol{q})\,(e(\boldsymbol{p})-e(\boldsymbol{k}))\,d^3\boldsymbol{p}.$$

- Towards wave-turbulence, AQNM, via (Sokotsky, Plemelj, Lebesgue ?)

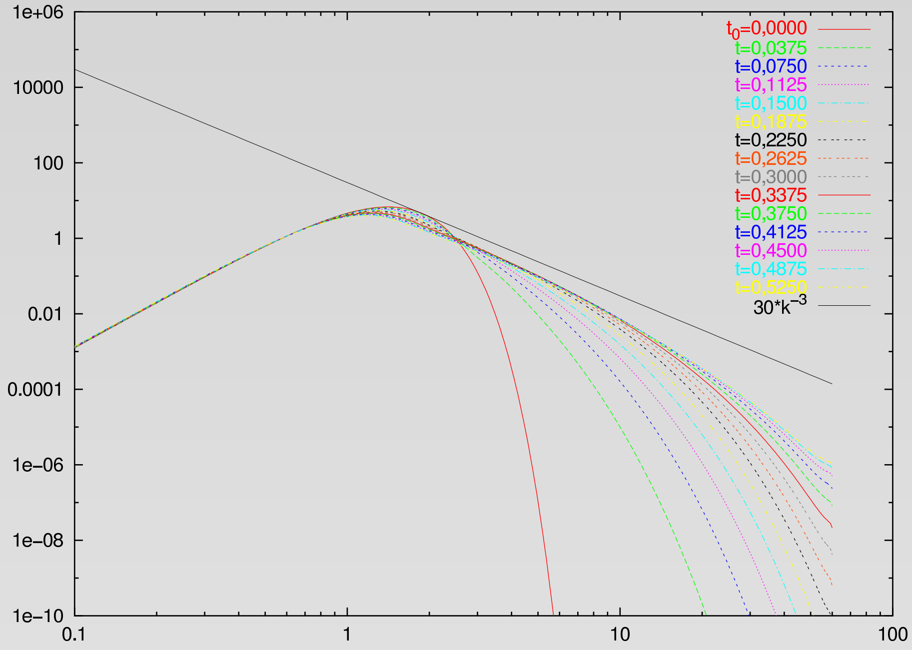
$$\frac{1}{\mu+\imath x} \rightarrow \pi\delta(x) + \mathcal{P}\left(\frac{1}{x}\right) \quad \text{when} \quad \mu \rightarrow 0$$

$$T^{(e)} = \sum_{s',s''=\pm 1} \int\int_{S_{s's''}} \pi \frac{A(k,s'p,s''q)}{s'\boldsymbol{C}_g(\boldsymbol{p}) - s''\boldsymbol{C}_g(\boldsymbol{q})}$$

$$\times e(\boldsymbol{q})\,(e(\boldsymbol{p})-e(\boldsymbol{k}))\,d^2\boldsymbol{p}$$

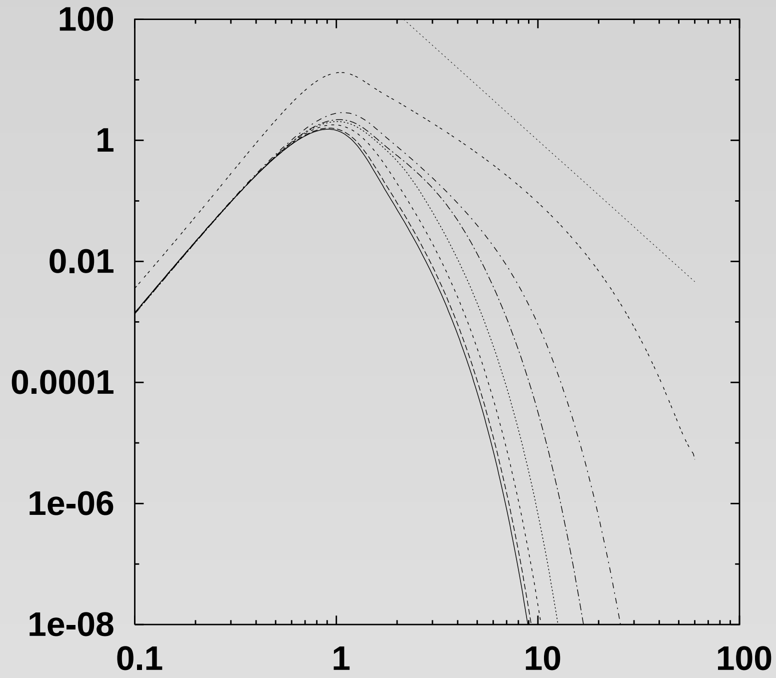
Results. Statistical theory

From classical EDQNM (isotropic, no rotation, Orszag 1970, 1976, Leith 1971) to EDQNM(3) $\longrightarrow$ (A) QNM energy equation (Bellet *et al.*, JFM, 2006)



Angle-dependent spectrum

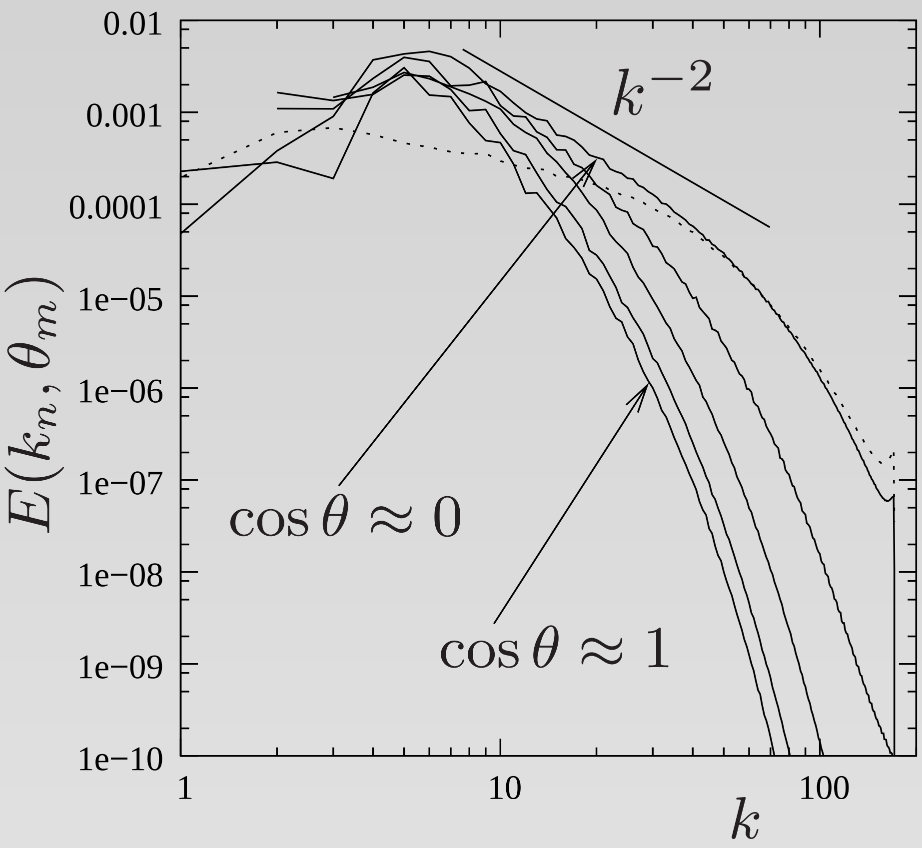
- Isotropy breaking by spectral transfer $T^{(e)}(\boldsymbol{k})$: directional anisotropy:

