Weak vs strong turbulence in relativistic field theory

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Turbulence in reheating

End of inflation : homogenous scalar field creates particles

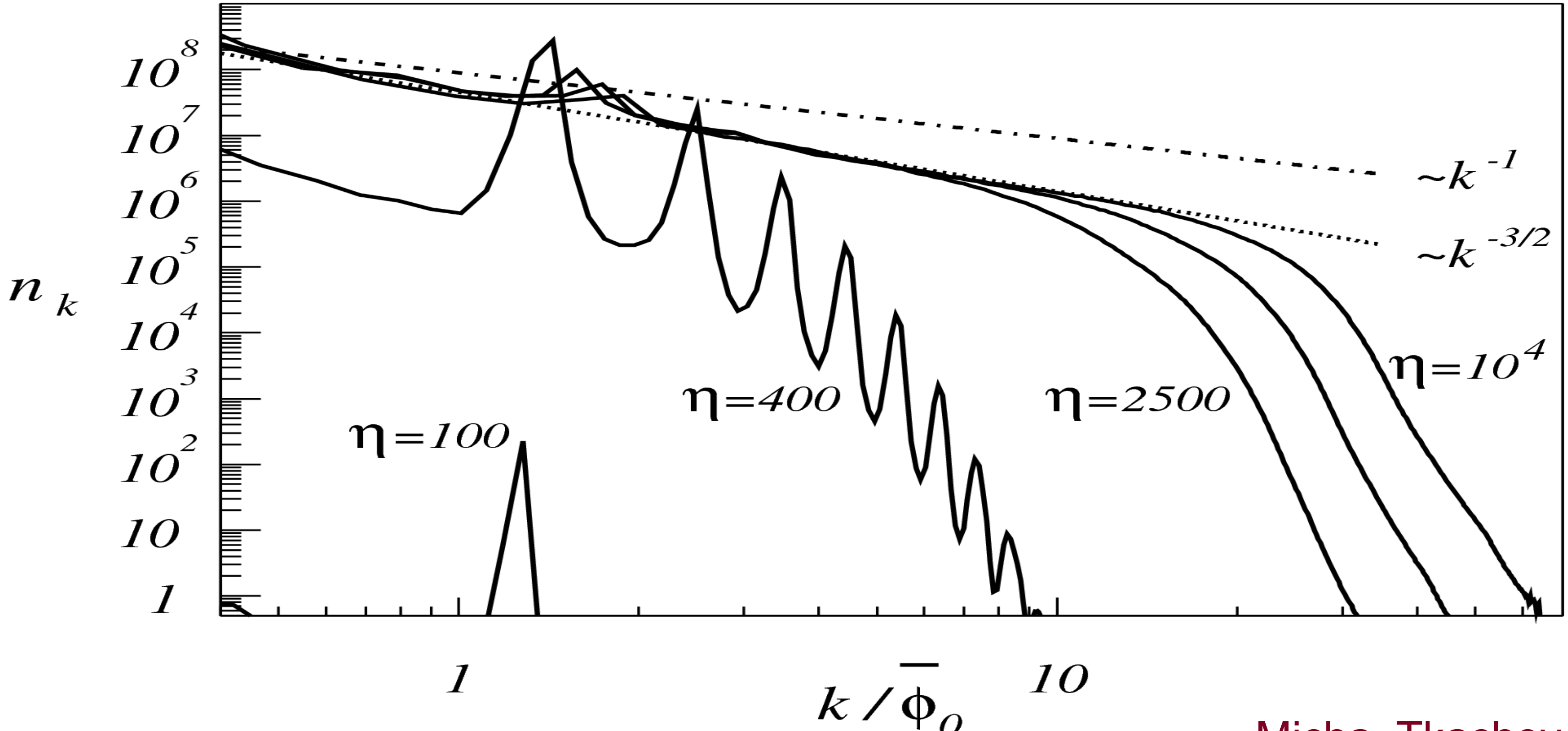
Parametric resonance

Klein-Gordon eq: $(\partial_t^2 - \partial_x^2)\varphi(x,t) + \lambda\varphi^3(x,t) = 0$

Initial condition: Highly occupied zero mode

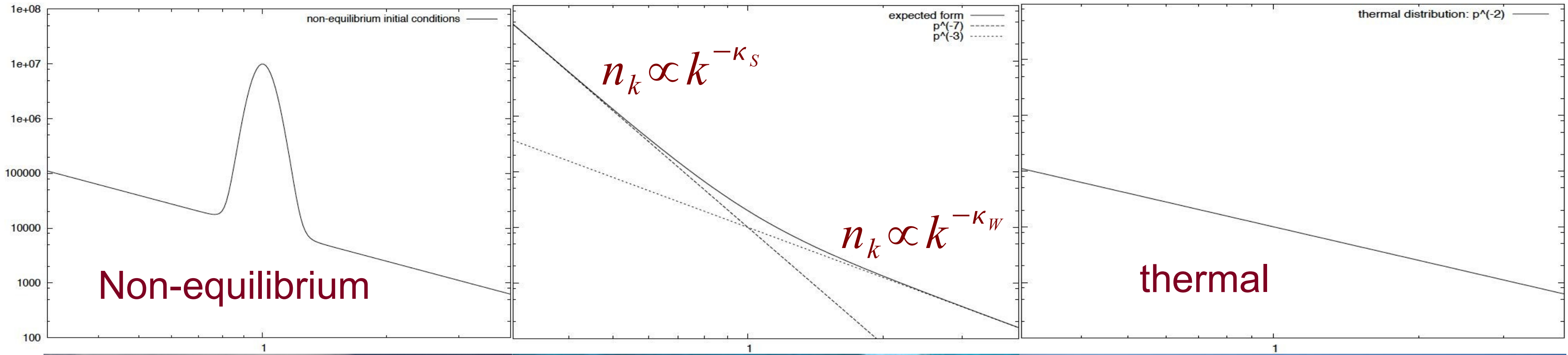
Unoccupied modes with $k > 0$

Turbulent spectrum emerges   exponent: weak wave turbulence



Micha, Tkachev (2004)

