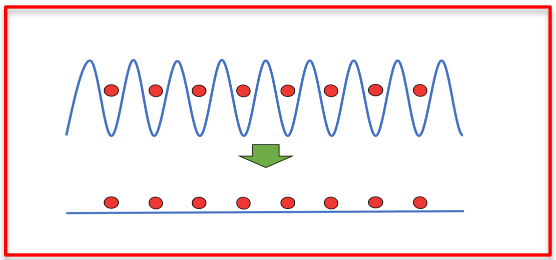
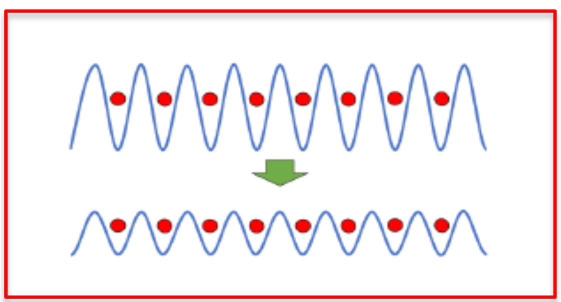
Quench dynamics of an optical lattice - local and global aspects

Bosons released in optical traps



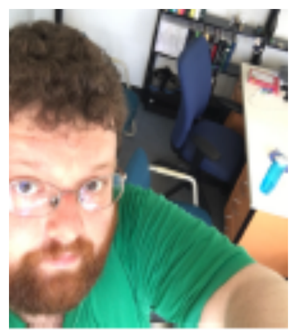
Time evolution:
The Lieb-Liniger Hamiltonian



Time evolution:
The Sine-Gordon Hamiltonian

RUTGERS

Natan Andrei
Rutgers university



Garry Goldstein
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SISSA

Non Equilibrium Universality in Many Body Physics
KITP, Sept 2021

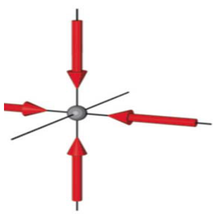
Non equilibrium dynamics of quantum systems - Quantum Quench of Optical Lattices

- **Nonequilibrium – the new frontier : Old and new questions**

- **Many experiments: cold atom systems, nano-devices, molecular electronics**
- **Isolated systems – effects not washed out by coupling to environment**
- **Fine control of parameters**
- **Many systems described by integrable Hamiltonians**

- Standard nonequilibrium protocol: Quantum Quench

e.g. release bosons in optical traps



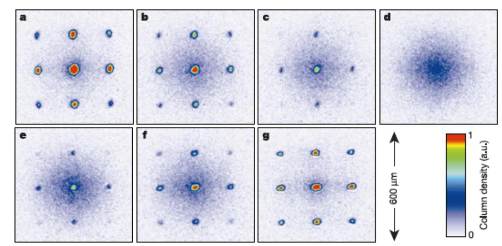
Bloch et al 2008

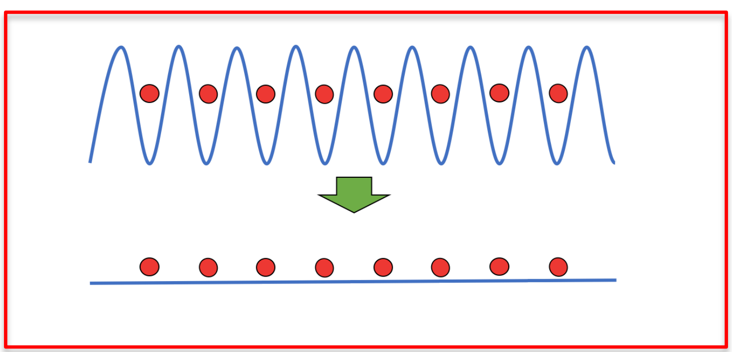
Greiner et al 2002

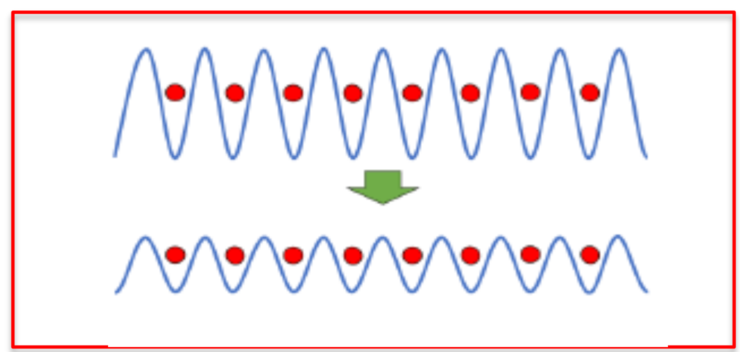
SF

Sudden raise of potential

Mott







Mott

SF

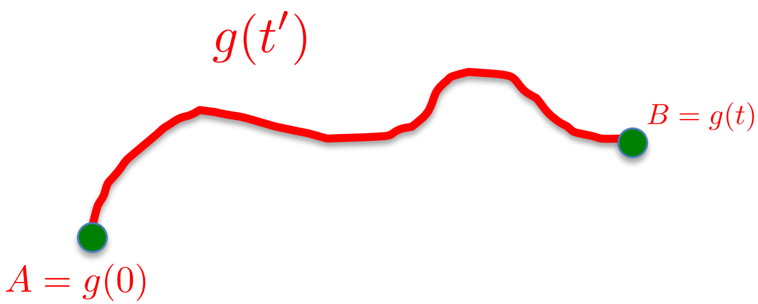
The Lieb-Liniger model

The sine-Gordon model

Time evolution - Quench protocol in isolated systems

Quench protocol

- **Isolated system** with Hamiltonian $H(t) = H[g(t)]$ depends on "work parameter" $g(t)$
- Initial state, $|\Phi_i\rangle$, typically ground state of $H(0) = H[A]$
- Evolve initial state under $H(t') = H[g(t')]$ from $t' = 0$ to $t' = t$



$$|\Phi_i, t\rangle = Te^{-i\int_0^t H(t')dt'}\,|\Phi_i\rangle \quad \Longrightarrow \quad |\Phi_i, t\rangle = e^{-iHt}|\Phi_i\rangle$$

Sudden quench

- Process depends on initial state and on Hamiltonian
- *Local characteristics*: evolution of local observables, correlations, thermalization, ETH ..
- *Global characteristics*: **quantum work,** spread of entanglement ..

Time Evolution and the Bethe Ansatz

How to time evolve the initial state?

