PDF Method Approach to Fully Developed Turbulence: Vorticity Statistics

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Overview

- introduction (features of vorticity and velocity fields, DNS)
- overview LMN hierarchy
- single-point vorticity statistics
- two-point vorticity statistics

Introduction

Closure Problem of Turbulence

- consequence of nonlinearity
- moment equation: coupling to higher order moments
- PDFs: coupling to multi-point PDFs

Possible Strategies

- analyze the statistical equations
- model unclosed terms
- use DNS / experiments to gather information

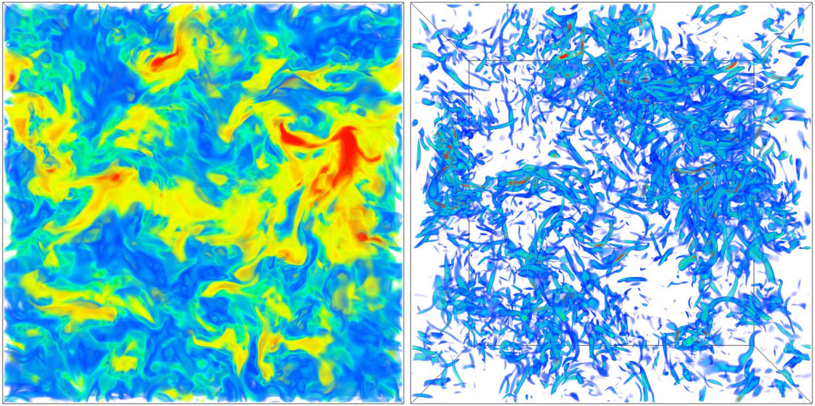
Basic Equations

velocity $\boldsymbol{u}(\boldsymbol{x},t)$, vorticity $\boldsymbol{\omega}(\boldsymbol{x},t) = \nabla \times \boldsymbol{u}(\boldsymbol{x},t)$

$$\frac{\partial}{\partial t}\boldsymbol{\omega} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \mathsf{S}\boldsymbol{\omega} + \nu\Delta\boldsymbol{\omega} + \nabla\times\boldsymbol{F}$$

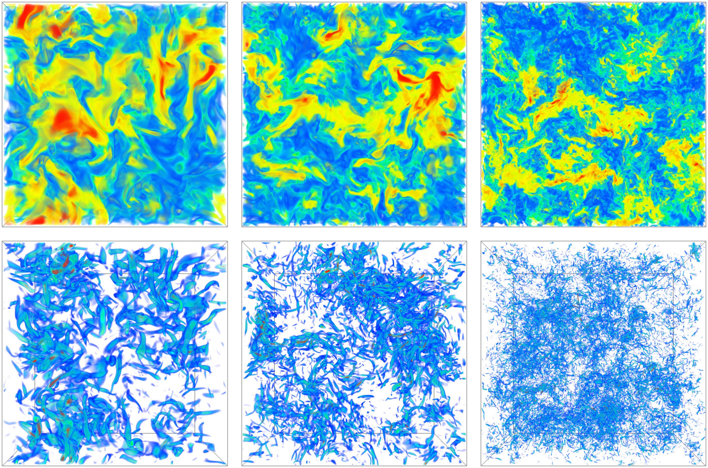
- different dynamical influences:
  - nonlinear advection
  - viscous dissipation
  - vortex stretching
  - forcing

Velocity and Vorticity: Fields ($R_\lambda = 112$)



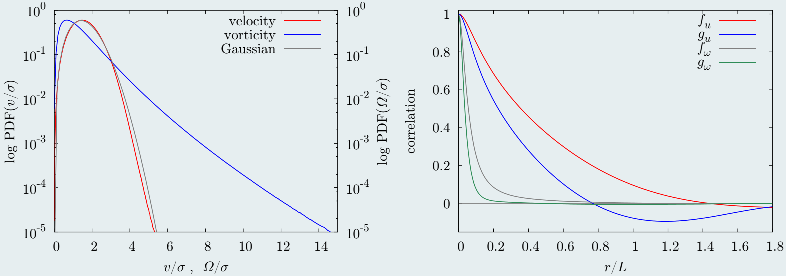
velocity field vorticity field

Fields: Reynolds number dependence ($R_\lambda = 76, 112, 225$)