Scaling analysis from 2PI formalism



Berges, Rothkopf, Schmidt (2008)
Scheppach, Berges, Gasenzer (2010)

$F(x,y) = \{\Phi(x), \Phi(y)\}$ Statistical function

$\rho(x,y) = [\Phi(x), \Phi(y)]$ Spectral function

Scaling ansatz:

$$F(s^z \omega, s\, p) = s^{-2-\kappa} F(\omega, p)$$
$$\rho(s^z \omega, s\, p) = s^{2-\eta} \rho(\omega, p)$$

Cascade in wave turbulence

Boltzmann equation:

$$\frac{\partial n_p}{\partial t} = C(t, p) = gain - loss$$

By integration one finds the flux in momentum space

Power law ansatz + constant flux → exponents

Using 2PI formalism

$$F(x, y) = \frac{1}{2}\{\Phi(x), \Phi(y)\}$$ Statistical function: occupation

$$\rho(x, y) = i[\Phi(x), \Phi(y)]$$ Spectral function: available modes

In equilibrium: Fluctuation dissipation theroem: $F(\omega, p) = -i(n_{eq}(\omega) + \frac{1}{2})\rho(\omega, p)$

In general, we define: $F_p(X) = -i(n_p(X) + 1/2)\rho_p(X)$

(using Wigner coord)

Weak wave turbulence

Free spectral function: $\rho_p^0 = 2\pi i\, sgn(p_0)\,\delta(p_0^2 - \omega_p^2)$

Defining "spectral particle number" $F_p(X) = -i\left(n_p(X) + \frac{1}{2}\right)\rho_p(X)$

We can write the particle number as: $\int_0^\infty \frac{d\,p_0}{2\pi} 2\,p_0\,\rho_p\,n_p(t) = n_{\boldsymbol{p}}(t)$

EOM: $\int_0^\infty \frac{d\,p_0}{2\pi} 2\,p_0 \frac{\partial F_p(t)}{\partial t} = \frac{\partial n_{\boldsymbol{p}}}{\partial t} = C(t,p)$

Loop expansion, lowest order of 2PI equations for a 4 vertex:

$$C(t,p) = \int d\,\Omega^{2\Leftrightarrow 2}(p,l,q,r)\left[(1+n_p)(1+n_l)\,n_q\,n_r - n_p\,n_l(1+n_r)(1+n_q)\right]$$

Phase space integrations: $d\,\Omega^{2\Leftrightarrow 2}(p,l,q,r) = \lambda^2 \frac{N+2}{18\,N^2}\int_{lpr}(2\pi)^{d+1}\delta^{(d)}(\boldsymbol{p}+\boldsymbol{l}-\boldsymbol{q}-\boldsymbol{r})$

$$\delta(\omega_p + \omega_l - \omega_q - \omega_r)\frac{1}{16\,\omega_p\,\omega_l\,\omega_q\,\omega_r}$$

Description valid when $1 \ll n_p \ll \frac{1}{\lambda}$

Equivalent to Boltzmann eq.

Weak wave turbulence

Local interactions in Fourier space:

$$\partial_t(\omega n_{\boldsymbol{p}}) + \nabla_p j_{\boldsymbol{p}}^{ener} = 0$$

Energy cascade

$$\partial_t n_{\boldsymbol{p}} + \nabla_{\boldsymbol{p}} j_{\boldsymbol{p}}^{part} = 0$$

Particle number cascade

Integrated on a sphere = flux

$$\int_0^k d^d p\, \partial_t(\omega_p n_{\boldsymbol{p}}) \propto A(k) = \int^k dp\, p^{d-1} \omega_{\boldsymbol{p}} \partial_t n_{\boldsymbol{p}}(t)$$

Using the scaling ansatz $n_{\boldsymbol{p}} \propto p^{-\kappa_W}$ $\omega_{\boldsymbol{p}} \propto p$

Plug in collision integral: $A(k) \propto k^{d+1-3\kappa+2\mathrm{d}-5}$

Well known Sakharov results $\kappa_W^{ener} = d - \frac{4}{3}$ $\kappa_W^{part} = d - \frac{5}{3}$

Similarly for interaction with 3-vertex: $\kappa_W^{ener} = d - \frac{3}{2}$ $\kappa_W^{part} = d - 1$

Cascade in strong turbulence

Relativistic field theory: particle number is not conserved
No unique particle number definition

In practice: something that works for free theory

$$\int_0^\infty \frac{d\,p_0}{2\pi} 2\,p_0 \rho_p(t) n_p(t) = n_{\mathbf{p}}^{eff}(t)$$

Effective particle number

When $n \gg \frac{1}{\lambda}$ one uses 1/N resummation

$$C^{NLO} = \int d\,\Omega^{2\Leftrightarrow 2}(p,l,q,r)[(1+n_p)(1+n_l)n_q n_r - n_p n_l (1+n_q)(1+n_r)] + \\ +(1\Leftrightarrow 3)+(0\Leftrightarrow 4)$$

Phase space integration
Resummation hides here in momentum dependent effective coupling

$$d\,\Omega^{2\Leftrightarrow 2}(p,l,q,r) = \frac{\lambda}{18N}\int_0^\infty \frac{dp^0\,dl^0\,dq^0\,dr^0}{(2\pi)^{4-(d+1)}} \int_{\boldsymbol{lqr}} \delta^{d+1}(p+l-q-r)\rho_p \rho_l \rho_q \rho_r \\ (\lambda_{eff}(p+l)+\lambda_{eff}(p-q)+\lambda_{eff}(p-r))$$

