Spectral Reduction: A Statistical Description of Turbulence

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2D Turbulence

- 2D Navier–Stokes vorticity equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q} \, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}},$$

  where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ and

$$\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q}) \, \delta(\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q})$$

  is antisymmetric under permutation of any two indices.

- Energy E_0 and enstrophy Z_0 on the fine grid:

$$E_0 \doteq \frac{1}{2} \int d\boldsymbol{k} \, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2}, \qquad Z_0 \doteq \frac{1}{2} \int d\boldsymbol{k} \, |\omega_{\boldsymbol{k}}|^2 .$$

- First consider $\nu_{\boldsymbol{k}} = 0$. Conservation of E_0 and Z_0 follow from:

$$\frac{1}{k^2} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \qquad \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{q},$$

$$\frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \qquad \text{antisymmetric in} \qquad \boldsymbol{k} \leftrightarrow \boldsymbol{p}.$$

Spectral Reduction

- Introduce a coarse-grained grid indexed by K.
- Define new variables

$$\Omega_{\boldsymbol{K}} = \langle \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{\Delta_{\boldsymbol{K}}} \int_{\Delta_K} \omega_{\boldsymbol{k}} \, d\boldsymbol{k},$$

where $\Delta_{\boldsymbol{K}}$ is the area of bin $\boldsymbol{K}$.

- Evolution of $\Omega_{\boldsymbol{K}}$:

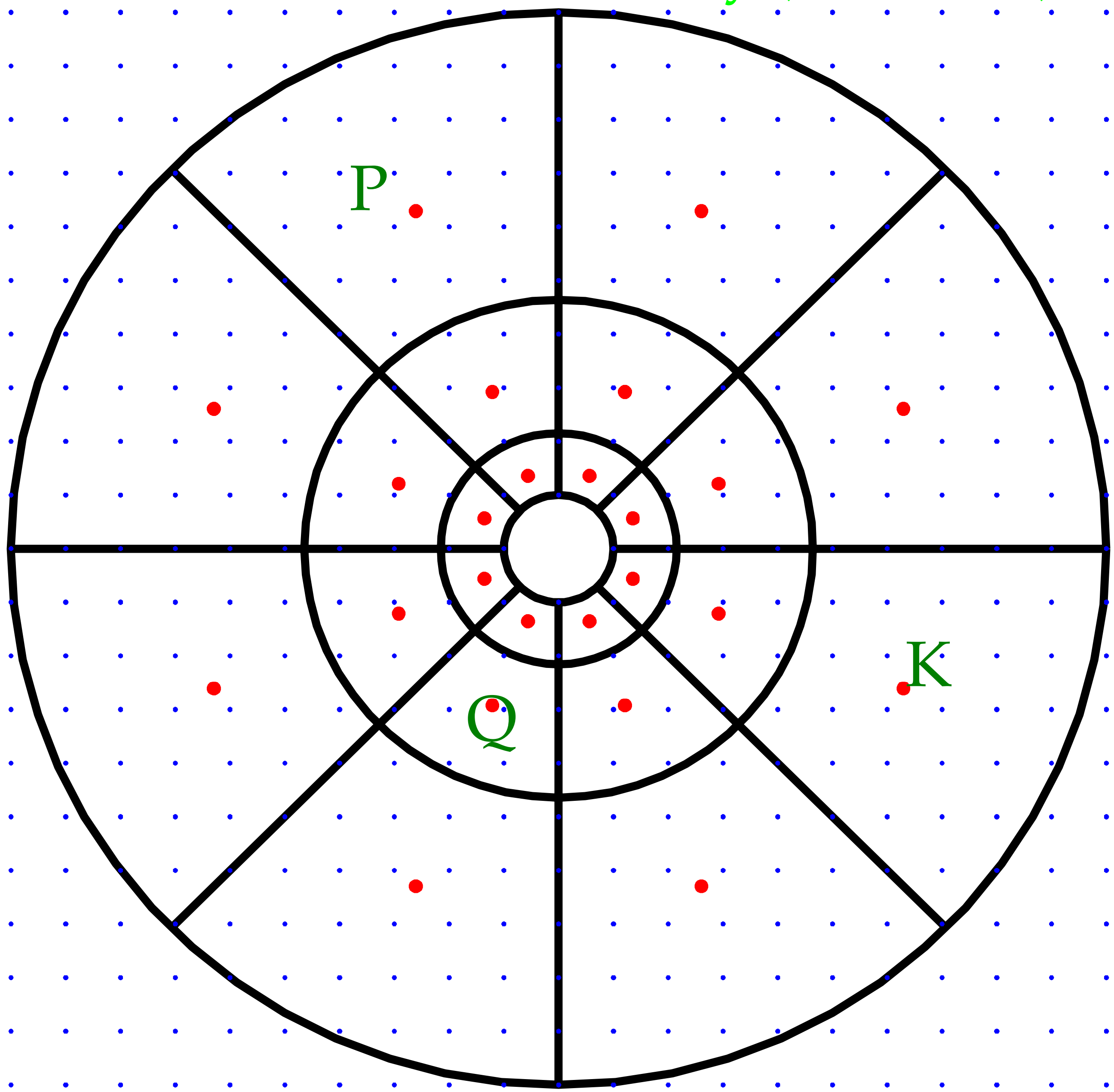
$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* \right\rangle_{\boldsymbol{KPQ}},$$