A given state $|\Phi_0\rangle$ can be time evolved using partition of the unity $I = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|$

$$|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle \quad \longrightarrow \quad |\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle = \sum_\lambda e^{-i\epsilon_\lambda} |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$$

If H integrable $\longrightarrow$ eigenstates $|F^\lambda\rangle$ are known via the *Bethe-Ansatz*

- Use Bethe Ansatz to study quench evolution and nonequilibrium
- New technology is necessary:

- **Standard approach: PBC (or OBC) $\longrightarrow$ Bethe Ansatz eqns $\longrightarrow$ spectrum $\longrightarrow$ thermodynamics**

- **Non equilibrium entails additional difficulties:**

i. Compute overlaps (form factors) ii. Compute matrix elements iii. Sum over complete basis

Much work in this direction: quench action *J. S. Caux et al, Essler et al, Calabrese et al*

ii. Contour approach - does not required computation of overlaps

Yudson's contour representation (infinite volume)

Instead of $|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$ introduce (directly in infinite volume):

- Contour representation of $|\Phi_0\rangle$

$$\boxed{|\Phi_0\rangle = \int_\gamma d^N |F^\lambda\rangle(F^\lambda|\Phi_0\rangle}$$

V. Yudson, sov. phys. JETP (1985)

Computed S-matrix of Dicke model

with: $|F^\lambda\rangle$ Bethe eigenstate, labeled by momenta $\{\lambda\}= \lambda_1 \ldots \lambda_N$

$|F^\lambda)$ obtained from Bethe eigenstate by setting $S=I$ in one quadrant $\theta(\vec{x}) = \theta(x_1 > x_2 > \cdots .x_N)$

γ contour in momentum space $\{\lambda\}$ determined by **pole structure** of $S(\lambda_i - \lambda_j)$

- In the infinite volume limit momenta $\{\lambda\}$ are not quantized (no Bethe Ansatz equations)

- Can generalize approach to *finite volume* (Goldstein, NA), allows application to thermodynamic systems with finite density

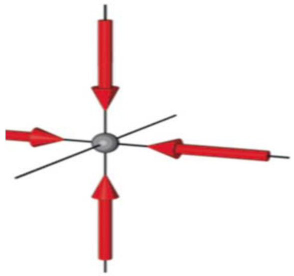
- The time evolution follows

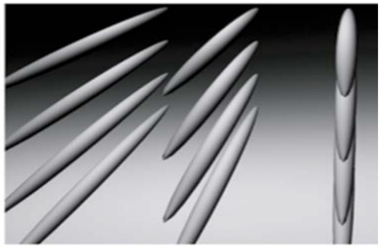
$$\boxed{|\Phi_0, t\rangle = \int_\gamma d^N\lambda\, e^{-iE(\lambda)t}|F^\lambda\rangle(F^\lambda|\Phi_0\rangle}$$

Ultracold Atoms – the Lieb Liniger model

Gas of neutral atoms moving on the line and interacting with short-range interaction $V(x_1 - x_2) = c\delta(x_1 - x_2)$

$$\boxed{H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c\sum_{j<l} \delta(x_j - x_l)}$$

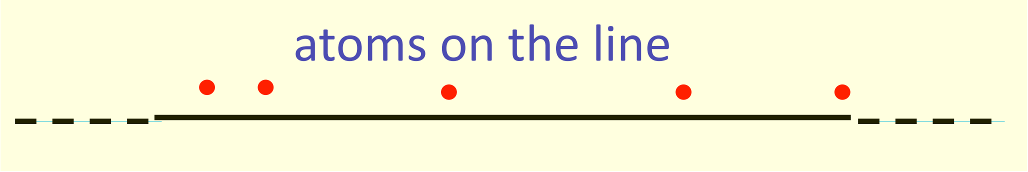




Bloch et al '08

$c > 0$ repulsive
$c < 0$ attractive

Can be tune by Feshbach resonance



Comment*:*

- Very short range interaction. Valid for low densities,

$$l = L/N \gg l_{\text{Van der Vaals}}$$

- The description of physics depends on the scale of observation

Bosonic system – BA solution

The N-boson eigenstatestate (Lieb-Liniger '63)

$$|F^\lambda\rangle = |\lambda_1, \cdots, \lambda_N\rangle = \int_y \prod_{i<j} Z^y_{ij}(\lambda_i - \lambda_j) \prod_j e^{i\lambda_j y_j}\, b^\dagger(y_j)|0\rangle = \int_y F^\lambda(y_1 \cdots y_N) \prod_j b^\dagger(y_j)|0\rangle$$

- **Eigenstates labeled by Momenta** $\lambda_1, \cdots, \lambda_N \begin{cases} \textit{real} & c>0 \\ \textit{n-strings} & c<0 \end{cases}$

- **Dynamic factor:** $Z^y_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} = \begin{cases} 1 & y_i > y_j \\ S^{ij} & y_i < y_j \end{cases}$

  - The 2-particle S-matrix: $S_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$ enters when the particles cross

  - poles of the S-matrix at: $\lambda_i = \lambda_j + ic$

- **The energy eigenvalues**

$$H|\lambda_1, \cdots, \lambda_N\rangle = \sum_j \lambda_j^2\, |\lambda_1, \cdots, \lambda_N\rangle$$

- ***Thermodynamics***: *impose PBC* $\longrightarrow$ *BA eqns* $\longrightarrow$ *momenta* $\longrightarrow$ Energy eigenvalues

- ***Dynamics*** *(infinite volume): momenta unconstrained*

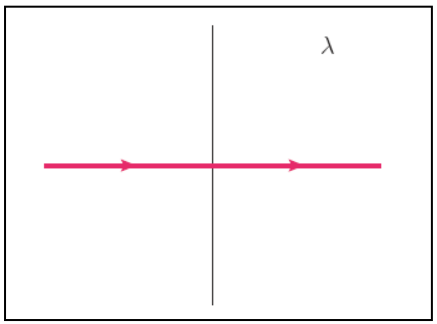
Bosonic system - contour representation

The contour representation $|\Phi_0\rangle = \int_\gamma d^N\lambda\, |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$ **takes the form:**

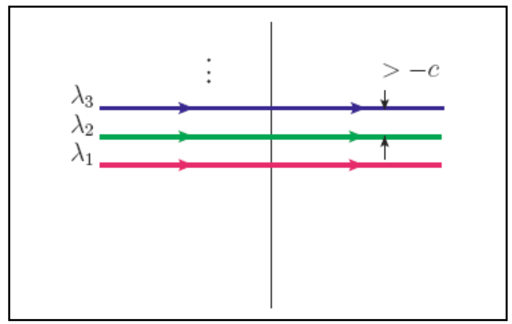
$$|\Phi_0\rangle = \int_x \Phi_0(\vec{x}) b^\dagger(x_N)\cdots b^\dagger(x_1)|0\rangle = \qquad F^\lambda(y)$$

$$= \int_{x,y}\int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle$$

$$(F^\lambda|\Phi_0\rangle$$



Repulsive c > 0



Attractive c < 0,

contour accounts for strings, bound states

It time evolves to:

$$\boxed{|\Phi_0, t\rangle = \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\lambda_j^2 t} e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle}$$

- **Expression contains full information about the dynamics of the system**

What to calculate?