Cascade in strong turbulence

Scaling ansatz
(similar to critical phenomena)

$$F(s^z \omega, s\, p) = s^{-2-\kappa} F(\omega, p)$$
$$\rho(s^z \omega, s\, p) = s^{2-\eta} \rho(\omega, p)$$

Using only classical approximation $1 \ll n$

Similarly to the weak case:

Using Fourier space continuity equation $\partial_t(\omega\, n_{\boldsymbol{p}}) + \nabla_p\, j^{ener}_{\boldsymbol{p}} = 0$

Plug in “collision integral” $A(k) \propto k^{d-k_S+2z-\eta}$

Exponents for strong turbulence

$$\kappa_S^{ener} = d + 2z - \eta \qquad \kappa_S^{part} = d + z - \eta$$

Scaling exponents from constant cascade

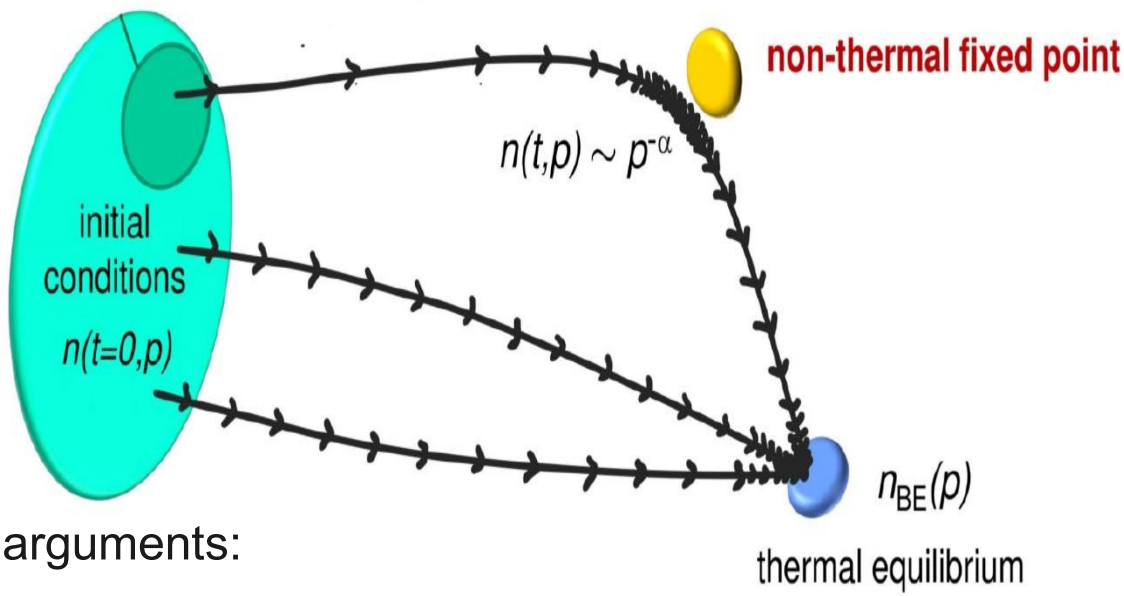
Cascades in O(N) scalar fields

Weak turbulence $1 \ll n_p \ll \frac{1}{\lambda}$ $\kappa_W^{ener} = d - \frac{3}{2}$ $\kappa_W^{part} = d - 1$

Strong turbulence $n_p > \frac{1}{\lambda}$ $\kappa_S^{ener} = d + 2z - \eta$ $\kappa_S^{part} = d + z - \eta$

Berges,Sexty (2010)

Universality:
no dependence on coupling, and N



Agrees with calculations using stationarity arguments:

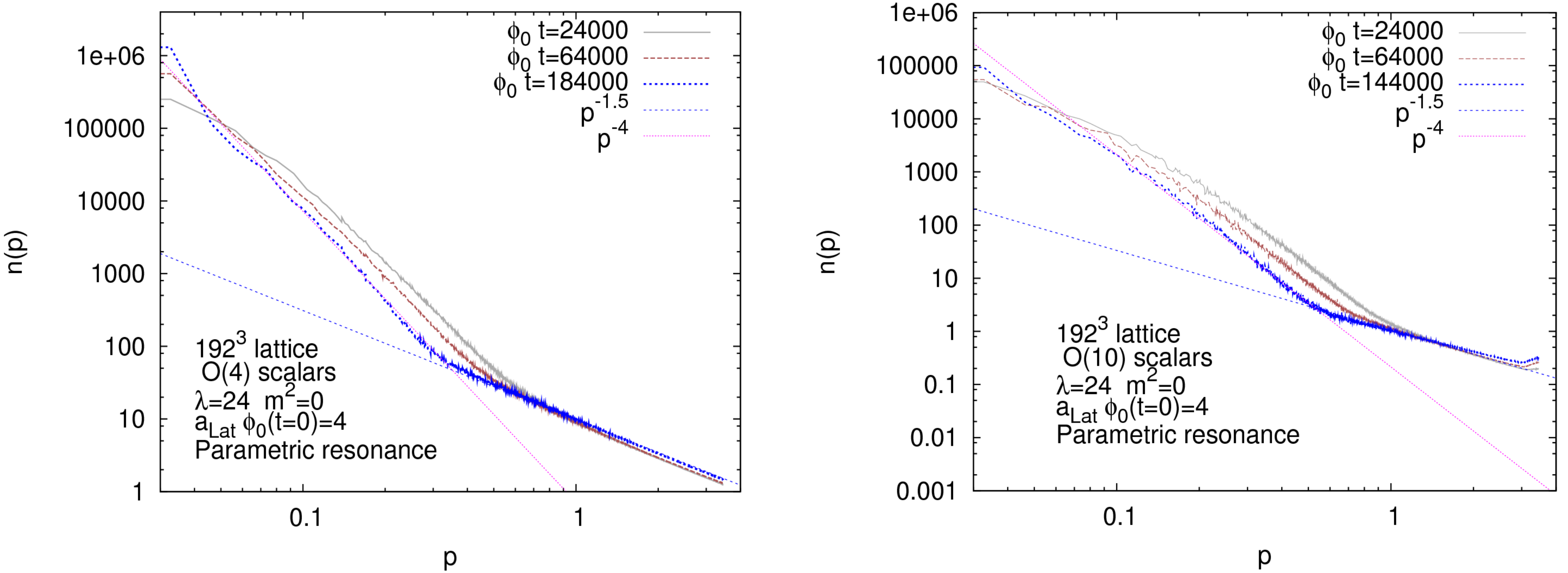
Berges, Hoffmeister (2008)
Scheppach, Berges, Gasenzer (2009)