where $\langle f \rangle_{\boldsymbol{KPQ}} = \dfrac{1}{\Delta_{\boldsymbol{K}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}}} \displaystyle\int_{\Delta_K} d\boldsymbol{k} \int_{\Delta_P} d\boldsymbol{p} \int_{\Delta_Q} d\boldsymbol{q} \, f.$

- Approximate $\omega_{\boldsymbol{p}}$ and $\omega_{\boldsymbol{q}}$ by bin-averaged values $\Omega_{\boldsymbol{P}}$ and $\Omega_{\boldsymbol{Q}}$:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^*.$$

Wavenumber Bin Geometry (3 x 8 bins)



- On the coarse grid, define the energy E and enstrophy Z

$$E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2}\Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2\Delta_{\boldsymbol{K}}.$$

- Enstrophy is still conserved since

$$\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{P}.$$

- But energy conservation has been lost!

$$\frac{1}{K^2}\left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \qquad \text{NOT antisymmetric in} \qquad \boldsymbol{K} \leftrightarrow \boldsymbol{Q}.$$

- Reinstate both desired symmetries with the modified coefficient

$$\frac{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}}}{Q^2}.$$

- Energy and enstrophy are now simultaneously conserved.

Properties

- We call the forced-dissipative version of this approximation *Spectral Reduction* (SR):

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \left\langle \nu_{\boldsymbol{k}} \right\rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}}}{Q^2} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}.$$

- SR conserves both energy and enstrophy and reduces to the exact dynamics in the limit of small bin size.
- It has the same general structure and symmetries as the original equation and in this sense may be considered a *renormalization*.
- SR obeys a Liouville Theorem; in the inviscid limit, it yields statistical-mechanical (equipartition) solutions.

Moments

- Q. How accurate is Spectral Reduction?
- A. For large bins, the *instantaneous* dynamics of SR is inaccurate.
- However: the equations for the *time-averaged* (or ensemble-averaged) moments predicted by SR closely approximate those of the exact bin-averaged statistics.
  Eg., time average the exact bin-averaged enstrophy equation:

$$\overline{\frac{\partial}{\partial t}\left\langle |\omega_{\boldsymbol{k}}|^2 \right\rangle_{\boldsymbol{K}}} + 2\,\mathrm{Re}\left\langle \nu_{\boldsymbol{k}} \overline{|\omega_{\boldsymbol{k}}|^2} \right\rangle_{\boldsymbol{K}} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \overline{\omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*} \right\rangle_{\boldsymbol{KPQ}},$$

  where the bar means time average and $\langle \cdot \rangle_{\boldsymbol{K}}$ means bin average.

- Time-averaged quantities such as $\overline{|\omega_{\boldsymbol{k}}|^2}$ and $\overline{\omega_{\boldsymbol{k}}^* \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^*}$ are generally *smooth* functions of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ on the four-dimensional surface defined by the triad condition $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$.