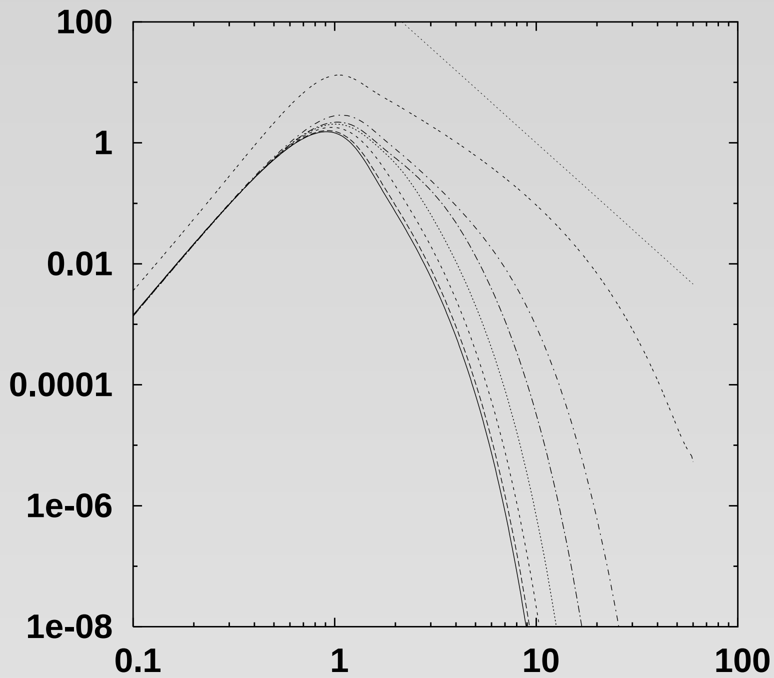


$$4\pi k^2 e(\boldsymbol{k}, t_f) = 4\pi k^2 e(k, \underbrace{\cos\theta}_{k_{||}/k}, t_f)$$

- Spherical averaging $\longrightarrow E(k, t_f)$, prefactor $E \sim \frac{\Omega}{t} k^{-3}$, not 2D !

AQNM and DNS





512^3 DNS by Liechtenstein *et al.*, JOT, 2005

More on computational aspects for AQNM

Numerical solution of inertial wave-turbulence theory equations with finite cut-off k_{max}

- Anisotropic AQNM statistical model: a difficult computation ! Accurate capture of resonant surfaces (not so simple)
- Completely different grid (vs DNS) embedded into 300^3 volume: constant step for Lnk, $\Delta k/k = constant$, polar-spherical for angles
- Any direction in $\boldsymbol{k}$-space vs. *analytical results* $e \sim k_{\|}^{-1/2} k_{\perp}^{-7/2}$ (Galtier 2003), $e \sim k_0^{-1/2} k_{\|}^{-1/2} k_{\perp}^{-3}$ (CC *et al.*, NJP, 2004), only at $k_{\perp} \ll k_{\|}$
- A new result for the radial (spherically-integrated) energy spectrum $E(k) \sim \frac{\Omega}{t} k^{-3}$, even if $k^2 e \sim k^{-2}$ at smallest $k_{\|}/k$.

Open problems, existing "vulgate"

- The answer to the Sheakespirian question, 2D or not 2D ... is not so simple. Contradictory alledgments by, e.g. P. Bartello (from DNS ☹), semi-empirical analysis using CB (Nazarenko & Schekochihin, JFM), ... Of course inertial wave turbulence *disagrees* with complete 2D, because of the integrable singularity, $k_{\|}^{-1/2}$ vs. Dirac $\delta(k_{\|})$

- Results of a vulgate: $E \sim \sqrt{\Omega\varepsilon}k^{-2}$, with new partly anisotropic form $E_{\perp}(k_{\perp}) \sim k_{\perp}^{-2}$ (Mininni *et al.*, CB by N-S above)

- What about the moderate Rossby range ? Interest of an "Oszmidov"-type wavenumber cutoff $k_{ri} \sim \left(\varepsilon\Omega^{-3}\right)^{-1/2}$, with re-isotropization and $k^{-5/3}$ recovered

- We tried to match all that with a generalized EDQNM(3) fully axisym model/code absorbing the A (asymptotic) EDQNM limit, valid at any Rossby number.

Realizability, renormalization of “bare” dispersion law from a dynamical ED $\longrightarrow$ John Bowman. $e^{3\nu(t-t')}$ (symbolic) $\rightarrow < \tilde{G} > (t,t')$ in the cubic two-time term

The (different ? hope complementary !) strategy in my team:

$$\int_{t_0}^{t} V(t,t'')G^0(\boldsymbol{k},t,t'')G^0(\boldsymbol{p},t,t'')G^0(\boldsymbol{q},t,t'')S^{QN}(\boldsymbol{k},\boldsymbol{p},t'')dt'',$$

as the cubic *single-time term.*

-) with G the true basic, possibly tensorial and breaking anisotropy, Green’s function, e.g. $G_{ij}(\boldsymbol{k},t,t'') = \Re\left(N_i N_j^* e^{2\imath\Omega(k_\parallel/k)(t-t')}\right)$, “RDT”.