Numerical simulations

3D simulations using O(4) and O(10) scalars

$$\kappa_S^{part} = d+1 \qquad \kappa_W^{ener} = d - \frac{3}{2}$$

Assuming $z = 1, \quad \eta = 0$



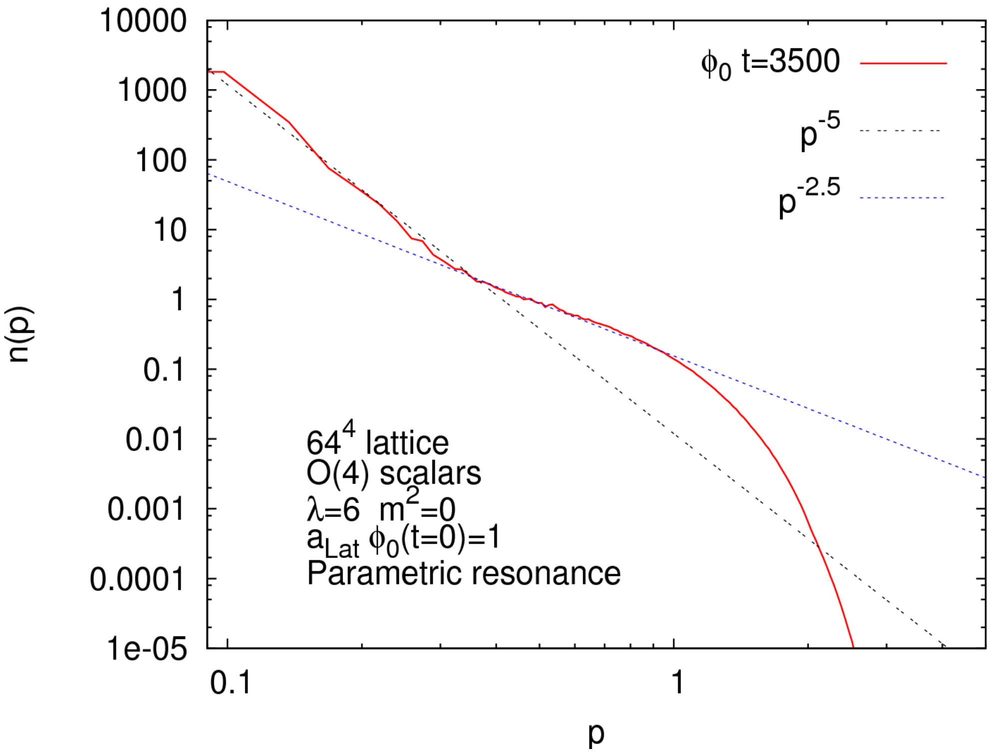
Berges,Sexty (2010)

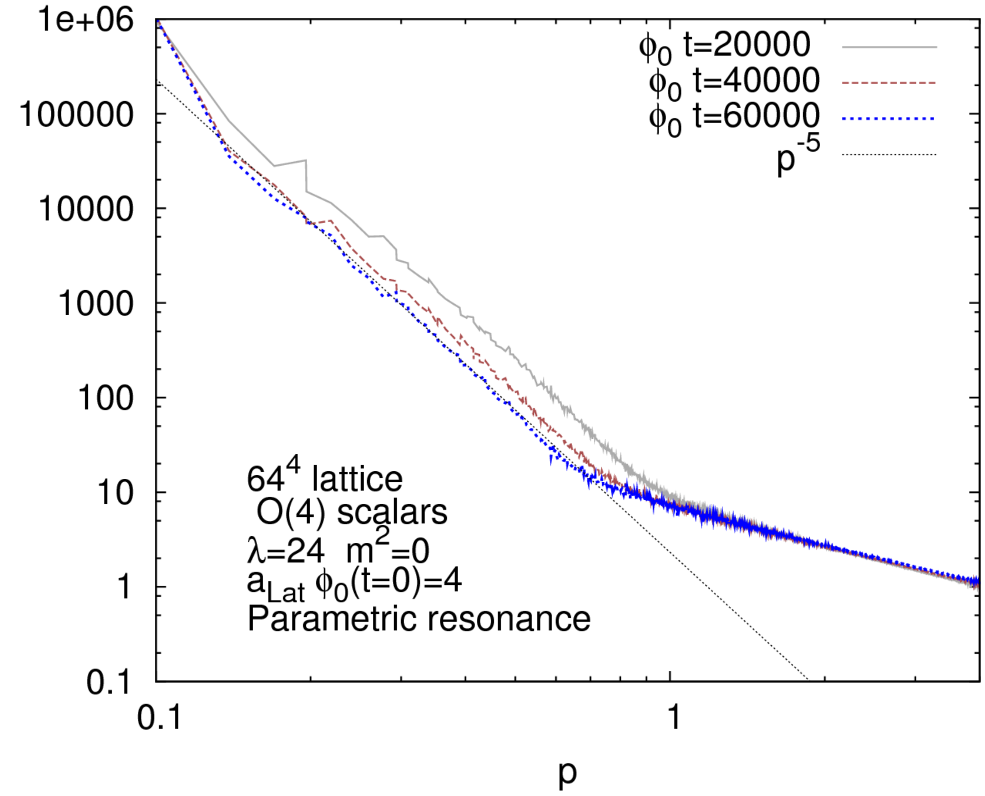
Numerical simulations

4D simulations using O(4) scalars

$$\kappa_S^{part} = d+1 \qquad \kappa_W^{ener} = d - \frac{3}{2}$$

Assuming $z=1, \quad \eta=0$





Berges,Sexty (2010)