- **We shall study local correlation functions:**

1. Evolution of the density $\rho(x) = b^\dagger(x) b(x)$

$$C_1(x,t) = \langle \Phi_0, t | b^\dagger(x) b(x) | \Phi_0, t \rangle = \int dx_1..dx_N \, |\Phi_0(x_1,..,x_N,t)|^2 \sum_j \delta(x - x_j)$$

- The probability to find the bosons at point x at time t if at time $t=0$ they started with wave function $\Phi_0(x_1,..,x_N)$

- Can be measured by Time of Flight experiments

- **competition** between quantum broadening and attraction

2. Evolution of noise correlation

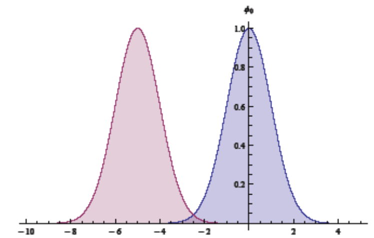
$$C_2(x,y,t) = \frac{\langle \Phi_0, t | \rho(x) \rho(y) | \Phi_0, t \rangle}{C_1(x,t) C_1(y,t)} - 1$$

- Time dependent Hanbury Brown - Twiss effect

Evolution of a bosonic system: density

- Consider an initial state:

$$\Phi_0(x_1, x_2) = \frac{1}{(\pi\sigma^2)^{\frac{1}{4}}} e^{-\frac{(x_1)^2+(x_2+a)^2}{2\sigma^2}}$$

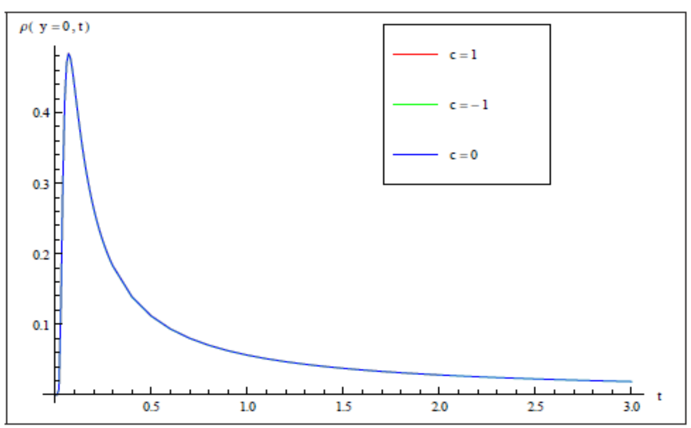


- Its evolution is:

$$\Phi_0(y_1, y_2, t) = \int_x \Phi_0(x_1, x_2)\frac{1}{4\pi i t} e^{i\frac{(y_1-x_1)^2}{4t}+i\frac{(y_2-x_2)^2}{4t}}[1 - c\sqrt{\pi i t}\theta(y_2-y_1)e^{\frac{i}{8t}\alpha^2(x,y,t)}\mathrm{erfc}(\frac{i-1}{4}\frac{i\alpha(x,y,t)}{\sqrt{t}})]$$

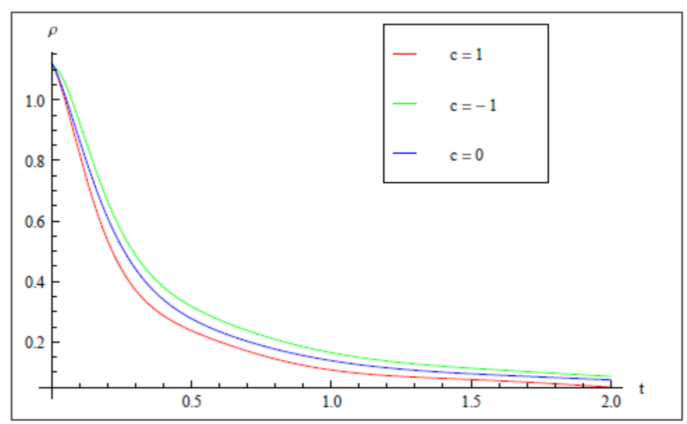
- Compute the evolution of the density $\rho(0, t)$

i. Initial condition: $a \gg \sigma$



No interaction effects- small initial overlap, then density too low

ii. Initial condition: $a \ll \sigma$

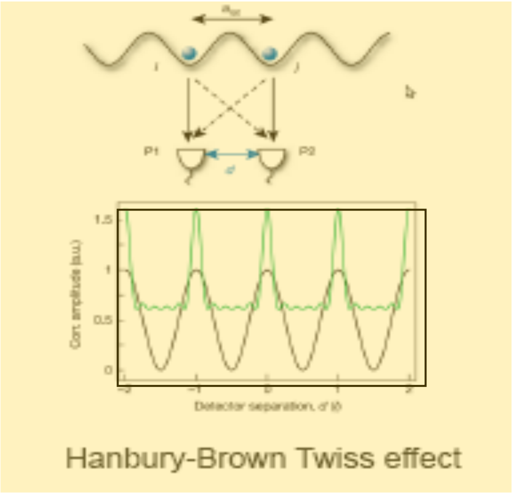


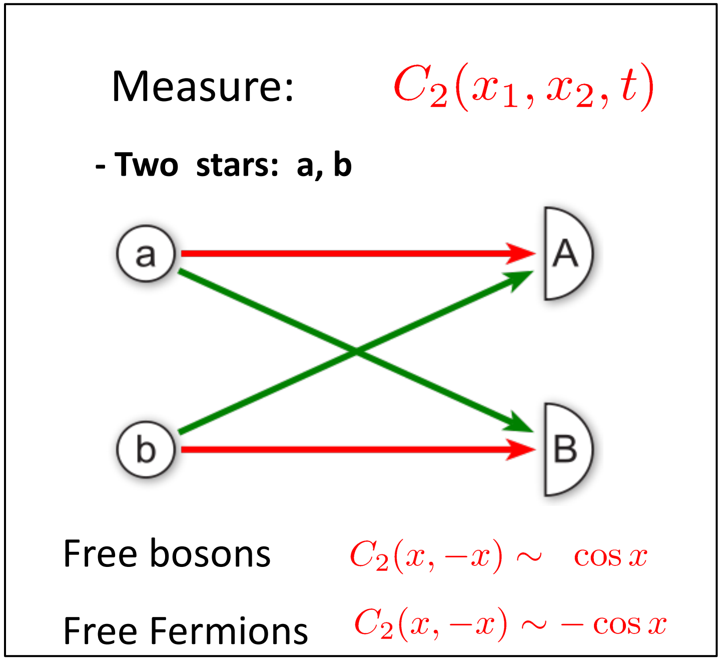
Strong interaction effects: initial overlap

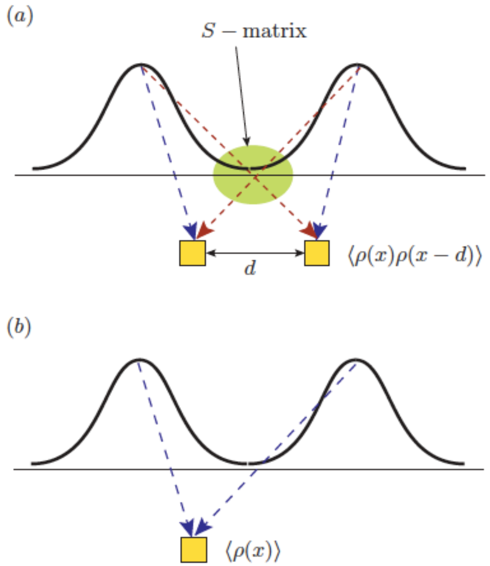
The Hanbury Brown – Twiss effect

- Hanbury brown-Twiss effect for two bosons

Bloch et al.
RMP '08







Time dependent Hanbury Brown - Twiss effect

- Time dependent
- Many bosons
- More structure: main peaks, sub peaks
- Effects of interactions?