→ increasing Reynolds number →

Basic Statistical Properties



- velocity: nearly Gaussian, long-range correlations
- vorticity: strongly non-Gaussian, (comparably) short-range correlated

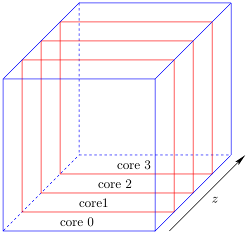
DNS

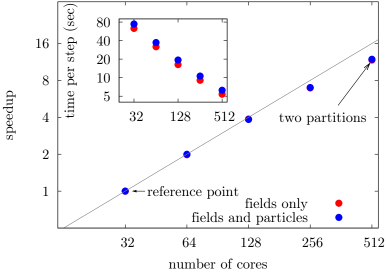
$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t}(\boldsymbol{k},t) + \nu\, k^2\, \tilde{\boldsymbol{\omega}}(\boldsymbol{k},t) = i\boldsymbol{k} \times \mathcal{F}\{\boldsymbol{u}(\boldsymbol{x},t) \times \boldsymbol{\omega}(\boldsymbol{x},t)\} + \tilde{\boldsymbol{F}}(\boldsymbol{k},t)$$

- standard dealiased pseudospectral method
- periodic boundary conditions
- RK3 with adaptive time-stepping (Courant-Friedrichs-Levy criterion)
- Lagrangian tracers: tricubic interpolation, 3rd order multistep predictor-corrector
- MPI parallel, slab domain decomposition via FFTW

DNS: parallelization

- slab domain decomposition
- descent scaling





LMN Hierarchy: Intro

- systematic way to derive evolution equations for PDFs
- introduced to hydrodynamics by Lundgren, Monin and Novikov (late 60's)

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Concept

- derive evolution equation for fine-grained PDF $\hat{f}(\boldsymbol{\Omega};\boldsymbol{x},t)=\delta(\boldsymbol{\omega}(\boldsymbol{x},t)-\boldsymbol{\Omega})$
- use hydrodynamical equations
- obtain PDF evolution by ensemble averaging: $f(\boldsymbol{\Omega};\boldsymbol{x},t)=\langle\hat{f}(\boldsymbol{\Omega};\boldsymbol{x},t)\rangle$
- exploit statistical symmetries: homogeneity, isotropy, stationarity

LMN Hierarchy: Recent Works

Classic Works

- velocity: Lundgren, Monin, Liubimov/Ulinich, ...
- vorticity: Novikov

Modeling Aspects

- reacting flows: Pope et al., Klimenko, Bilger, Minier, ...

Recent Theoretical Works

- velocity: Tatsumi et al., Hosokawa, Yakhot, Wilczek et al., ...
- vorticity: Novikov, Wilczek/Friedrich
- passive scalar: Ching, Yakhot, ...

...and, of course, ...many others!

LMN Hierarchy

$$\frac{\partial}{\partial t} f_1 = \mathcal{C}_1\left[f_2, \boldsymbol{F}_1\right]$$

$$\frac{\partial}{\partial t} f_2 = \mathcal{C}_2\left[f_3, \boldsymbol{F}_1, \boldsymbol{F}_2\right]$$

$$\vdots$$

$$\frac{\partial}{\partial t} f_{N-1} = \mathcal{C}_{N-1}\left[f_N, \boldsymbol{F}_1, \dots, \boldsymbol{F}_{N-1}\right]$$

$$\frac{\partial}{\partial t} f_N = -\sum_{i=1}^{N} \nabla_{\boldsymbol{x}_i} \cdot \left\{ \left\langle \boldsymbol{u}_i \middle| \boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_N \right\rangle f_N \right\} - \sum_{i=1}^{N} \nabla_{\boldsymbol{\Omega}_i} \cdot \left\{ \left\langle \mathrm{S}_i \boldsymbol{\omega}_i + \nu \Delta_{\boldsymbol{x}_i} \boldsymbol{\omega}_i + \nabla_{\boldsymbol{x}_i} \times \boldsymbol{F}_i \middle| \boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_N \right\rangle f_N \right\}$$

LMN Hierarchy

- What kind of information can be extracted from this framework?
- Connection between statistics and dynamical features?
- Connection between statistics and local structure of the field?