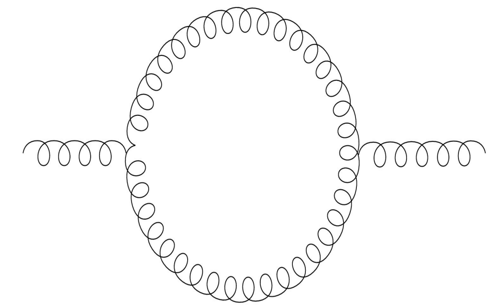
Turbulence in gauge theories

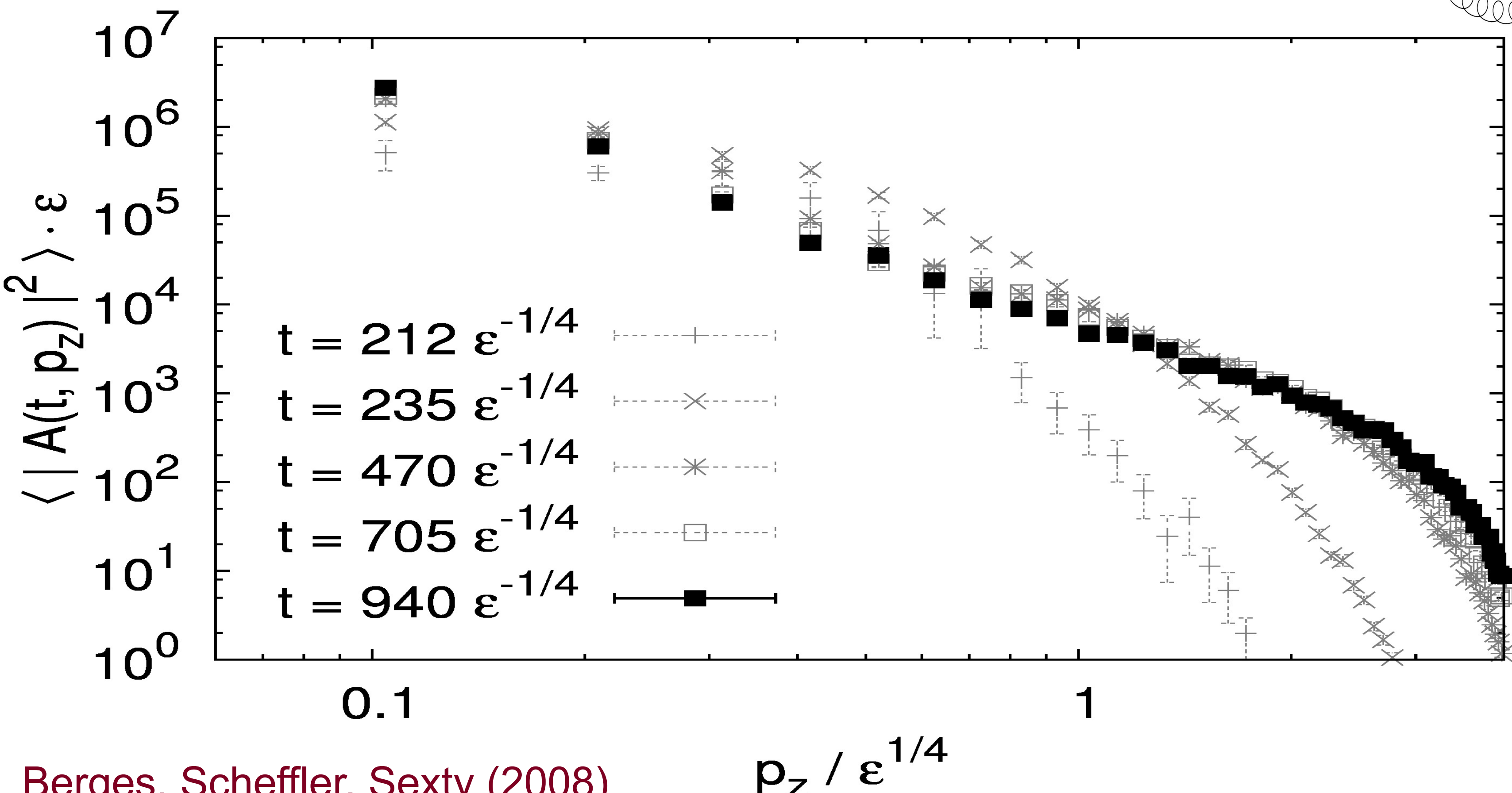
Anisotropic plasma in heavy-ion collisions
Weibel instability populates low modes