- repulsive bosons evolve into fermions
- attractive bosons evolve to a condensate

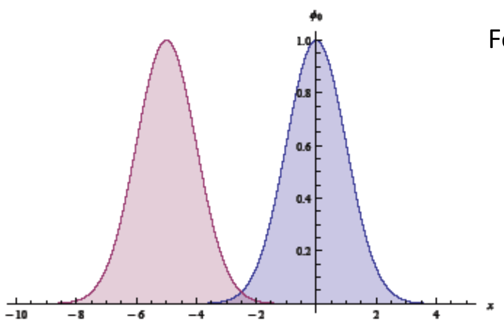
Evolution of repulsive bosons into fermions: HBT

Density - Density correlation: long time asymptotics - repulsive: (Iyer, NA)

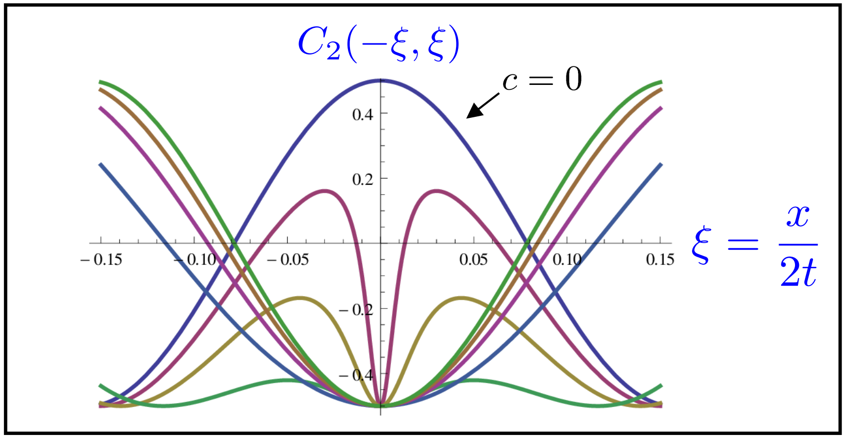
- **Bosons turn into fermions as time evolves (for any $c > 0$)**
- **Can be observed in the noise correlations: (dependence on t only via $\xi = x/2t$)**

$$C_2(x_1, x_2, t) \to C_2(\xi_1, \xi_2) = \frac{\langle \rho(\xi_1)\rho(\xi_2)\rangle}{\langle \rho(\xi_1)\rangle\langle \rho(\xi_2)\rangle} - 1,$$

$c/a = 0, .3, \cdots, 4$



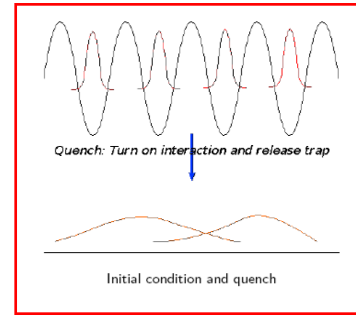
Fermionic correlations evolve



- **Fermionic dip develops for any repulsive interaction on time scale set by** $1/c^2$

Evolution of a bosonic system: noise correlations

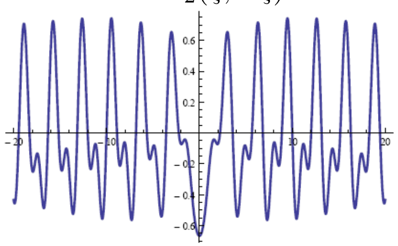
Noise correlations – starting from a lattice



Repulsive bosons

3 particles

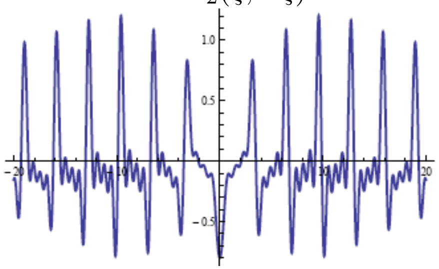
$C_2(\xi, -\xi)$



ξa

5 particles

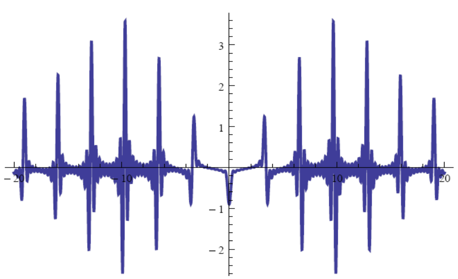
$C_2(\xi, -\xi)$



ξa

10 particles

$C_2(\xi, -\xi)$



ξa

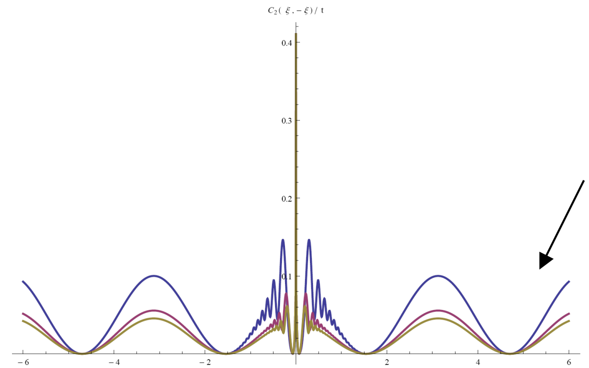
Fermionic dip as $\xi \to 0$

Structure emerges at $\xi a = \sigma$

Attractive bosons

2 particles

$C_2(\xi, -\xi)/t$



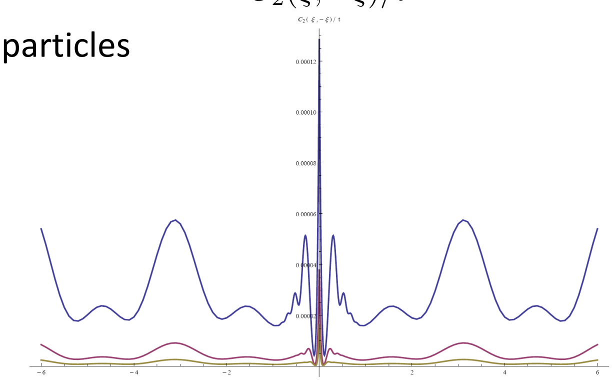
ξa

Blue - short times
Violet - longer
Magenta - longest

central peaks increase with time
- weight in the bound states increases

3 particles

$C_2(\xi, -\xi)/t$



ξa

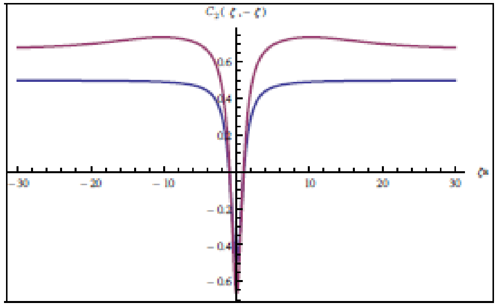
Blue - short times
Violet - longer
Magenta - longest

peaks diffuse – momenta redistribute

Evolution of a bosonic system: noise correlation

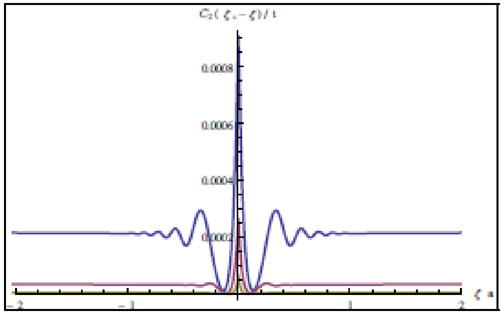
Noise correlations – starting from a condensate:

Repulsive bosons



Two (blue) and three bosons,

Attractive bosons



Three bosons, at times: $tc^2 = 20; 40; 60$

Dynamic Fermionization

Long time asymptotics - repulsive interaction

Recall: $|\Phi_0, t\rangle = \int_\gamma d^N\lambda\, e^{-iE(\lambda)t}|F^\lambda\rangle(F^\lambda|\Phi_0\rangle$

- **Bosons turn into fermions as time evolves (for any $c > 0$)**

D. Jukic et al '08

$$|\Phi_0, t\rangle = \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\Sigma_j \lambda_j^2 t - \lambda_j(y_j - x_j)} \prod_j b^\dagger(y_j)|0\rangle$$

$$= \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\sqrt{t}\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic\sqrt{t}} e^{-i\Sigma_j \lambda_j^2 - \lambda_j(y_j - x_j)/\sqrt{t}} \prod_j b^\dagger(y_j)|0\rangle$$

$$\to \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) e^{-i\Sigma_j \lambda_j^2 - \lambda_j(y_j - x_j)/\sqrt{t}} \prod_{i<j} sgn(y_i - y_j) \prod_j b^\dagger(y_j)|0\rangle$$

$$= e^{-iH_0^f t} \int_{x,k} \mathcal{A}_x\, \theta(\vec{x})\Phi_0(\vec{x}) \prod_j c^\dagger(x_j)|0\rangle.$$

$\mathcal{A}_x$ antisymmetrizer

where

$$H_0^f = -\int_x c^\dagger(x)\partial^2 c(c)$$

- In the long time limit repulsive bosons for any $c > 0$ propagate under the influence of Tonks – Girardeau Hamiltonian (hard core bosons ~ free fermions)

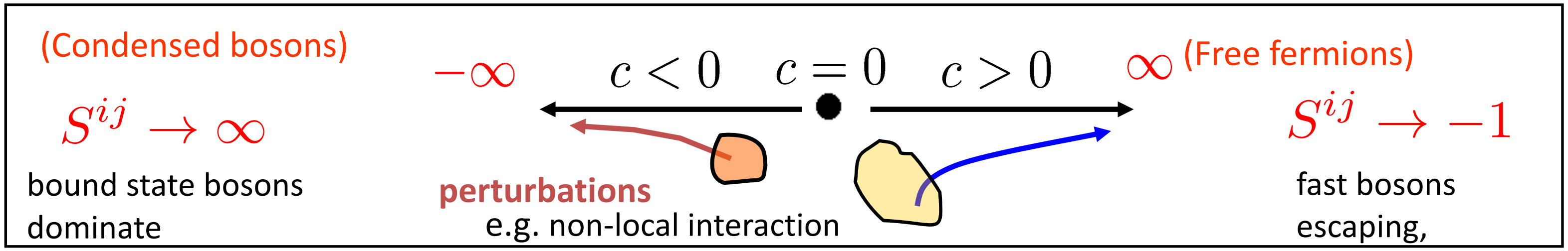
- Argument valid for any initial state Φ_0

- Experimentally observed recently - *Greiner Group 2015, Weiss group 2020*

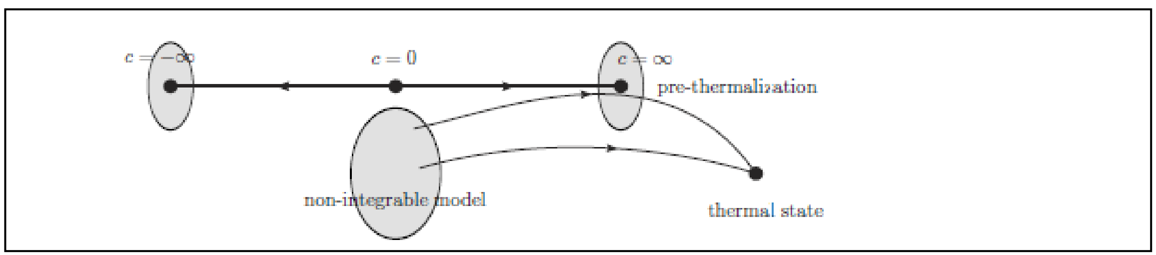
- **Scaling argument fails for attractive bosons** (instead, they form bound states)

Evolution of a bosonic system

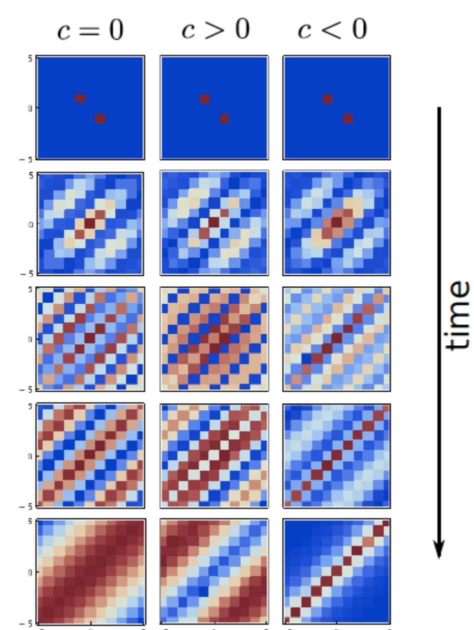
- Conclusion: coupling constant *"effectively evolves"* with time $t \sim \ln(D_0/D)$



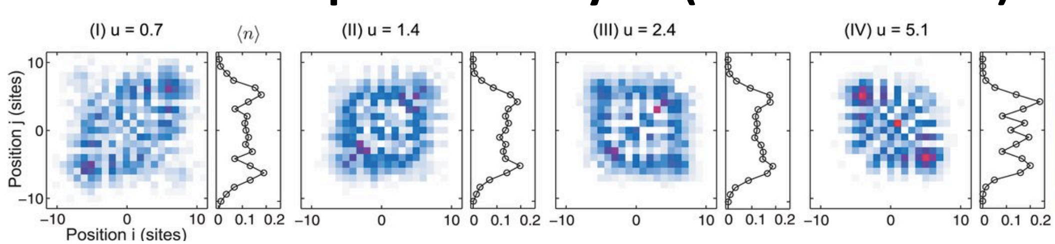
- **What is beyond** $c = \infty$ **?** **Thermalization**



(Iyer, NA)



- **Fermionization also occurs on the lattice: *Bose Hubbard model* (not integrable)**
- **Observed experimentally** **(***Greiner et al. '15***)**

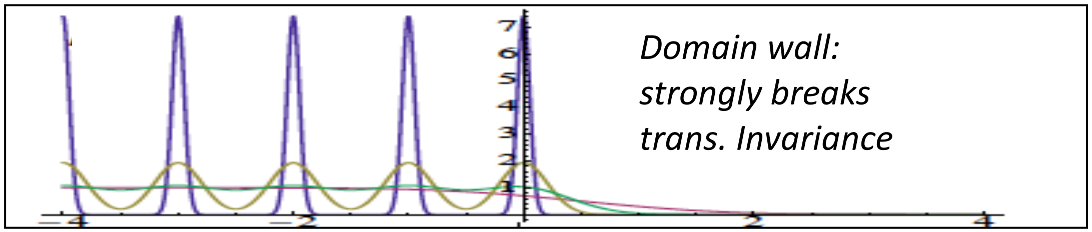


Evolution to NESS: Domain wall (no thermalization)