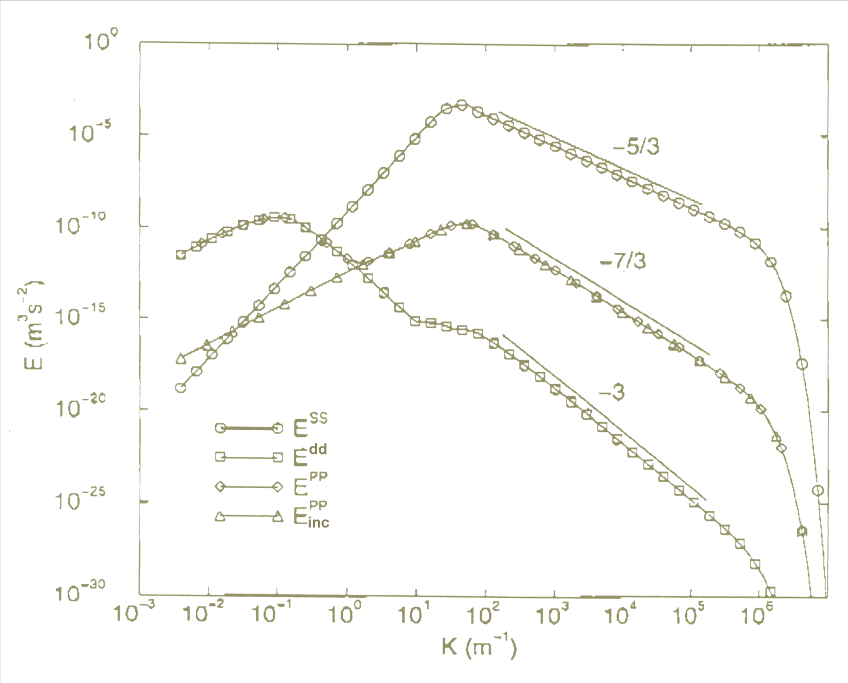
-) S^{QN} exact and quadratic in terms of $e, Z, \mathcal{H}$. No special care for renornalizing the “ED” in $V(t,t')$

Three versions: EDQNM(1) $t=t''$ in both G and S^{QN}; EDQNM(2) $t=t''$ only in Ω^{QN}; EDQNM(3) $t=t''$ only in the slow part of $e, Z(t'') = Z_{slow}(t)e^{4\imath\Omega(k_\parallel/k)t''}, \mathcal{H}$. Consistent with the most general “rapid/slow” decomposition, recovering *exact realizable* three-wave turbulence theory at any unimportant V.

What about anomalous exponents ? other open issues

- A “soft” concensus $< (\delta u)^n >\sim (\varepsilon r)^{n/3}$ (K41) should become $< (\delta u)^n >\sim r^{n/2}$
- e.g. from oversimplified (wrong ?) isotropized $E(k) \sim k^{-2} \rightarrow < (\delta u)^2 > (r) \sim r$, but nothing serious about a modified $4/5$ law, except perhaps in the team of F. Moisy (excellent experimentalist, accepting to look at theory)
- nothing serious from empirical approach to “anomalous exponent” and classical pontificating about ESS ...
- Boundary effects, Ekman pumping, complex shear layer with both jettal and vortical structures

An example of strong renormalization needed: weakly compressible turbulence at low Mach number.



Stably-stratified turbulence with and without rotation

Rotating stratified equations: governing equations

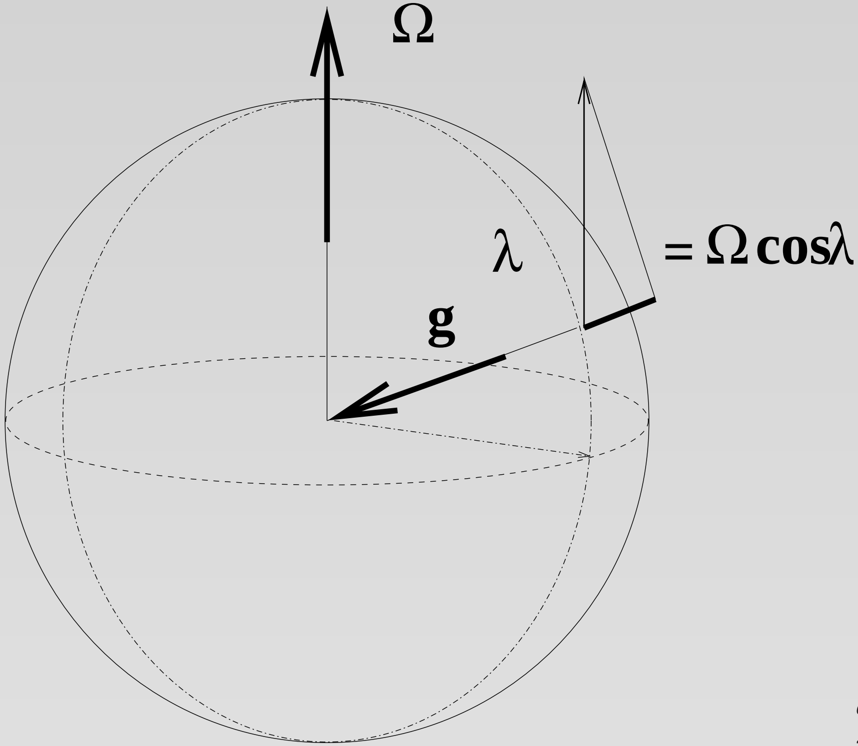
$$\frac{\partial u_i}{\partial t} + \underbrace{f \epsilon_{i3j} u_j}_{\text{Coriolis}} - \underbrace{b \delta_{i3}}_{\text{buoyancy}} + \frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}, \quad \frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial b}{\partial t} + \underbrace{N^2 u_3}_{\text{stratification}} = P_r \nu \nabla^2 b - u_j \frac{\partial b}{\partial x_j}$$

2 external parameters N and f (frequencies)

Valid for a liquid or a gas. P_r characterizes the diffusivity of the stratifying agent (temperature, salt)