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Three Different Aspects of Single-Point Statistics

- conditional balance equation
- shape of PDF (+ a comparison to velocity statistics)
- connection to local conditional structure

Two-Point Enstrophy Statistics

- scale-dependent probability fluxes

Single-Point Statistics: Strategy

PDF equation from LMN hierarchy

↓ statistical symmetries

2 versions: conditional balance & stationary solution with unclosed terms

↓ DNS results & interpretation

shape of PDF

Single-Point Statistics: PDF Equation

$$\frac{\partial}{\partial t} f_1 = -\nabla_{\boldsymbol{\Omega}_1} \cdot \left[\left\langle \mathrm{S}_1 \boldsymbol{\omega}_1 + \nu \Delta_{\boldsymbol{x}_1} \boldsymbol{\omega}_1 + \nabla_{\boldsymbol{x}_1} \times \boldsymbol{F}_1 \middle| \boldsymbol{\Omega}_1 \right\rangle f_1 \right]$$

Evolution of PDF is given by conditional

- vortex stretching
- vorticity diffusion (dissipation)
- external forcing

Single-Point Statistics: Statistical Symmetries

- PDF:

$$\tilde{f}(\Omega) = 4\pi\Omega^2 f(\boldsymbol{\Omega})$$

- conditional averages:

$$\left\langle S_{ij} \middle| \boldsymbol{\Omega} \right\rangle = \frac{1}{2}\Sigma(\Omega)\left(3\frac{\Omega_i\,\Omega_j}{\Omega^2} - \delta_{ij}\right) \qquad \Sigma(\Omega) = \left\langle \hat{\omega}\mathrm{S}\hat{\omega} \middle| \boldsymbol{\Omega} \right\rangle$$

$$\left\langle \nu\Delta\boldsymbol{\omega} \middle| \boldsymbol{\Omega} \right\rangle = \Lambda(\Omega)\,\hat{\boldsymbol{\Omega}} \qquad \Lambda(\Omega) = \left\langle \nu\hat{\boldsymbol{\omega}}\cdot\Delta\boldsymbol{\omega} \middle| \boldsymbol{\Omega} \right\rangle$$

$$\left\langle \nabla\times\boldsymbol{F} \middle| \boldsymbol{\Omega} \right\rangle = \Phi(\Omega)\hat{\boldsymbol{\Omega}} \qquad \Phi(\Omega) = \left\langle \hat{\boldsymbol{\omega}}\cdot(\nabla\times\boldsymbol{F}) \middle| \boldsymbol{\Omega} \right\rangle$$

- result (version 1):

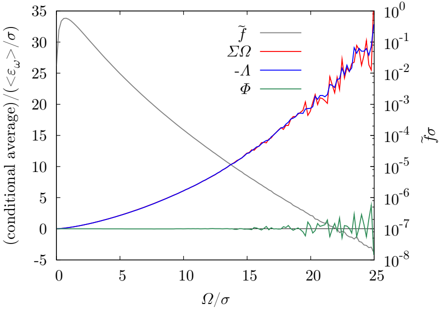
$$\boxed{\frac{\partial}{\partial t}\tilde{f} = -\frac{\partial}{\partial\Omega}\left(\Sigma\,\Omega + \Lambda + \Phi\right)\tilde{f}}$$

- effectively one-dimensional problem!

Single-Point Statistics: DNS results

▷ stationary turbulence:

$$0 = \Sigma(\Omega)\,\Omega + \Lambda(\Omega) + \Phi(\Omega)$$



- ▷ balance of enstrophy production and dissipation
- ▷ external forcing negligible:
  enstrophy production internal mechanism

cf: Novikov et al. Mod. Phys. Lett. B 1994, Wilczek et al. PRE 2009

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Single-Point Statistics: Conditional Dissipation Tensor

▷ homogeneity: re-expression of diffusive term → conditionally averaged dissipation tensor

$$\left\langle \nu \frac{\partial \omega_i}{\partial x_k} \frac{\partial \omega_j}{\partial x_k} \middle| \boldsymbol{\Omega} \right\rangle = \mu(\Omega)\, \delta_{ij} + \left[\lambda(\Omega) - \mu(\Omega)\right] \frac{\Omega_i \Omega_j}{\Omega^2}$$