Field theory with local SU(2) symmetry

Weak wave turbulence analysis: $\kappa = \frac{5}{3}$, $\kappa = \frac{4}{3}$

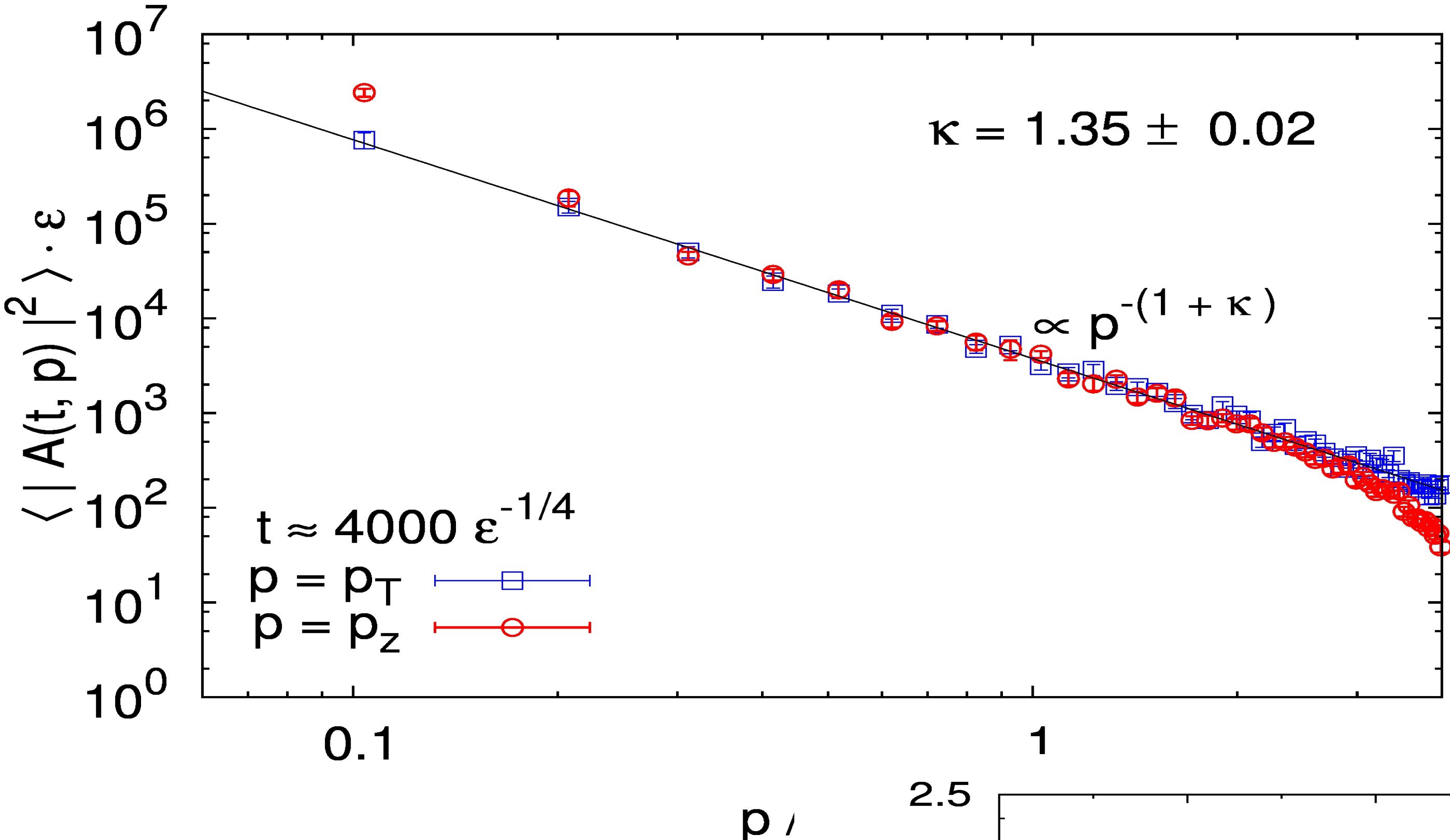
Cascade of: energy particle





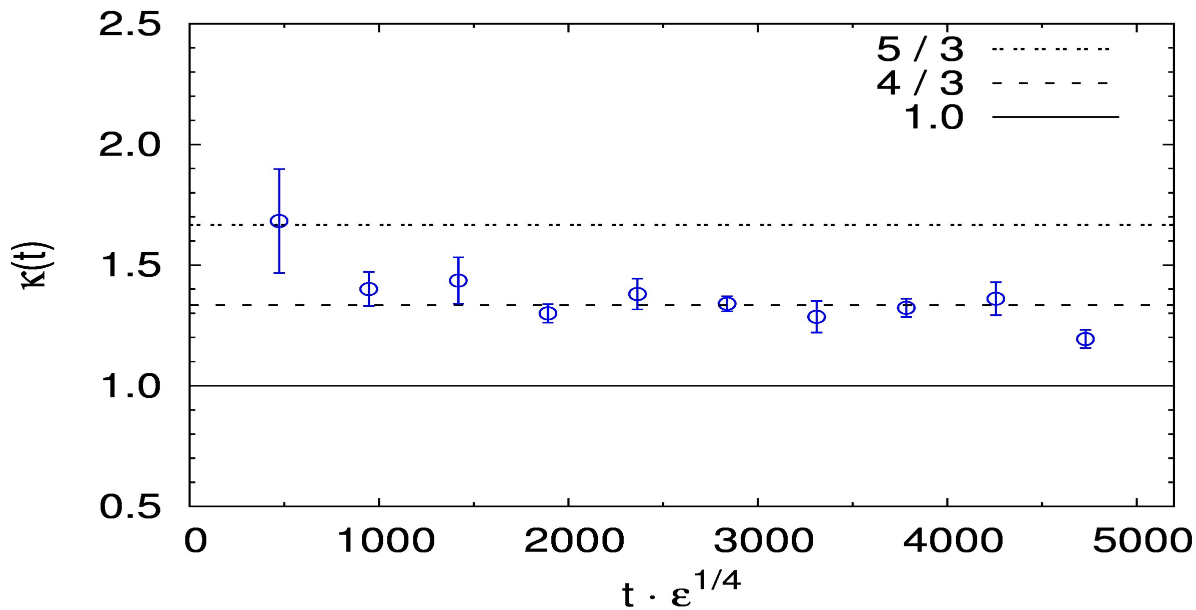
Berges, Scheffler, Sexty (2008)

Turbulence in gauge theories



Exponent given by weak turb.