- Mean Value Theorem for integrals: for some $\boldsymbol{\xi} \in \boldsymbol{K}$,

$$\overline{|\Omega_{\boldsymbol{K}}|^2} = \overline{|\omega_{\boldsymbol{\xi}}|^2} \approx \overline{|\omega_{\boldsymbol{k}}|^2} \qquad \forall \boldsymbol{k} \in \boldsymbol{K}.$$

- To good accuracy these statistical moments may therefore be evaluated at the characteristic wavenumbers $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{Q}$:

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \right\rangle_{\boldsymbol{KPQ}} \overline{\Omega^*_{\boldsymbol{K}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}}.$$

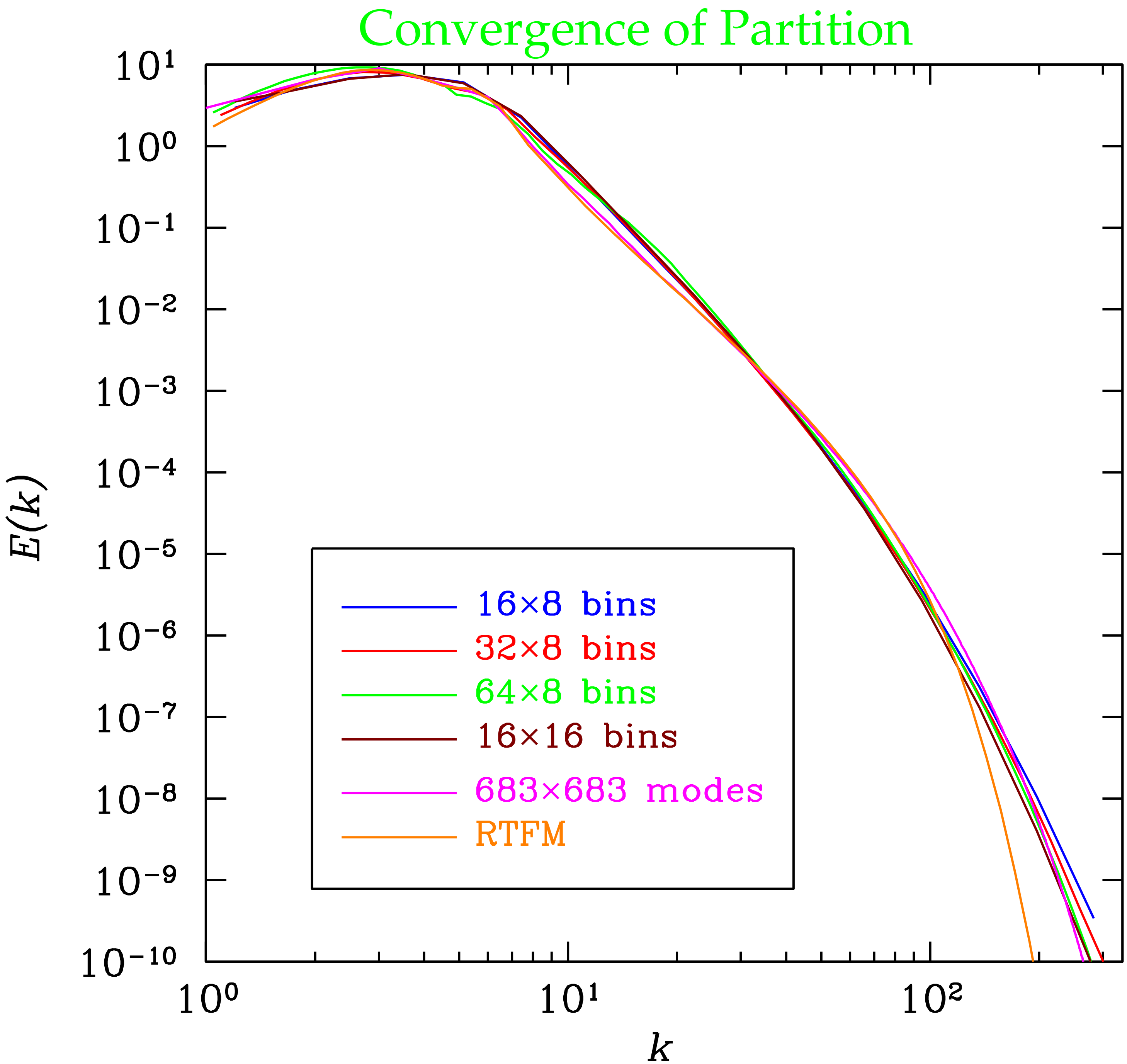
To the extent that the wavenumber magnitude q varies slowly over a bin:

$$\overline{\frac{\partial}{\partial t}|\Omega_{\boldsymbol{K}}|^2} + 2\,\mathrm{Re}\,\langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \overline{|\Omega_{\boldsymbol{K}}|^2} = 2\,\mathrm{Re} \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}} \Delta_{\boldsymbol{Q}} \frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2} \overline{\Omega^*_{\boldsymbol{K}} \Omega^*_{\boldsymbol{P}} \Omega^*_{\boldsymbol{Q}}}.$$