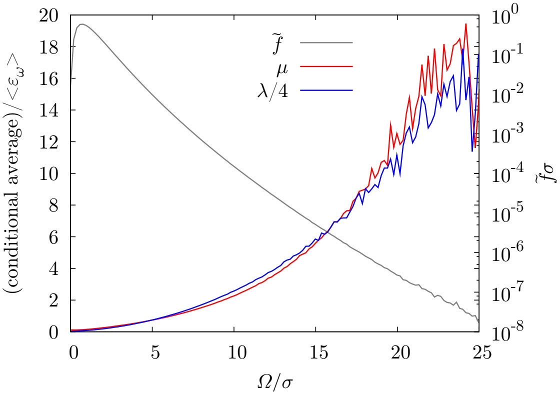
$$\mathrm{Tr}(\mathrm{D}) = \lambda(\Omega) + 2\mu(\Omega)$$

$$\hat{\boldsymbol{\Omega}} \mathrm{D} \hat{\boldsymbol{\Omega}} = \lambda(\Omega)$$

▷ result (version 2):

$$\boxed{\frac{\partial}{\partial t} \tilde{f} = -\frac{\partial}{\partial \Omega} \left(\Sigma\, \Omega + \Phi - \frac{2\mu}{\Omega} \right) \tilde{f} - \frac{\partial^2}{\partial \Omega^2} \lambda \tilde{f}}$$

Single-Point Statistics: Eigenvalues of CEDT

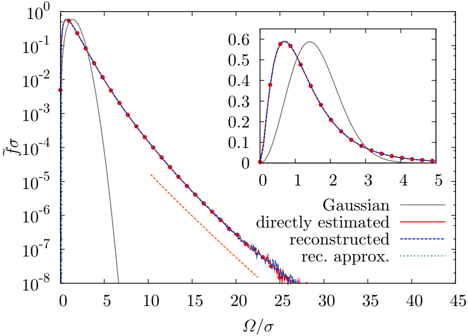


- ▷ pronounced statistical correlations
- ▷ enstrophy dissipation tensor not isotropic (→ structures)

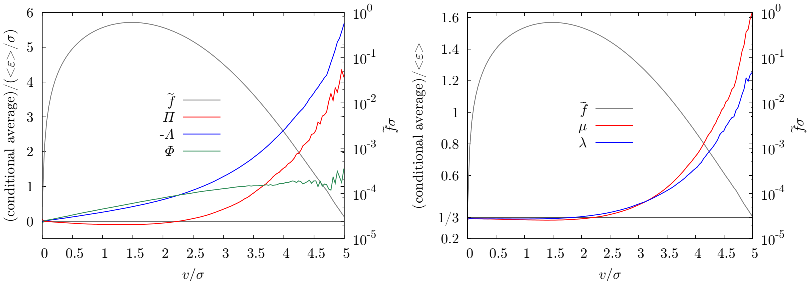
Single-Point Statistics: Stationary Solution

▷ stationary solution:

$$\tilde{f}(\Omega) = \frac{\mathcal{N}}{\lambda(\Omega)} \exp \int_{\Omega_0}^{\Omega} \mathrm{d}\Omega' \, \frac{-\Sigma(\Omega')\,\Omega' - \Phi(\Omega') + \frac{2}{\Omega'}\mu(\Omega')}{\lambda(\Omega')}$$



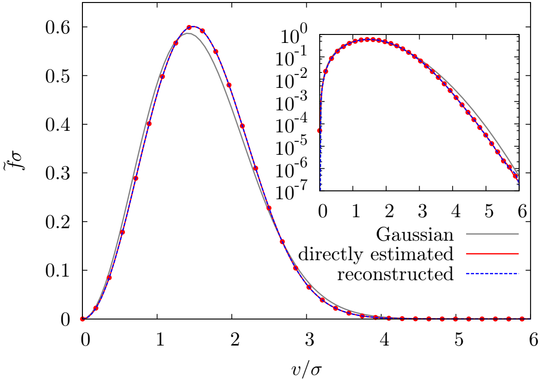
Excursion: Velocity Statistics



- ▷ pressure term plays prominent role
- ▷ external forcing matters
- ▷ dissipation tensor approx. isotropic

cf: Wilczek et al. EPL (2011)

Excursion: Velocity Statistics



- ▷ interplay of correlations leads to slightly sub-Gaussian tail

Comparison: Velocity vs. Vorticity

Velocity

- Gaussian core (e.g. ϵ and $\boldsymbol{v}$ weakly correlated)
- sub-Gaussian tails due to stronger correlations for large $\boldsymbol{v}$
- reason: scale separation of velocity and its gradients