Quasistationarity observed



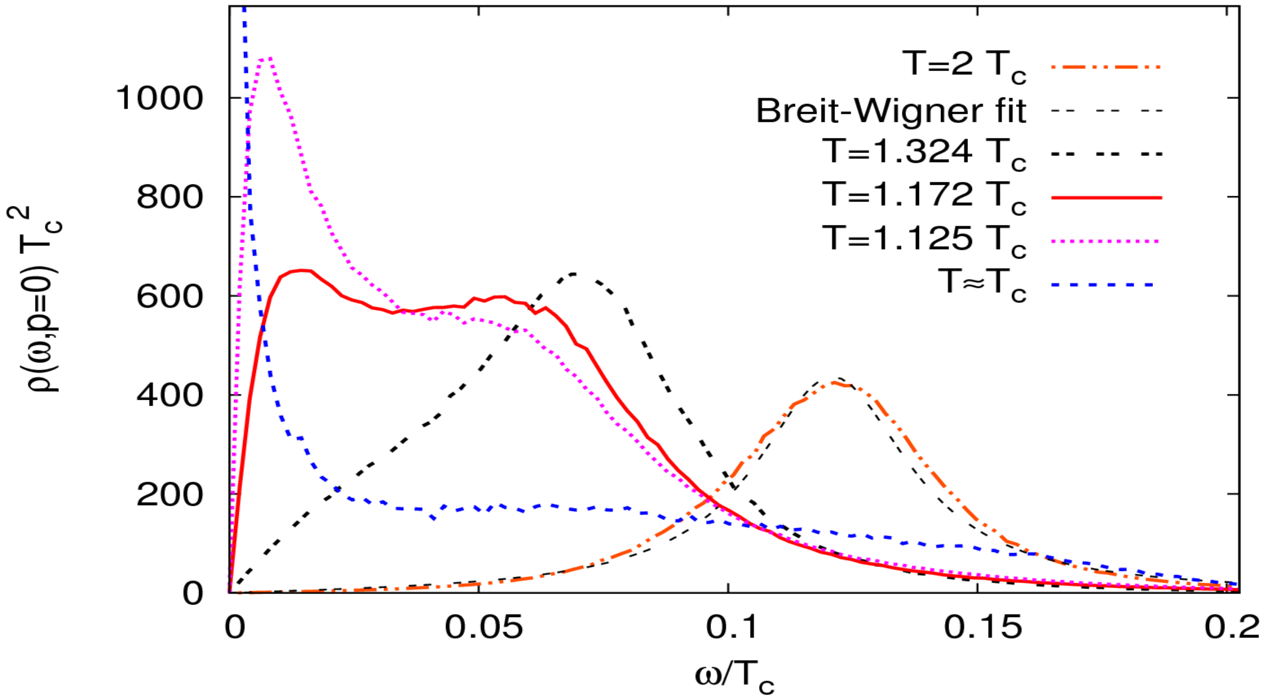
Spectral function

To measure all exponents one needs to measure the spectral function

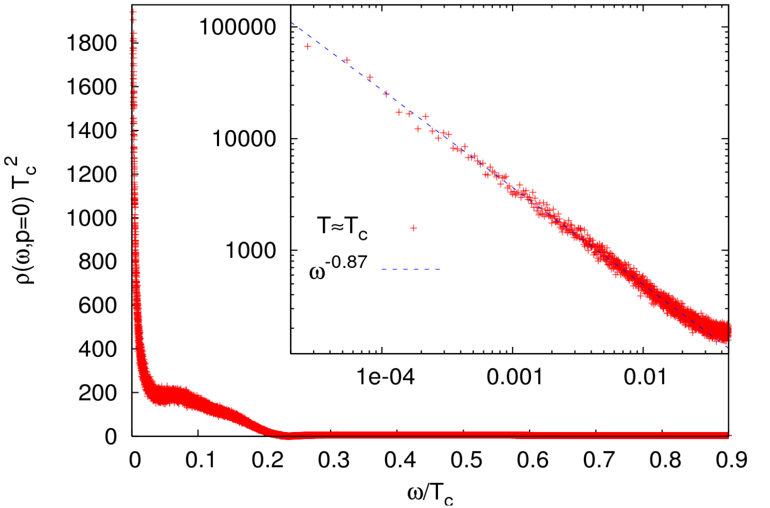
Turbulence ≈ critical phenomena

Single component scalar in 2D in thermal equilibrium
Ising universality class

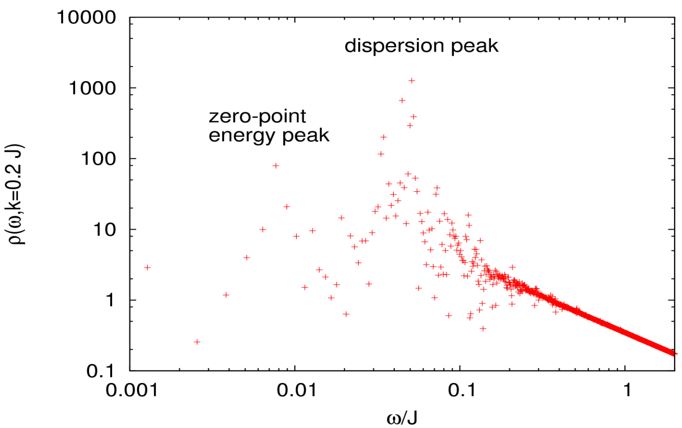
At the critical point spectral function develops “second peak”