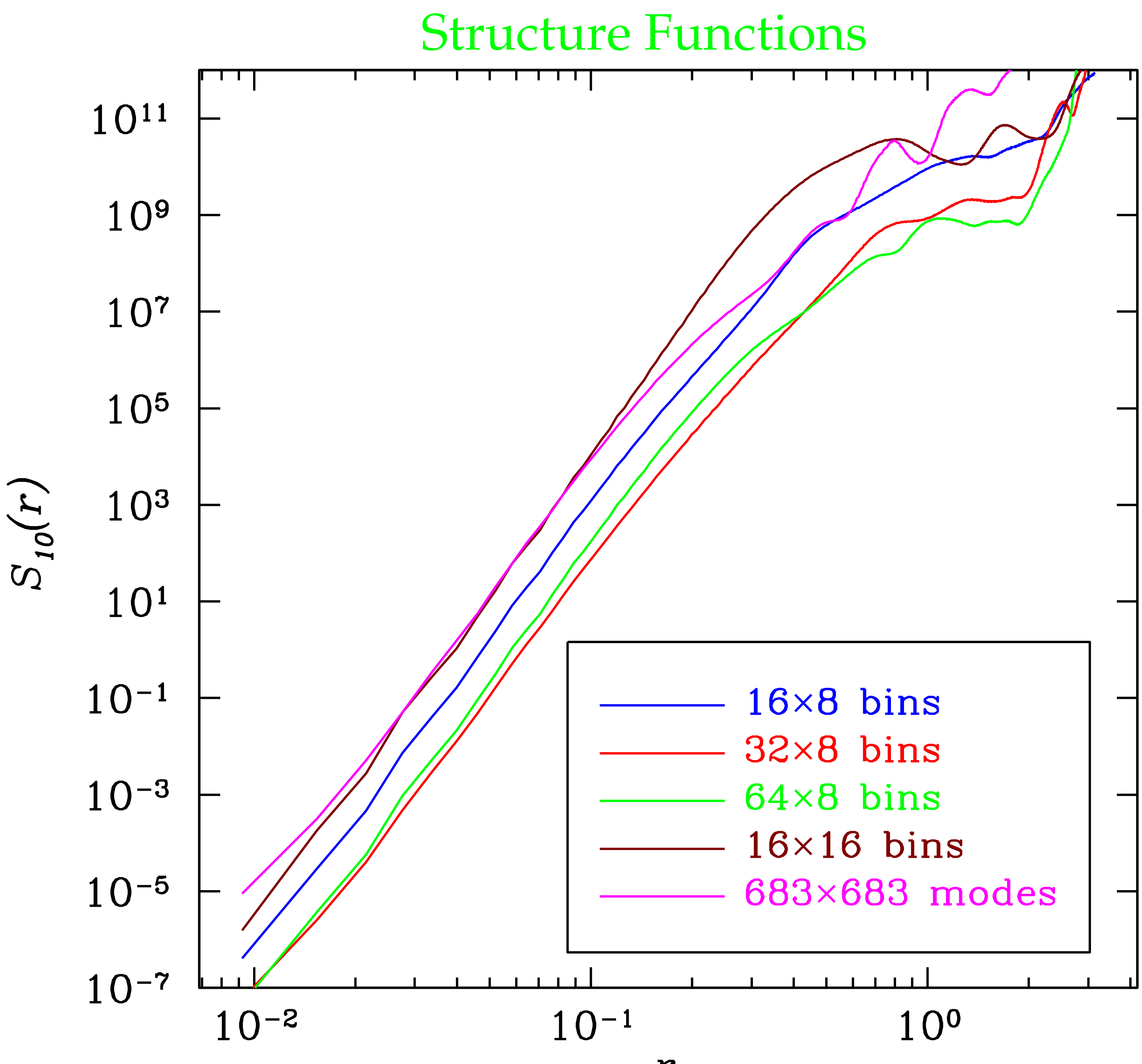
- But this is precisely the time-average of the SR equation!

Convergence

- The previous argument suggests that Spectral Reduction can indeed provide an accurate statistical description of turbulence, even when each bin contains many statistically independent modes.
- As the wavenumber partition is refined, one expects the solutions of the time-averaged SR equations to converge to the exact statistical solution.
- An object-oriented C^{++} program `(Triad)` has been developed to implement and test Spectral Reduction.