Vorticity

- stretched exponential tails
- con. ave. depend strongly on $\boldsymbol{\Omega}$
- reason: localized quantity, vorticity and gradients vary on comparable scales

Connection to Local Structure: Conditional Vorticity Field

▷ analysis of PDF equation: unclosed terms can be calculated from conditional vorticity field:

$$\boxed{\left\langle \boldsymbol{\omega}_0 \middle| \boldsymbol{\Omega}_1, \boldsymbol{r} \right\rangle = a(r, \Omega_1, \gamma)\, \widehat{\boldsymbol{\Omega}}_1 + b(r, \Omega_1, \gamma)\, \hat{\boldsymbol{\lambda}} + c(r, \Omega_1, \gamma)\, \hat{\boldsymbol{\tau}}}$$

$$\widehat{\boldsymbol{\Omega}}_1 = \frac{\boldsymbol{\Omega}_1}{\Omega_1}$$

$$\hat{\boldsymbol{\lambda}} = \frac{\hat{\boldsymbol{r}} - (\hat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{\Omega}}_1)\widehat{\boldsymbol{\Omega}}_1}{\left[1 - (\hat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{\Omega}}_1)^2\right]^{\frac{1}{2}}}$$

$$\hat{\boldsymbol{\tau}} = \widehat{\boldsymbol{\Omega}}_1 \times \hat{\boldsymbol{\lambda}}$$

▷ a, b and c projections of $\left\langle \boldsymbol{\omega}_0 \middle| \boldsymbol{\Omega}_1, \boldsymbol{r} \right\rangle$ on $\widehat{\boldsymbol{\Omega}}_1$, $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\tau}}$

Connection to Local Structure

- unclosed terms can be calculated from conditional vorticity field:

$$\Lambda(\Omega_1) = \lim_{r\to 0} \nu\, \widehat{\boldsymbol{\Omega}}_1 \cdot \Delta_{\boldsymbol{r}} \left[a\, \widehat{\boldsymbol{\Omega}}_1 + b\, \hat{\boldsymbol{\lambda}} \right]$$

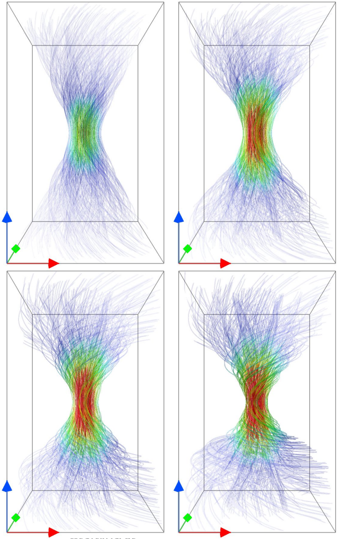
$$\Sigma(\Omega_1) = \frac{3}{4\pi} \int \mathrm{d}\boldsymbol{r}\, \frac{\gamma\,(1-\gamma^2)^{\frac{1}{2}}\, c}{r^3}$$

- dissipative terms related to a and b
- enstrophy production related to c
- a and b related by solenoidality

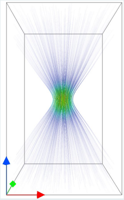
- recall version 1 (Φ is negligible):

$$\boxed{\frac{\partial}{\partial t}\tilde{f} = -\frac{\partial}{\partial \Omega}\left(\Sigma\,\Omega + \Lambda + \Phi\right)\tilde{f}}$$

Connection to Local Structure: DNS Results



- localized twisted structure
- twist increases with magnitude of vorticity
- Gaussian random field:



- twist non-Gaussian feature!

cf: Novikov et al. Fluid Dyn. Res. 1993,

Mui et. al PRE 1996

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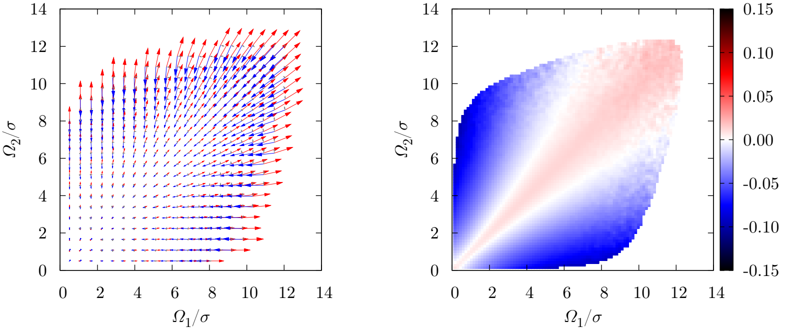
Two-Point Enstrophy Statistics