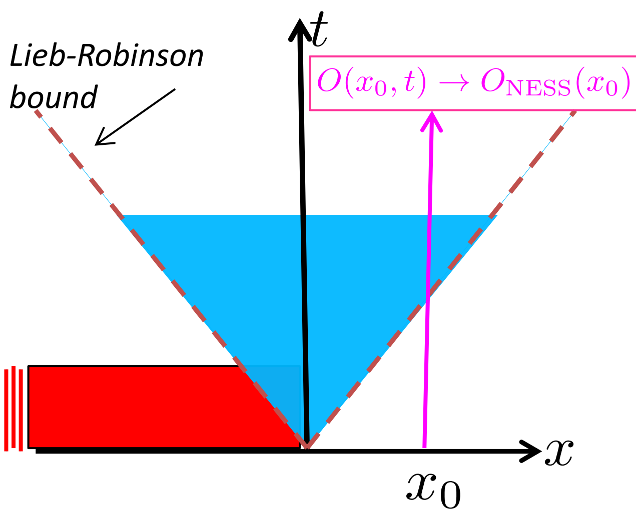
Example*: time evolution from a non-trans. invariant initial state (no equilibration)*

$$|\Psi(t=0)\rangle = \prod_{j=0}^{\infty}\int_{-\infty}^{\infty}\varphi(x+jl)\,b^{\dagger}(x)\,|0\rangle$$

with: $\varphi(x) = \dfrac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$



- System evolves to NESS



$\rho(x,t) \rightarrow$ (G Goldstein, NA)

$\frac{1}{l}\left(1+\sum_{s=1}^{\infty}e^{-\pi^2(8t^2/\sigma+\sigma/2)s^2/l^2}\cos(2\pi sx/l)\right)$ Outside the cone $\quad x \ll -\sqrt{8t^2/\sigma}$

$\frac{1}{l}\left(\frac{1}{2}-\frac{4\pi}{cl}\right)$ *Nonequilibrium steady state (NESS)* *Independent of* x, t *- interaction effects* $\quad \sigma \ll |x| \ll t$

$$\frac{1}{l}\left(\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)+\right.$$
$$+\frac{16}{\pi cl}e^{-x^2/A}\left(\frac{1}{2}\sqrt{\pi}\frac{x}{\sqrt{A}}Erfc\left(\frac{x}{\sqrt{A}}\right)-\frac{1}{2}e^{-x^2/A}\right)$$
$$\left.+\frac{16}{\pi cl}\frac{\pi}{2}\left(1-\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)\right)Erfc\left(\frac{x}{\sqrt{A}}\right)\right)$$

$x \sim t$

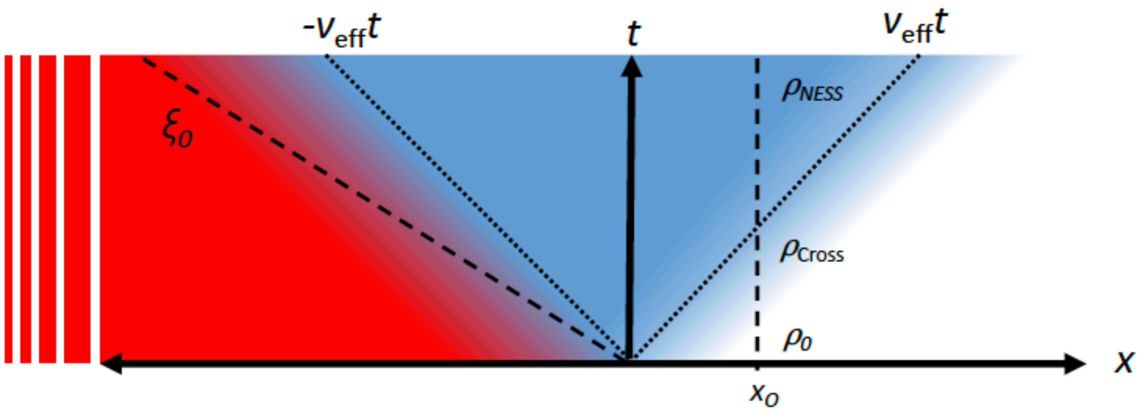
$A \equiv 8t^2/\sigma+\sigma/2$

Crossover regime

0 *Outside the cone* $\quad x \gg \sqrt{8t^2/\sigma}$

Evolution to NESS: Domain wall (no thermalization)

The domain wall quench:



- **Nonequilibrium Steady state** (NESS) : order of limits $L \to \infty$ *vs*. $t \to \infty$

- Evolution along space-time rays $x/t = \xi_0$

- RG flow in time, *dynamical fermionization*

On the left, outside the "light cone, $$\rho(x,t) = \frac{1}{\delta}\left[1 + \sum_{s=1}^{\infty} e^{-\sigma(t)\frac{\pi^2 s^2}{\delta^2}} \cos\left(\frac{2\pi s x}{\delta}\right)\right]$$

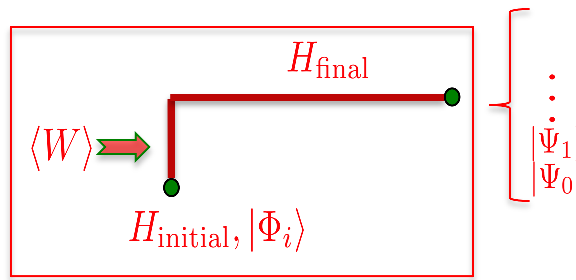
Same result as from TG gas (Tonks-Girardeau $c \to \infty$), cf Hanbury Brown – Twiss effect

Global Aspects: Nonequilibrium Thermodynamics

Quantum work

- During the quench energy is pumped into the system - $W = E_{\rm final} - E_{\rm initial}$: **work** is done

First law of thermo' $dE = dQ + dW$ but system isolated: $dQ = 0$, so $dE = dW$



Work – *random variable* : involves two measurements - initial and final energies,
- at initial time - yielding $|\Phi_i\rangle$ and ϵ_i with probability $P_i = 1$
- at final time - yielding $|\Psi_n\rangle$ and E_n with probability $P_{i\to n} = |\langle \Psi_n | \Phi_i \rangle|^2$

- For a sudden quench the work distribution: *(Talkner et al '07 "Not an observable")*

$$P(W) = \sum_n \delta\left(W - (E_n - \epsilon_i)\right) |\langle \Psi_n | \Phi_i \rangle|^2$$

Nonequilibrium Dynamics – Loschmidt amplitude

- Claim: Work distribution is related to Loschmidt Amplitude (*Talkner et al 07*)

$$\mathcal{G}(t) = \langle \Phi_i | \, e^{-iHt} \, | \Phi_i \rangle$$

$$P(W) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} e^{iWt + i\epsilon_i t} \mathcal{G}(t)$$