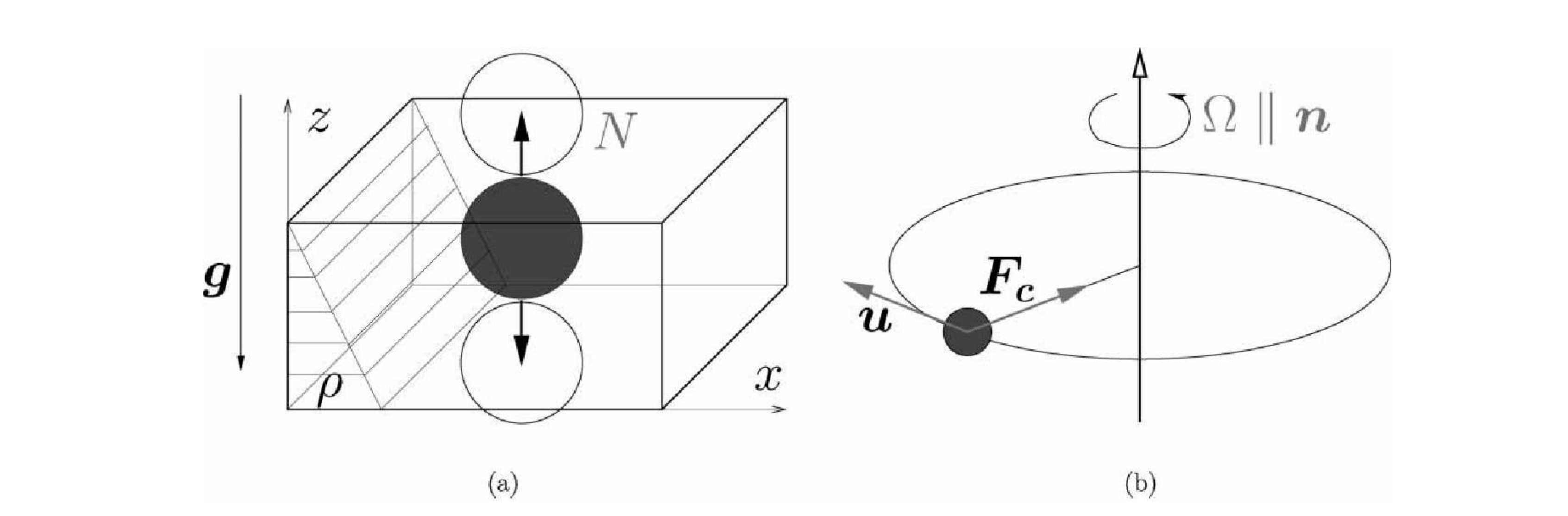
Geophysical context



$$2\Omega \rightarrow f = 2\Omega \cos \lambda$$

Stable stratification in ocean (under the mixed zone) and in atmosphere (temporary inversion in troposphere, stratosphere)

A first cartoon of linear effects



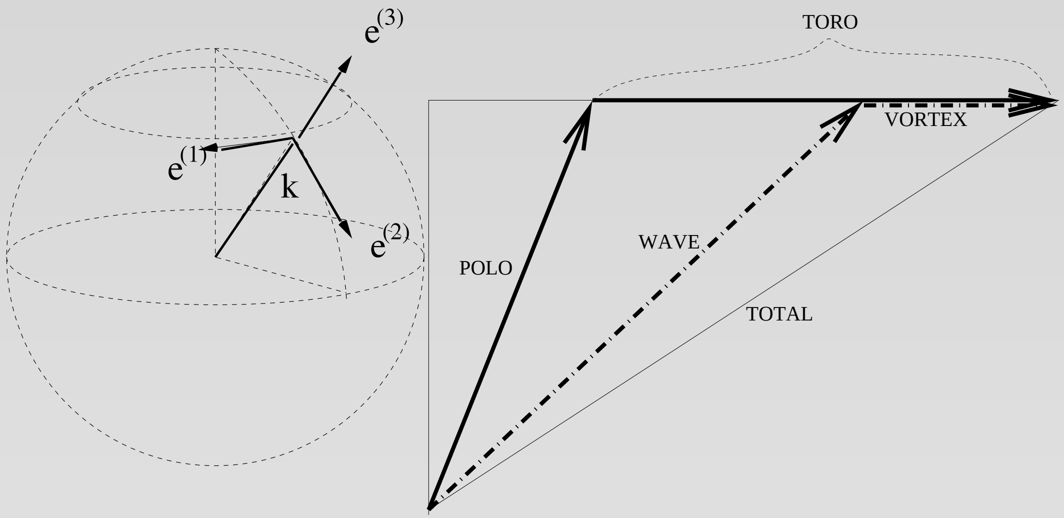
Fluctuating pressure ? incompressibility ?

Eigenmodes decomposition

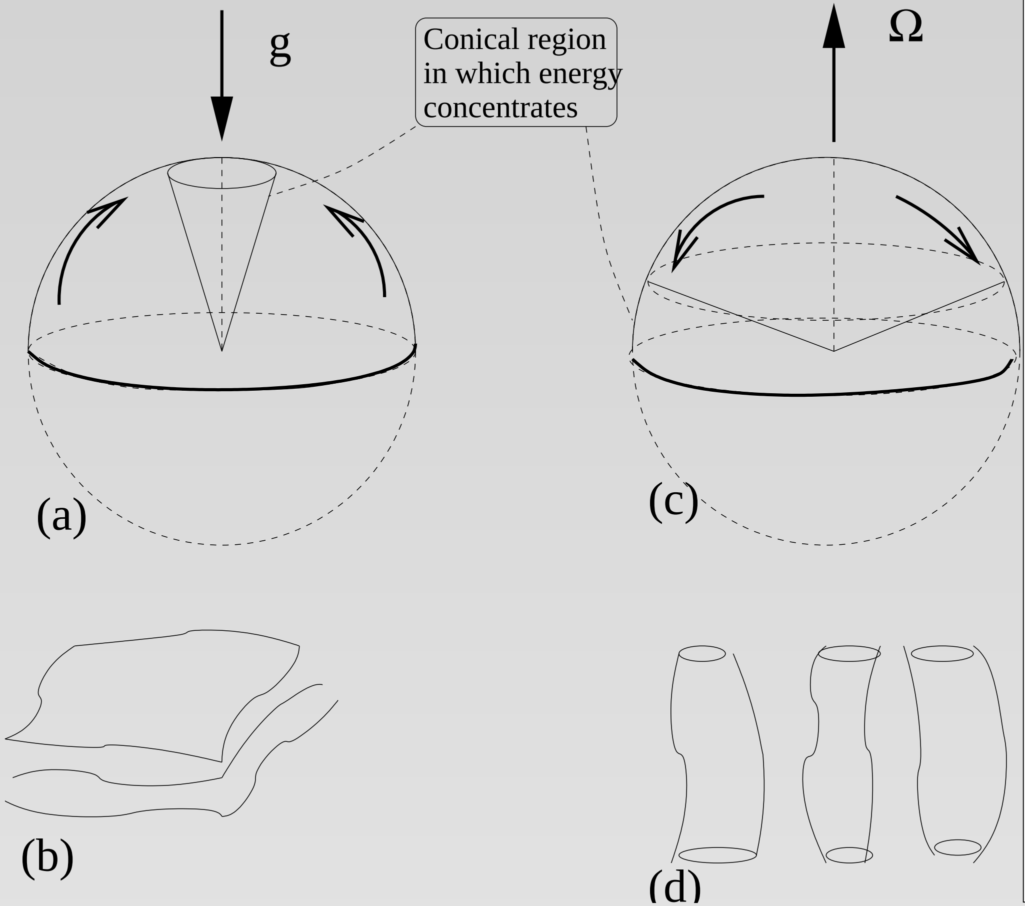
- incompressibility and pressure $\longrightarrow$ 3D Fourier space
$(\boldsymbol{u}, b)(\boldsymbol{x}, t) =$
$$\sum e^{\imath \boldsymbol{k}\cdot\boldsymbol{x}} \left(\underbrace{a_0 \boldsymbol{N}^{(0)}}_{\text{vortex (QG)}} + \underbrace{a_{+1}\boldsymbol{N}^{(1)} e^{\imath\sigma_k t} + a_{-1}\boldsymbol{N}^{(-1)} e^{-\imath\sigma_k t}}_{\text{wave (AG)}} \right)$$
- Dispersion law $\sigma_k = \sqrt{N^2 \sin^2\theta + f^2\cos^2\theta}$
- Linear dynamics: slow amplitudes $a_{0,\pm 1}$ are constant. General case
- Advantages $\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$, five $(u_1, u_2, u_3, b, p) \longrightarrow$ three (a_0, a_{+1}, a_{-1}).