Convergence of Partition
10^1
10^0
10^−1
10^−2
10^−3
10^−4
10^−5
10^−6
10^−7
10^−8
10^−9
10^−10
E(k)
10^0
10^1
10^2
k
16×8 bins
32×8 bins
64×8 bins
16×16 bins
683×683 modes
RTFM


Structure Functions
$S_{10}(r)$
r
10^{11}
10^{9}
10^{7}
10^{5}
10^{3}
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-2}
10^{-1}
10^{0}
16×8 bins
32×8 bins
64×8 bins
16×16 bins
683×683 modes

Noncanonical Hamiltonian Formulation

- Underlying *noncanonical* Hamiltonian formulation for inviscid 2D vorticity equation:

$$\dot{\omega}_{\boldsymbol{k}} = \int d\boldsymbol{q}\, J_{\boldsymbol{kq}} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}},$$

where

$$H \doteq \frac{1}{2} \int d\boldsymbol{k}\, \frac{|\omega_{\boldsymbol{k}}|^2}{k^2},$$

$$J_{\boldsymbol{kq}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{kpq}} \omega^*_{\boldsymbol{p}}.$$

- Leads to inviscid Navier–Stokes equation:

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \int d\boldsymbol{p} \int d\boldsymbol{q}\, \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega^*_{\boldsymbol{p}} \omega^*_{\boldsymbol{q}}.$$

Liouville Theorem

- Navier–Stokes:

$$J_{\boldsymbol{kq}} \doteq \int d\boldsymbol{p}\, \epsilon_{\boldsymbol{kpq}} \omega_{\boldsymbol{p}}^*$$

$$\Rightarrow \qquad \int d\boldsymbol{k}\, \frac{\delta \dot{\omega}_{\boldsymbol{k}}}{\delta \omega_{\boldsymbol{k}}} = \int d\boldsymbol{k} \int d\boldsymbol{q}\, \underbrace{\frac{\delta J_{\boldsymbol{kq}}}{\delta \omega_{\boldsymbol{k}}}} \frac{\delta H}{\delta \omega_{\boldsymbol{q}}} + J_{\boldsymbol{kq}} \frac{\delta^2 H}{\delta \omega_{\boldsymbol{k}} \delta \omega_{\boldsymbol{q}}} = 0.$$

$$\epsilon_{\boldsymbol{k}(-\boldsymbol{k})\boldsymbol{q}} = 0$$

- Spectral Reduction:

$$J_{\boldsymbol{KQ}} \doteq \sum_{\boldsymbol{P}} \Delta_{\boldsymbol{P}} \left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^*$$

$$\Rightarrow \qquad \sum_{\boldsymbol{K}} \frac{\partial \dot{\Omega}_{\boldsymbol{K}}}{\partial \Omega_{\boldsymbol{K}}} = \sum_{\boldsymbol{K},\boldsymbol{Q}} \underbrace{\frac{\partial J_{\boldsymbol{KQ}}}{\partial \Omega_{\boldsymbol{K}}}} \frac{\partial H}{\partial \Omega_{\boldsymbol{Q}}} + J_{\boldsymbol{KQ}} \frac{\partial^2 H}{\partial \Omega_{\boldsymbol{K}} \partial \Omega_{\boldsymbol{Q}}} = 0.$$

$$\left\langle \epsilon_{\boldsymbol{kpq}} \right\rangle_{\boldsymbol{K}(-\boldsymbol{K})\boldsymbol{Q}} = 0$$

Statistical Equipartition

- If the dynamics are *mixing*, the Liouville Theorem and the coarse-grained invariants

$$E \doteq \frac{1}{2}\sum_{\boldsymbol{K}} \frac{|\Omega_{\boldsymbol{K}}|^2}{K^2}\Delta_{\boldsymbol{K}}, \qquad Z \doteq \frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}},$$