- full two-point statistics numerically inaccessible (dimensionality)
- projection: two-point enstrophy (or magnitude of vorticity) statistics ($z_1 = \frac{1}{2}\boldsymbol{\omega}_1^2$):

$$\frac{\partial}{\partial t} f_2 + \frac{1}{r}\frac{\partial}{\partial r} r \left\{ \left\langle \delta u_l \middle| z_1, z_2, r \right\rangle f_2 \right\}$$
$$= - \frac{\partial}{\partial z_1} \left\{ \left\langle \boldsymbol{\omega}_1 \mathrm{S}_1 \boldsymbol{\omega}_1 + \nu \boldsymbol{\omega}_1 \cdot \Delta_{\boldsymbol{x}_1} \boldsymbol{\omega}_1 \middle| z_1, z_2, r \right\rangle f_2 \right\}$$
$$- \frac{\partial}{\partial z_2} \left\{ \left\langle \boldsymbol{\omega}_2 \mathrm{S}_2 \boldsymbol{\omega}_2 + \nu \boldsymbol{\omega}_2 \cdot \Delta_{\boldsymbol{x}_2} \boldsymbol{\omega}_2 \middle| z_1, z_2, r \right\rangle f_2 \right\}$$

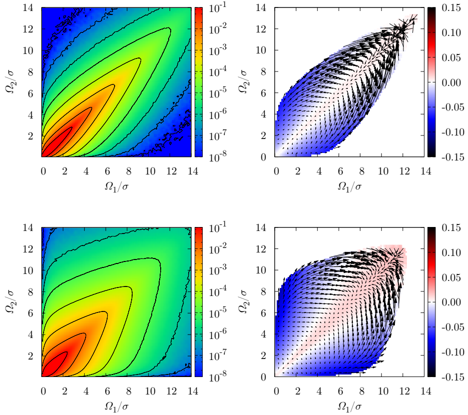
- enstrophy production and dissipation at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$
- scale transfer by longitudinal velocity increment

Two-Point Enstrophy Statistics



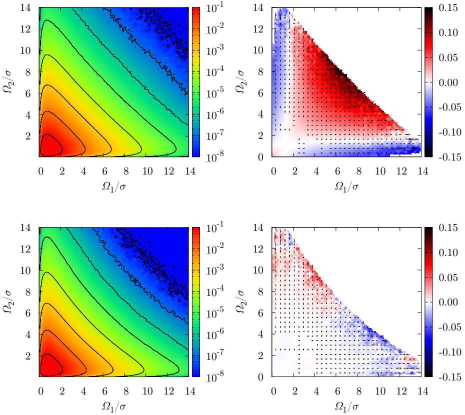
- amplification by enstrophy production, depletion by dissipation
- small net vector field (here: $r = 2.5\eta$)
- up- and downscale probability flux (due to $\langle \delta u_l \rangle = 0$)

Two-Point Enstrophy Statistics: Small Scales



- small-scale (1.2,2.5η) dynamics: leveling of vorticity
- amplification and decay
- typical "life cycle" of a vortex

Two-Point Enstrophy Statistics: Large Scales



- intermediate/large scales $(40, 320\eta)$: balance of enstrophy production and dissipation
- recovery of single-point statistics
- vanishing scale transfer

Summary

- LMN hierarchy: framework for a non-Equilibrium Statistical Mechanics of Turbulence
- DNS delivers information on unclosed terms
- single-point vorticity statistics: balance of enstrophy production and dissipation
- shape of PDF depends additionally on conditional enstrophy dissipation tensor
- enstrophy production and dissipation may be expressed in terms of conditional local structure of the vorticity field
- two-point vorticity statistics: scale transfer and local imbalance
- Ad: Interested in this method applied to RB convection? → NJP 13(1), 015002, 2011

References and Acknowledgements

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Visualizations produced by VAPOR, `www.vapor.ucar.edu`

Appendix: DNS parameters

N	R_λ	L	T	u_{rms}	ν	$\langle\varepsilon\rangle$	$k_{\max}\eta$
512	112	1.55	2.86	0.543	10^{-3}	0.103	2.03