ref. Cambon, Bartello, Smith & Waleffe, Morinishi, Kaneda ...etc

Geometric aspects of eigenmodes



Toroidal/poloidal (Craya-Herring, standard) and “Vortex/wave” ($f/N-$ depending)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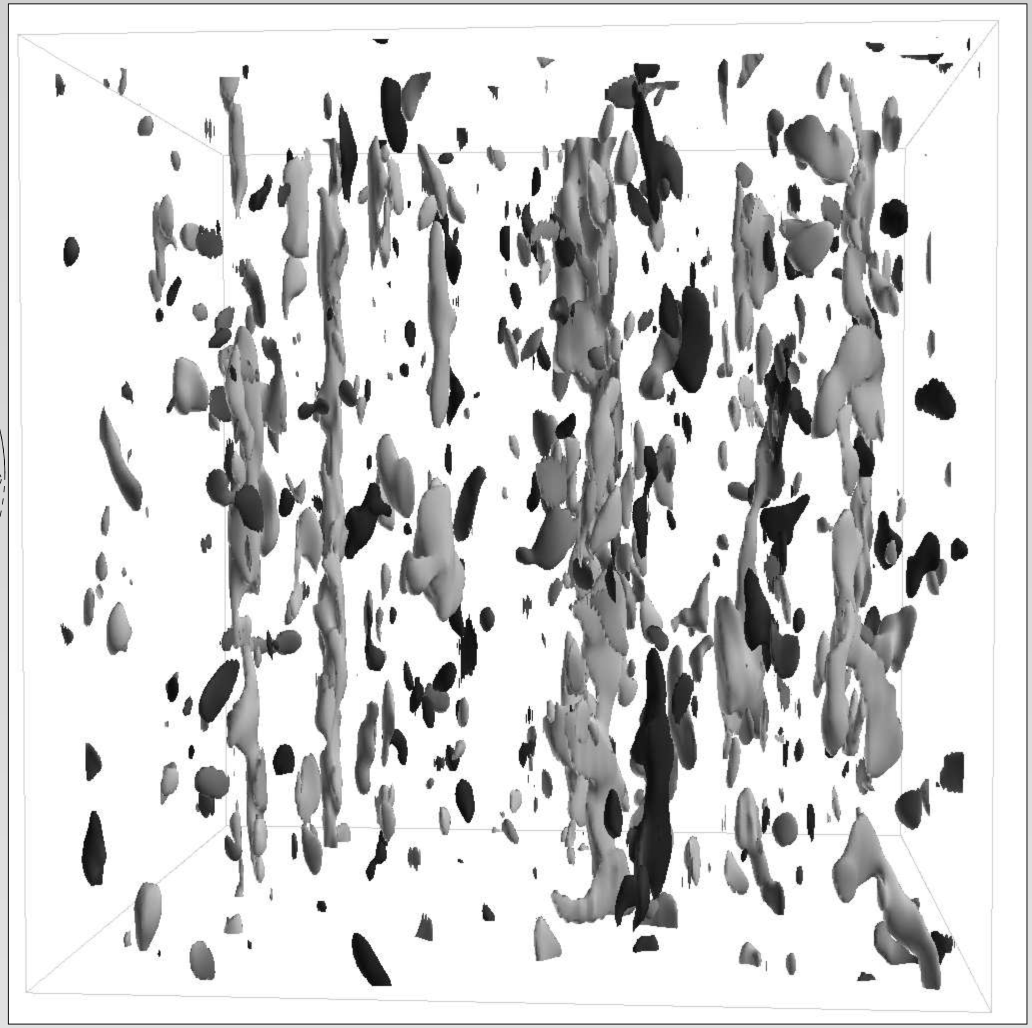
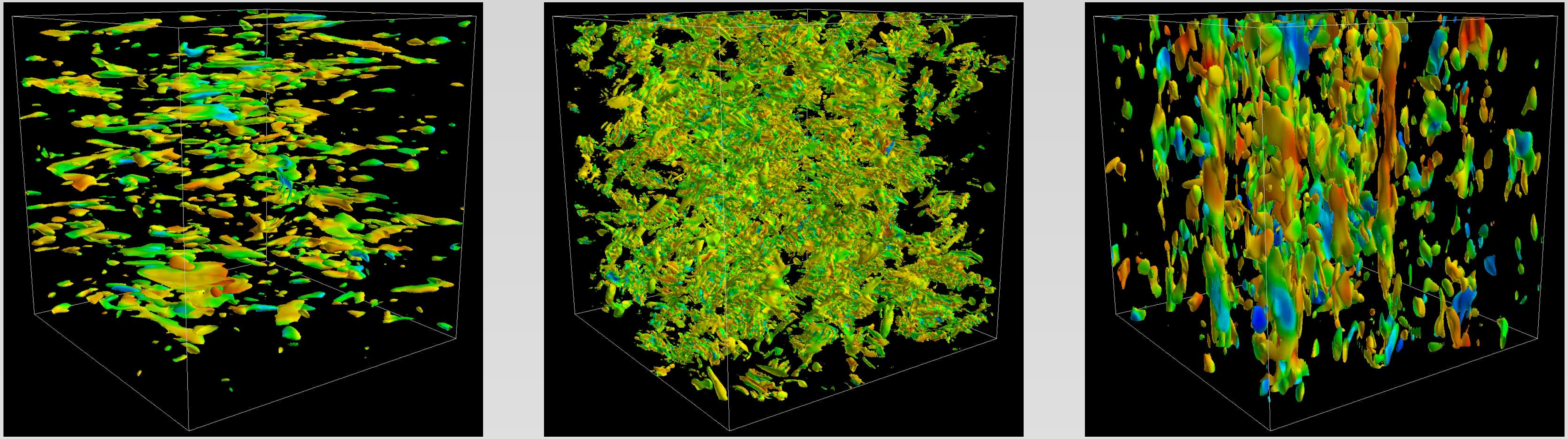
Conical region
in which energy
concentrates
Ω
(a)
(c)
(b)
(d)