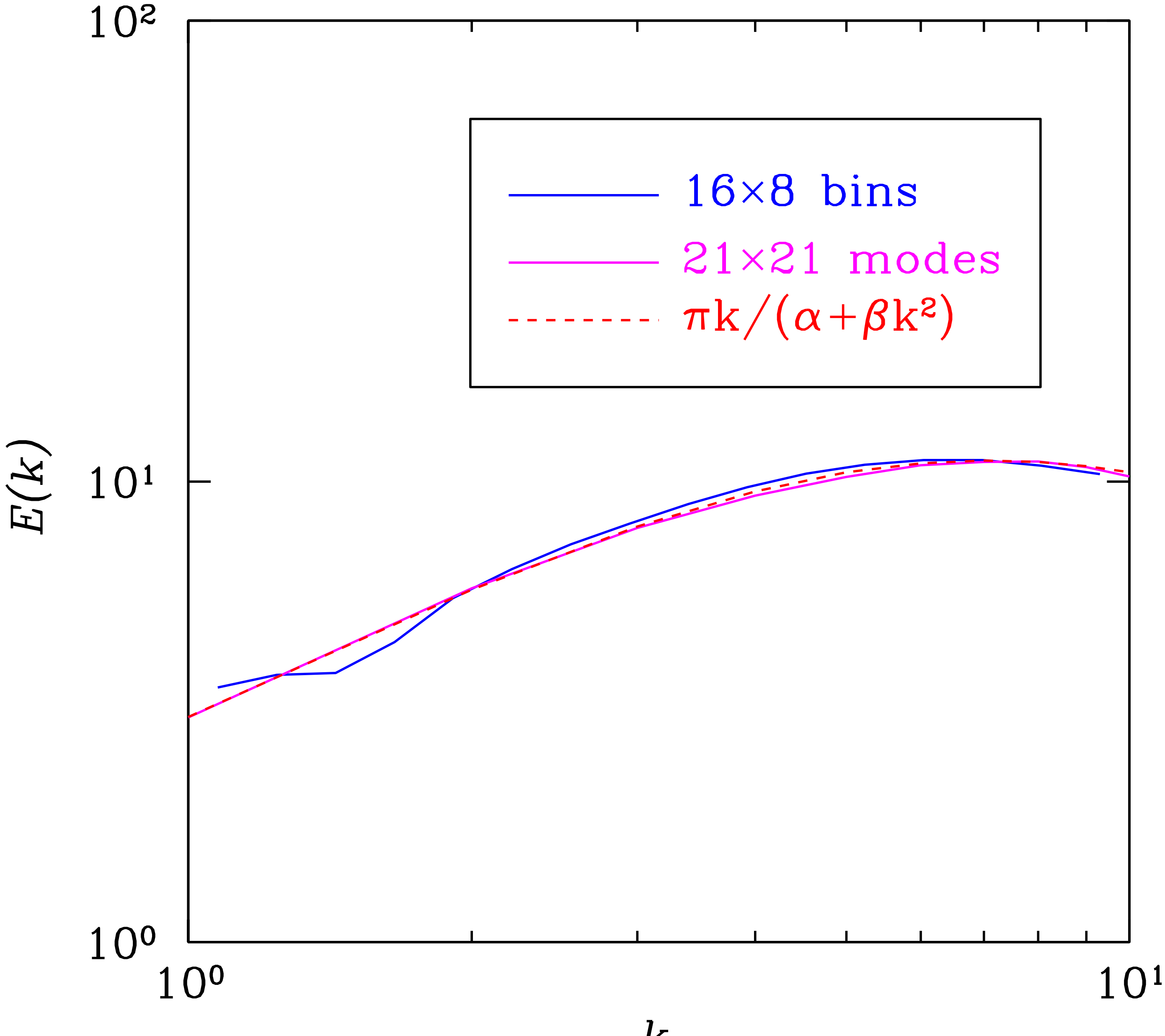
lead to statistical equipartition of $(\alpha/K^2+\beta)\,|\Omega_{\boldsymbol{K}}|^2\,\Delta_{\boldsymbol{K}}$.

- This is the correct equipartition only for uniform bins. However, for nonuniform bins, a rescaling of time by $\Delta_{\boldsymbol{K}}$:

$$\frac{1}{\Delta_{\boldsymbol{K}}}\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}}\,\Omega_{\boldsymbol{K}} = \sum_{\boldsymbol{P},\boldsymbol{Q}} \Delta_{\boldsymbol{P}}\Delta_{\boldsymbol{Q}}\,\frac{\langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}}}{Q^2}\,\Omega^*_{\boldsymbol{P}}\Omega^*_{\boldsymbol{Q}}.$$

yields the correct inviscid equipartition:

$$\left\langle |\Omega_{\boldsymbol{k}}|^2 \right\rangle = \frac{1}{\frac{\alpha}{K^2}+\beta}.$$