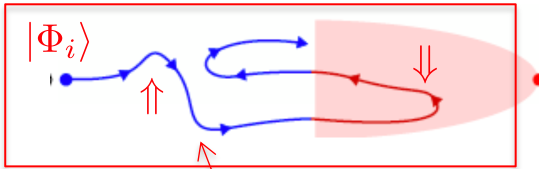
Related concepts:

Return Probability
Survival probability
Persistence

- The Loschmidt amplitude probes the full Hilbert space of states

$$\mathcal{G}(t) = \langle \Phi_i | \, e^{-iHt} \, | \Phi_i \rangle = \sum_n e^{-iE_n t} |\langle \Psi_n | \Phi_i \rangle|^2$$

starting from and weighted by overlaps with initial state $|\Phi_i\rangle$



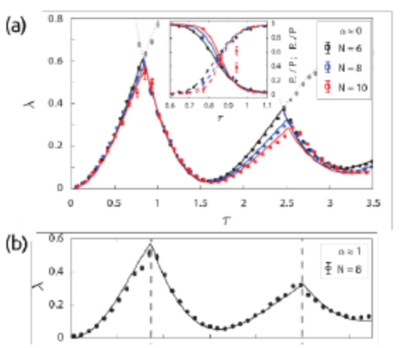
$e^{-iHt} \, |\Phi_i\rangle$

- May exhibit singularities Dynamical Quantum Phase Transition *(Kehrein, Heyl '18)* similarities to thermodynamic phase transitions, Fisher &Lee-Yang zeroes

$$\lambda(t) = \lim_{N \to \infty} \frac{1}{N} \log |\mathcal{G}(t)|^2$$

DQPT depends on initial state, not only on the Hamiltonian

Observation of DQPT in Ising model

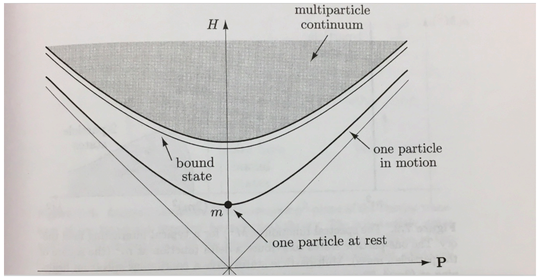


Jurevic et al '17

Nonequilibrium Thermodynamics -Work done in a quench

- $P(W)$ **has the form of the spectral function:**

Silva, Gambassi, Palmai, Sotiriadis, Mussardo, Calabrese, Goold …. : ' 08 –'18



- Defined for $W \geq \delta E = E_0 - \epsilon_i$

- A delta function at threshold $W = \delta E$
  weighted by the fidelity $\mathcal{F} = |\langle \Psi_0 | \Phi_i \rangle|^2$ transition from the initial state to the ground state of H_f

- If H_f is gapped and translation invariant - continuum of excited states into which $|\Phi_i\rangle$ can transition with $W = 2m + \delta E$ lowest threshold for the continuum.

- Power like behavior at threshold $P(W) \sim \theta(W - \delta E - 2m)(W - \delta E - 2m)^{\alpha}$

- Similarly the four-particle emission continuum threshold $W = 4m + \delta E$

- It there are bound states m_b then delta function may appear at $W = m_b + \delta E$

- **Work distribution directly measureable**

- ***Reversible and irreversible processes, entropy production, spread of entanglement***

- ***Fluctuation theorems*** : *Jarzynski ' 97* $\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$ *Cohen-Galavotti ' 95* $\frac{P(S)}{P(-S)} = e^{\frac{S}{k_B}}$
  Crooks ' 99 $\frac{P_F(W)}{P_B(-W)} = e^{\frac{W-\Delta F}{T}}$ *and more.*

- relate equilibrium and nonequilibrium, relate forward and backward evolution

Eigenstates and Yudson states

- The exact eigenstates of the Lieb-Liniger model – study finite size

$$|\{k_j\}\rangle = \int \mathrm{d}^N x \prod_{\substack{i,j=1 \\ i<j}}^{N} \frac{k_i - k_j - ic\,\mathrm{sgn}(x_i - x_j)}{k_i - k_j - ic} \prod_{l=1}^{N} e^{ik_l x_l} b^\dagger(x_l)|0\rangle \quad \longrightarrow |\{n_j\}\rangle$$

For PBC the momenta satisfy $\boxed{k_j = \frac{2\pi}{L} n_j - \frac{1}{L}\sum_{l\neq j} \varphi(k_j - k_l)}$ with $\varphi(x) = 2\arctan(x/c)$

The integer quantum numbers $\{n_j\}$ determine the momenta $\{k_j\}$, denote eigenstates $|\{n_j\}\rangle$

- Calculation of overlaps $\langle\{n_j\}|\Psi_i\rangle$ between initial state and Bethe eigenstates is very difficult!
  - Use Yudson representation

- Claim *(Yudson ; Goldstein, NA)* : Can rewrite the standard partition of the unity

$$I_N = \sum_{n_1<\cdots<n_N} \frac{|\{n_j\}\rangle\langle\{n_j\}|}{\mathcal{N}(\{n_j\})}$$ as $$I_N = \sum_{n_1,\ldots,n_N} \frac{|\{n_j\}\rangle(\{n_j\}|}{\mathcal{N}(\{n_j\})}$$

normalization

$$\mathcal{N}(\{n\}) = \det\left[\delta_{jk}\left(L + \sum_{l=1}^{N}\varphi'(k_j - k_l)\right) - \varphi'(k_j - k_k)\right]$$

in terms of Yudson states $|\{n\}) = \int \mathrm{d}^N x\, \theta(x_1 > \cdots > x_N) \prod_{l}^{N} e^{ik_l x_l}\, b^\dagger(x_l)\,|0\rangle$

The Loschmidt Amplitude

- This allows the computation of overlaps and therefore of time evolution of $|\Psi_i\rangle$:

$$|\Psi_i(t)\rangle = \left[\frac{4\pi\hbar}{m\omega}\right]^{\frac{N}{4}} \sum_{n_1,\ldots,n_N} \frac{e^{-\sum_{j=1}^{N}\left[\frac{\hbar}{2m\omega}(1+i\hbar\omega t)k_j^2+ik_j\bar{x}_j\right]}}{\mathcal{N}(\{n_j\})}|\{n_j\}\rangle$$

- The Loschmidt Amplitude:

$$\boxed{\mathcal{G}(t) = \langle\Phi_i|\, e^{-iHt}\,|\Phi_i\rangle = \langle\Phi_i|\Phi_i(t)\rangle = \left[\frac{4\pi}{m\omega}\right]^{\frac{N}{2}} \sum_{n_1,\ldots,n_N} e^{-\frac{1}{m\omega}\left[1+i\frac{\omega}{2}t\right]\sum_{j=1}^{N}k_j^2}\frac{G(\{n\})}{\mathcal{N}(\{n\})}}$$

with

$$G(\{n\}) = \det\left[e^{-ik_j(\bar{x}_j-\bar{x}_k)-i\theta(j-k)\varphi(k_j-k_k)}\right]$$

$$\mathcal{N}(\{n\}) = \det\left[\delta_{jk}\left(L+\sum_{l=1}^{N}\varphi'(k_j-k_l)\right)-\varphi'(k_j-k_k)\right]$$