512^3 DNS, Liechtenstein et al. 2005, without mean shear



STRATIFIED $2\Omega = f = N$ ROTATING

Main difference vs. conventional wave turbulence

- Role of the toroidal "strong" cascade to concentrate energy towards vertical wavevectors, in agreement with layering in physical space
- Once the concentration is well advanced, gravity wave-turbulence is of poor interest, because only "slow" waves are involved, given $\omega_k = Nk_\perp/k$, together with renormalized toroidal mode, in the VSHF vicinity.
- A stereoscopic, 3D view of anisotropic distribution, permitted in 3D Fourier space only (because $\boldsymbol{k} \cdot \hat{\boldsymbol{u}} = 0$)
- Completely different view, vs. Lindborg, Chomaz, Brethouwer and co.
- How to interpret "physically" the toroidal cascade ?

Systematic use of generalized Lin equation for anisotropic turbulence, and useful detailed (triad by triad) conservation laws

- "Pure" rotation. $a_+ a_+^* + a_- a_-^* \longrightarrow e$, $a_+ a_+^* - a_- a_-^* \longrightarrow \mathcal{H}$, $a_+ a_- \longrightarrow Z e^{2\imath\omega_k t}$, the set (e, Z, H) generates any tensorial covariance matrix for second-order statistics. Related conservation law (in terms of helical modes) of both energy and *helicity*

$$\dot{\xi}_s(\boldsymbol{k}) = G_{kpq}(s'p - s''q)\xi^*_{s'}(\boldsymbol{p})\xi^*_{s''}(\boldsymbol{q})$$

(Waleffe 1992, 1993), to compare to the well known 2D case $p^2 - q^2$ (Fiortjof, Kraichnan)

- Stably-stratified turbulence: Two energy spectra $e^{(tor)}$, $e^W = e^{(pol)+(pot)}$, Z' (imbalance)

$$\dot{u}^{(1)}(\boldsymbol{k}) = G'_{kpq}(p_\perp^2 - q_\perp^2)u^{(1)*}(\boldsymbol{p})u^{(1)*}(\boldsymbol{q})$$

at small Froude number.

- Rotating stably-stratified turbulence, idem replacing toroidal by QG, gravity waves by inertia-gravity waves, and p^2 by ω_p^2.

- Alfvénic MHD: One energy spectrum, (kinetic + magnetic), + helicity spectrum + 8 imbalance terms, including 2 “polarization” ones, the spectrum of magnetic helicity and the spectrum of the electromotive force.

advertising (generalized) EDQNM

- EDQNM (including fully anisotropic multimodal for me) is not popular and underestimated: too complex and not enough flexible for enginers, not “sexy” for physicisists (no obvious phenomenology, no intermittency (?), sometime considered as a degraded byproduct of DIA (?), etc ...)
- The basic closure assumption is simple, but at the complex level of triadic fourth-order terms

$$T^{IV}_{ss's''}(\boldsymbol{k},\boldsymbol{p},t) - T^{IV,QN}_{ss's''} = -(\eta^{ED}(k,t) + \eta^{ED}(p,t) - \eta^{ED}(\boldsymbol{q},t))T^{IV}_{ss's''}(\boldsymbol{k},\boldsymbol{p},t$$

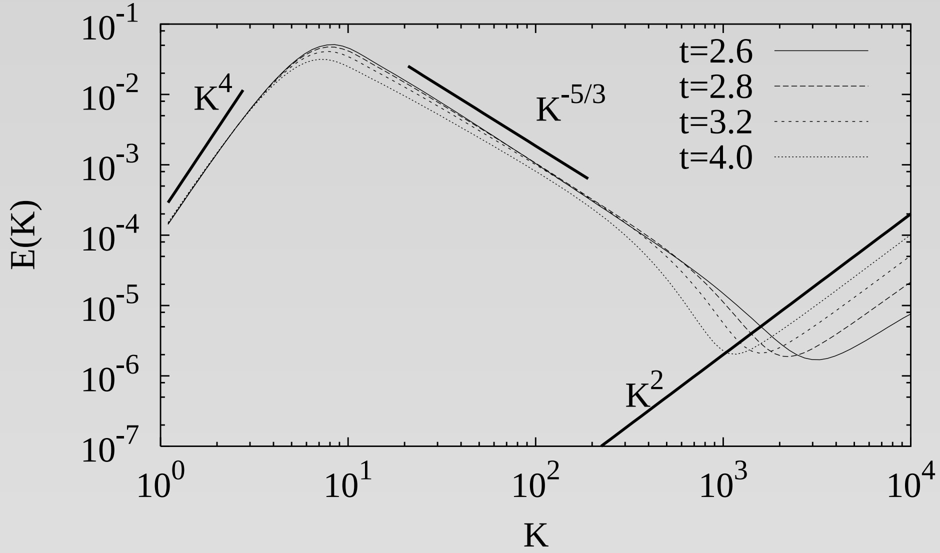
- Retaining the maximum of “true” linear operators

$$\dot{\xi}_s(\boldsymbol{k}) + \nu k^2 \xi_s(\boldsymbol{k}) + M_{ss'}\xi_{s'}(\boldsymbol{k}) = \int G_{ss's''}\xi_{s'}(\boldsymbol{p})\xi_{s''}(\boldsymbol{q})d^3\boldsymbol{p} \quad \rightarrow$$

$$\dot{S}_{ss's''} + \nu(k^2+p^2+q^2) + M_{ss_k}(\boldsymbol{k})S_{s_k s' s''} + M_{s's_p}(\boldsymbol{k})S_{ss_p s''} + M_{ss_q}(\boldsymbol{k})S_{ss's_q}$$

$$= T^{IV}_{ss's''}$$

Even basic isotropic EDQNM ...