Relaxation to equipartition

Stiffness Problem

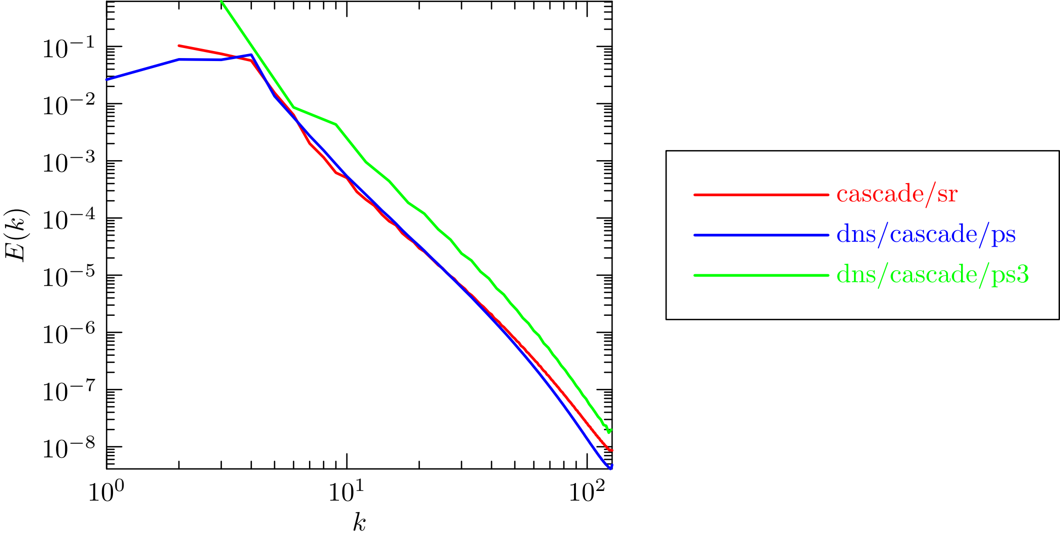
- The rescaling of time does not change the steady-state moment equations.
- It does affect the statistical trajectory of the system and the resulting statistical solution.
- However, the resulting system becomes numerically very stiff.
- Unsolved Problem: given an efficient numerical method for evolving the system of equations

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{S}(\boldsymbol{y}),$$

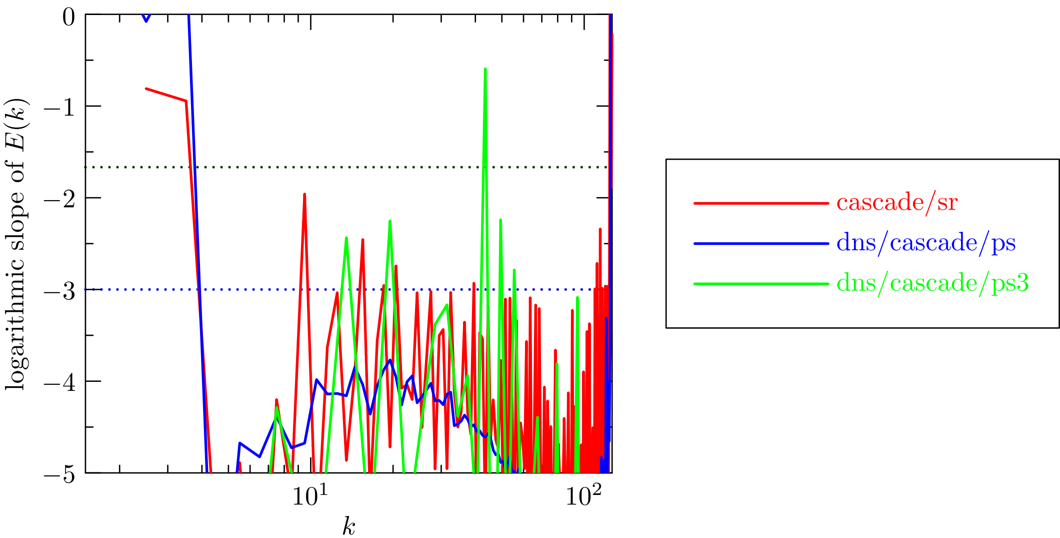
find an efficient numerical method to evolve

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{\Lambda}\boldsymbol{S}(\boldsymbol{y}),$$