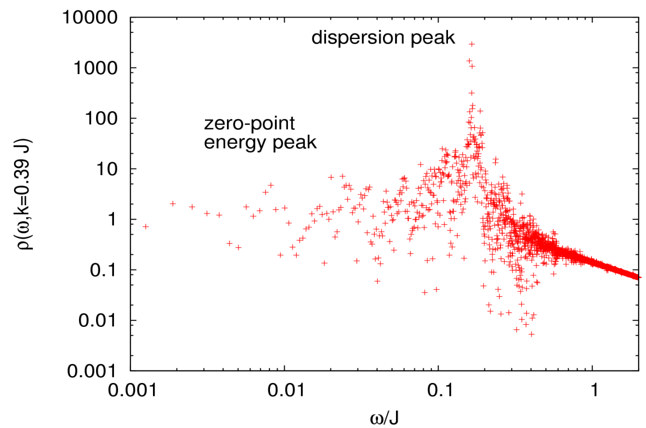


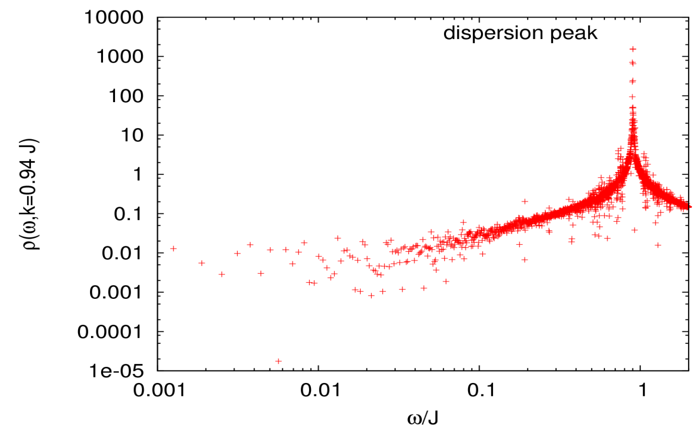
Berges, Schlichting, Sexty (2009)



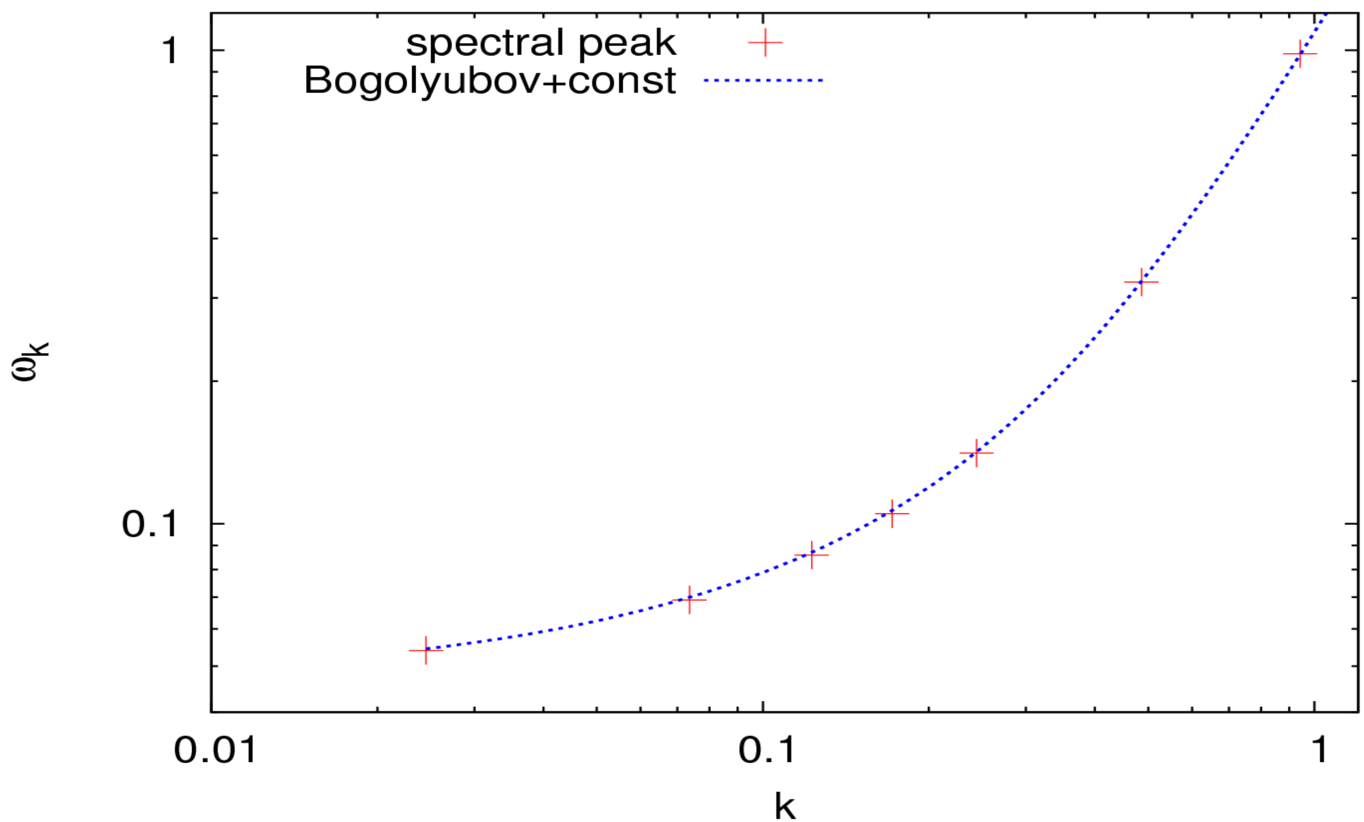
Spectral function in GPE simulation







Bogolyubov
spectrum peak present



Peak at zero?

256^2 lattice

$U = 3\times 10^{-5}, \quad N = 10^8$

Conclusions

Predictions from 2PI: stationarity + scaling ansatz =
weak and strong exponents

Cascades of particles and energy identified in weak and strong regimes

Numerical simulations confirm expectations
in scalar, GPE, and gauge field theories

Open questions:

Strong turbulence $\Leftrightarrow$ Kolmogorov scaling?
Measure spectral function $\eta, z = ?$
Other resummation = other exponents?
Can we see vortices?
Self similarity?