(Bos & Bertoglio, 2006) immediately following

Brachet (Chicowlas *et al.*)

Even basic isotropic EDQNM ... cont

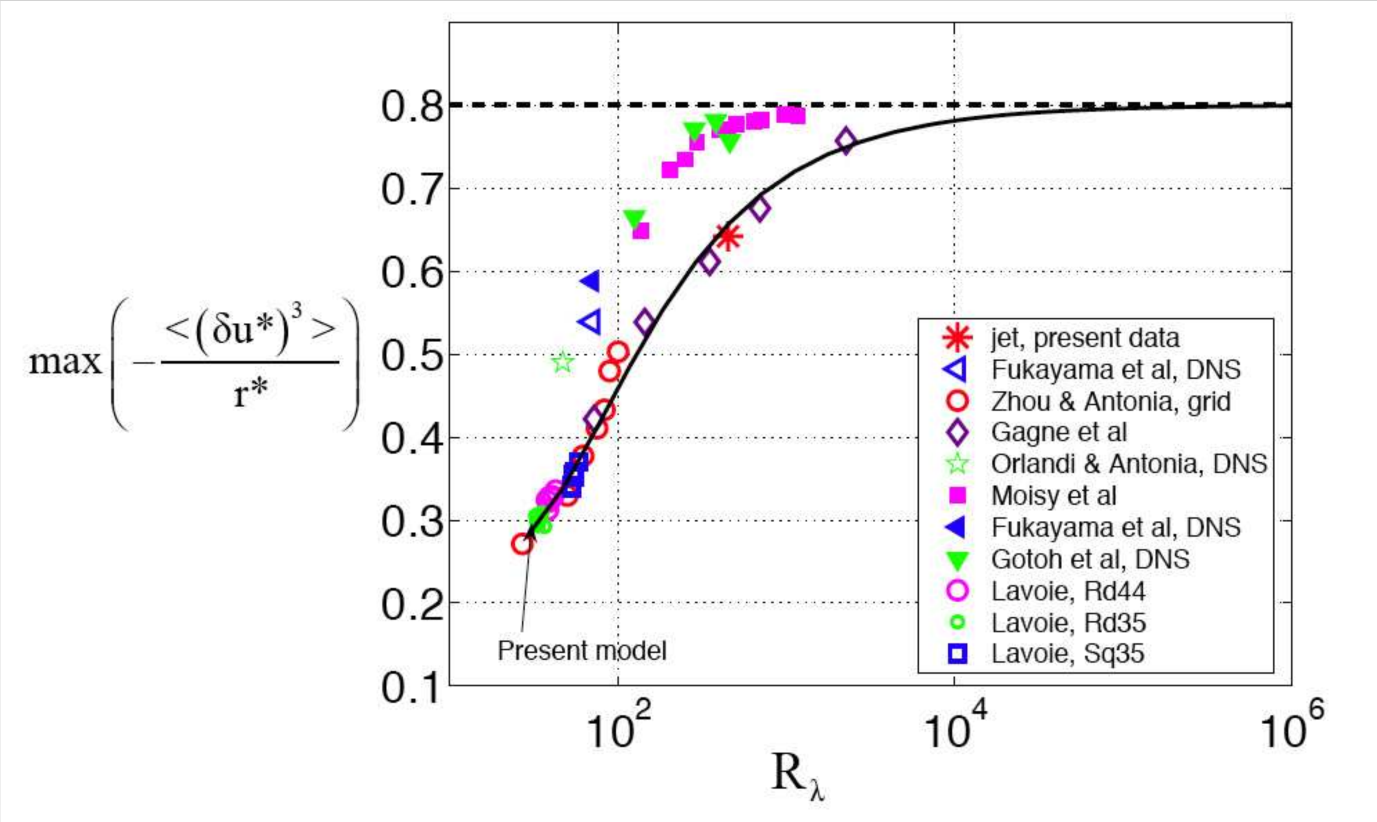
- Physical space

$$\underbrace{\frac{\partial f}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right)\left(R_{LL,L} + 2\nu\frac{\partial f}{\partial r}\right)}_{\text{Kàrmàn-Howarth (1938)}} \quad \rightarrow \underbrace{\langle(\delta u_L)^3\rangle = -\frac{4}{5}\varepsilon r}_{\text{(4/5) Kolmo 41}}$$

- Fourier space

$$\underbrace{\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k,t) = T(k,t)}_{\text{Lin equation (1949)}} \quad \rightarrow \quad \int_k^\infty T(p)dp = \varepsilon$$

Useful in BOTH 'weak' wave-turbulence (Kinetic equations) and strong turb.



Antonia (Cargèse 2007)

HIT, ordre 2 et 3

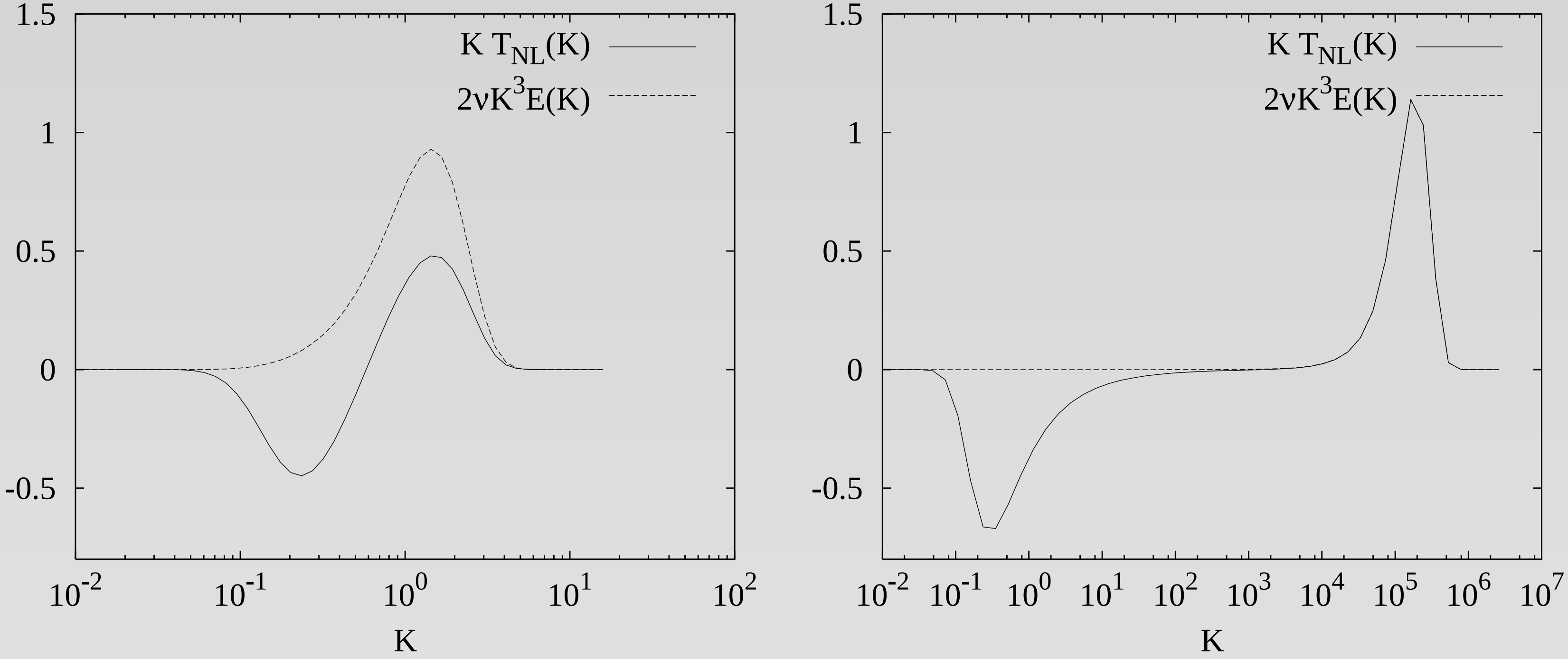
- All second-order (single- and two-point) statistics is derived from $E(k,t)$
- All third-order (single and two-point) statistics is derived from T(k,t)