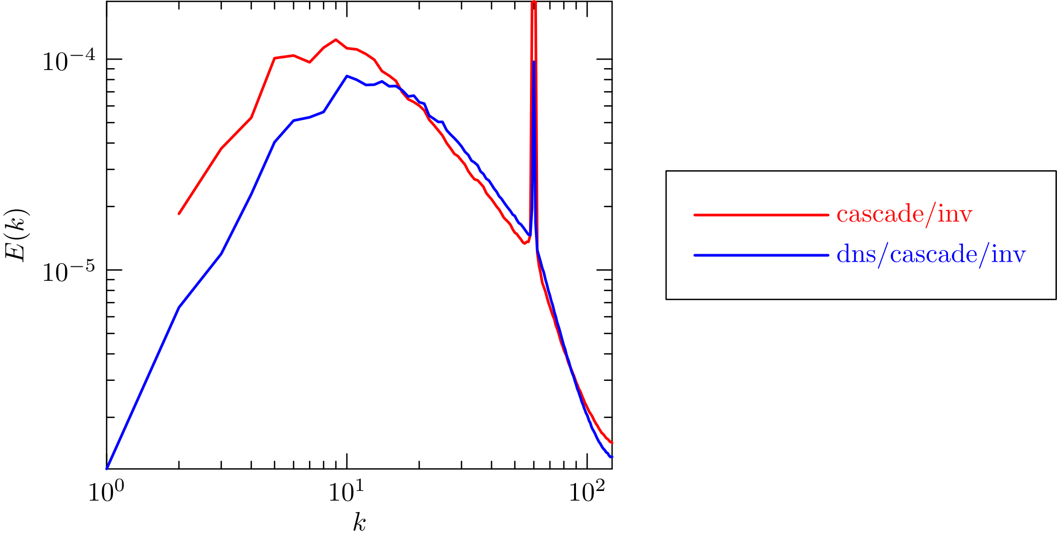
where $\boldsymbol{\Lambda}$ is a constant real diagonal matrix.



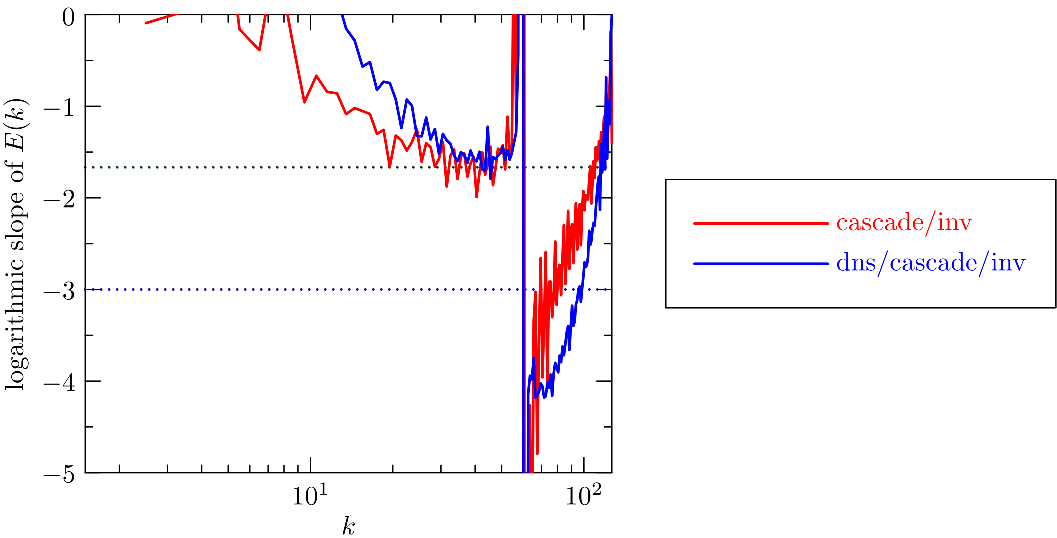
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$E(k)$
10^0
10^1
10^2
k
cascade/sr
dns/cascade/ps
dns/cascade/ps3



0
−1
−2
−3
−4
−5
logarithmic slope of E(k)
10^1
10^2
k
cascade/sr
dns/cascade/ps
dns/cascade/ps3



$E(k)$
10^{-4}
10^{-5}
10^0
10^1
10^2
k
cascade/inv
dns/cascade/inv



0
−1
−2
−3
−4
−5
logarithmic slope of $E(k)$
10^1
10^2
k
cascade/inv
dns/cascade/inv