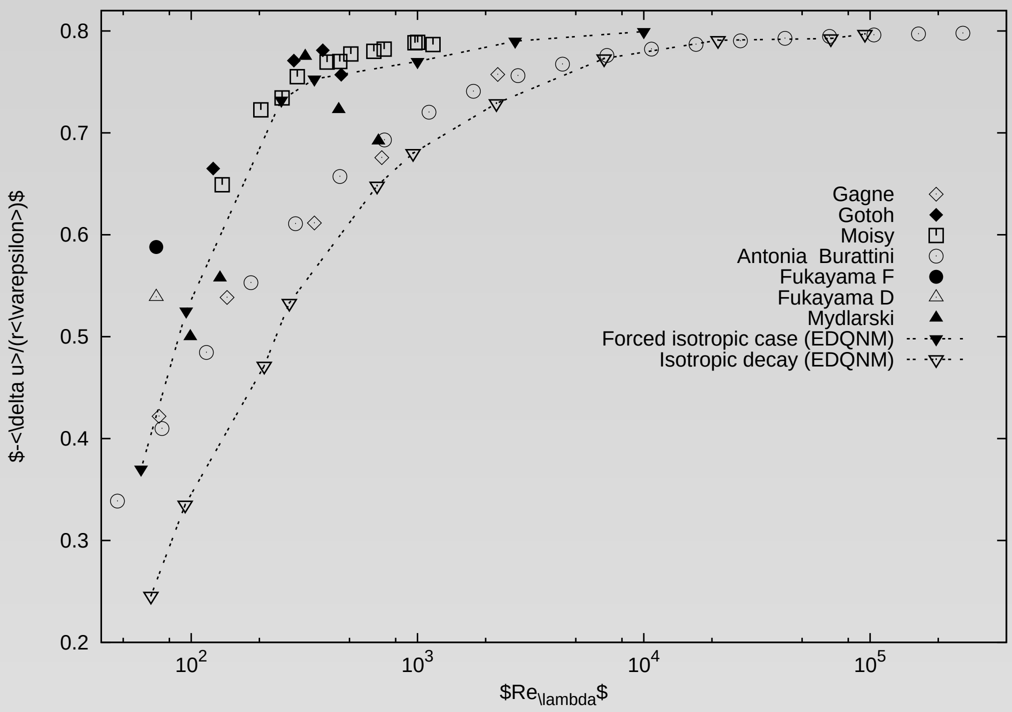
$$\langle (\delta u_L)^3 \rangle = 12r \int_0^\infty g_5(kr) T(k) dk$$

$$g_5(kr) = \frac{3(\sin kr - kr \cos kr) - (kr)^2 \sin kr}{(kr)^5} \rightarrow \frac{1}{15} - \frac{1}{210}(kr)^2$$

Recovering K41, among other things, not a scoop (Brachet & Frisch, Quian) but work remains to be done

Robust closures for the transfer term, from small ($R_\lambda = 30$) to huge ($R_\lambda = 10^5$) Reynolds number.





Fit by Burattini and Antonia. Recent (unpublished) EDQNM calculation by Joel Tchoufag.

What about the superfluid ?

J. Tchoufag & P. Sagaut, *Phys. Fluids* **22**, 125103 (2010)

- Two coupled Lin equation for the “normal” and the “superfluid” energy spectra
- A crucial interaction term, reflecting mutual friction, in agreement with L’vov, Nazarenko & Sbrbek, JLTP, 2006.
- Very interesting results at “high” and “low” temperature, with “weak” and “strong” mutual friction term.

$$\left(\frac{\partial}{\partial t} + 2\nu_n k^2\right) E_n(k,t) = T_n(k,t) + F_{ns}(k,t)$$

$$\frac{\partial E_s}{\partial t}(k,t) = T_s + F_{sn}(k,t)$$

Key points for the intercomponent energy transfers

$$F_{ns}(k,t) = \alpha_n \omega_0 \left(\frac{\omega_0}{\mu_k}(\alpha_s E_n + \alpha_n E_s) - E_n\right)$$

$$F_{sn}(k,t) = \alpha_n \omega_0 \left(\frac{\omega_0}{\mu_k}(\alpha_s E_n + \alpha_n E_s) - E_s\right)$$

with $\mu_k = (\nu_n + \nu_s)k^2 + \eta_n(k,t) + \eta_s(k,t) + (\alpha_s + \alpha_n)\omega_0$ system total relaxation time (inverse), $\omega_0 = k\ell^{-2}$ being the characteristic vorticity of the superfluid vortex tangle. $\eta = A\sqrt{\int_0^k p^2 E(p,t)dp}$.

- Wave-turbulence theory can be matched with generalized EDQNM
- Anisotropy is essential and requires a straightforward description, spectra in terms of two wavenumber components, directional and polarization anisotropy
- Results from second-order and cubic terms, in 3D Fourier space, have to be used to derive measurable (or fashionable) statistics (Reynolds stress components, various anisotropic length scales, structure functions up to the third order
- Towards fourth-order correlations, multi-point flatness terms ? (Bos & Rubinstein, in progress)