Other approach: *Quench action*
Perfetto *et al.*

- Exact for any $c, N, L.$ with $\rho = N/L$ and ω characterizing the initial state

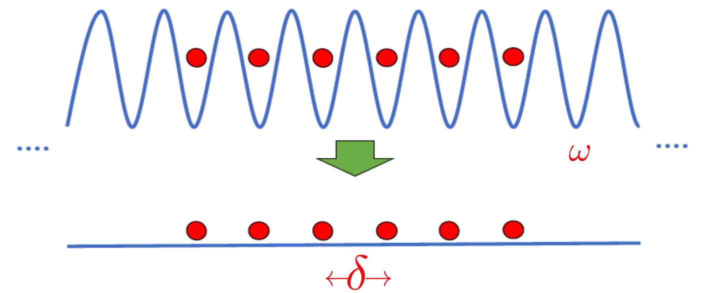
- Displays **recurrences** : $\tau = (1+2\rho/c)^2\, L^2/\pi\omega$

Quench of an optical lattice: *gapless scenario*

- **Quench protocol**: Release N bosons from a deep trap

$$\bar{x}_j = j\delta, \; j = 1 \cdots N$$

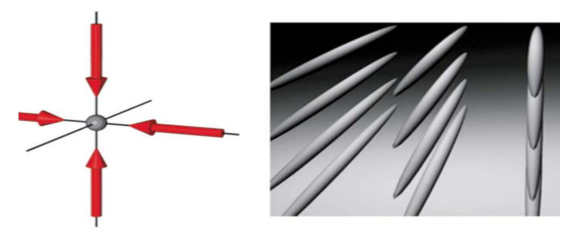
$$|\Psi_i\rangle = \int \mathrm{d}^N x \prod_{j=1}^{N} \left[\frac{m\omega}{\pi}\right]^{\frac{1}{4}} e^{-\frac{m\omega}{2}(x_j - \bar{x}_j)^2} b^\dagger(x_j)|0\,\rangle$$



- Bosons well described by the Lieb-Liniger Hamiltonian

$$H_{LL} = \frac{1}{2m} \int \mathrm{d}x \,[\, -b^\dagger(x)\partial_x^2 b(x) + c\, b^\dagger(x)b(x)b^\dagger(x)b(x)\,]$$

I. Bloch et al



Coupling: $c > 0$ repulsive, $c < 0$ attractive

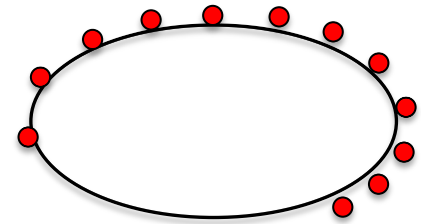
- Allow them to evolve $|\Psi(t)\rangle = e^{-itH_{LL}}|\Psi_i\rangle$ **Localized peaks broaden and bosons begin to collide**

- How to calculate evolution? Use partition of the unity

$$|\Psi_i(t)\rangle = e^{-itH_{LL}}|\Psi_i\rangle = e^{-itH_{LL}} \sum_n |n\rangle\langle n|\Psi_i\rangle = \sum_n e^{-iE_n t}|n\rangle\langle n|\Psi_i\rangle$$

The Loschmidt Amplitude (infinite volume)

Open system (partially filled lattice) $\rho = N/L \ll 1/\delta$



(i.) ***Repulsive interactions*** Simplifies for strong repulsion $c \gg m\omega$

$$\mathcal{G}(t) = \frac{1}{\left[1 + i\frac{\omega}{2}t\right]^{\frac{N}{2}}} \sum_P (-1)^P e^{-\frac{\omega\alpha_P^2}{4\left(1+i\frac{\omega}{2}t\right)}}$$

with $\alpha_P^2 = m\delta_{\text{eff}}^2 \|P\|^2/2$

$\delta_{\text{eff}} = \left[1 + \frac{2}{c\delta}\right]\delta$ Effective increased distance due to repulsion

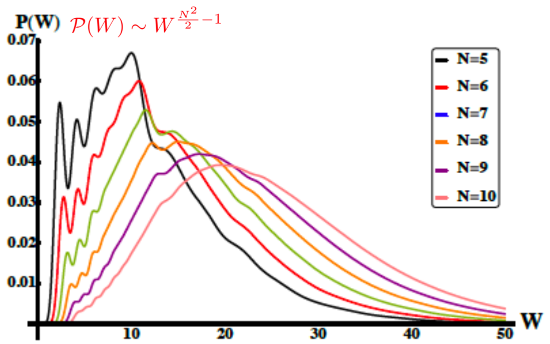
$\|P\|^2 = \sum_j^N (j - P(j))^2$ Measures how many particles were exchanged

Sum over permutations corresponds to a sum over particles exchanging positions: e.g $\|P\|^2 = 2$ corresponds to a neighboring pair exchanging positions, $\|P\|^2 = 8$ could be 4 nearest neighbor exchanges or 1 next nearest neighbor exchange.

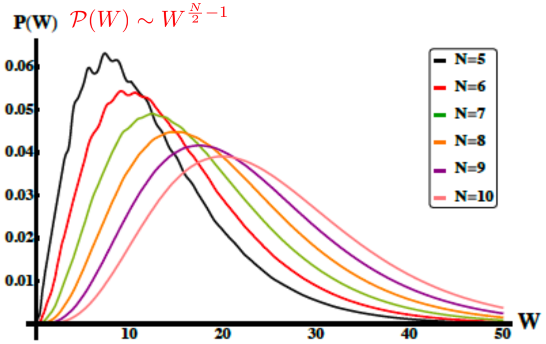
The Work Distribution (repulsive interactions, infinite volume)

- The work distribution is obtained from the Loschmidt amplitude

$$P(W) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} e^{iWt}\mathcal{G}(t) = \frac{e^{-\frac{2W}{\omega}}}{W}\left[\frac{2W}{\omega}\right]^{\frac{N}{2}} \sum_P (-1)^P \frac{J_{\frac{N-2}{2}}(2\sqrt{\alpha_P^2 W})}{[\alpha_P^2 W]^{\frac{N-2}{4}}}$$



Work distribution for repulsive bosons

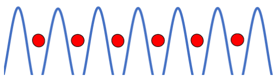


Work distribution for free bosons

$5 \leq N \leq 10$

$\delta/m = 2, \ \omega = 10$

- For large value of W (short times) no dependence on interaction (no overlaps initially)



- For small values of W strong effect of interactions: $\mathcal{P}(W) \sim W^{\frac{N^2}{2}-1}$ as opposed to: $\mathcal{P}(W) \sim W^{\frac{N}{2}-1}$

Indeed, $|\mathcal{G}(t)|^2 \to 1/t^{N^2}$ vs. $|\mathcal{G}(t)|^2 \to 1/t^{N}$ as $t \to \infty$. In 1-d even weak interaction have strong effect.

- Average work $\langle W \rangle = \int \mathrm{d}W\, W\mathcal{P}(W) = N\omega/4 = \langle \Psi_i | H | \Psi_i \rangle$

- Exponentiated work (yields all moments) $\langle e^{-\beta W} \rangle = \left(\frac{1}{1+\frac{\omega\beta}{2}}\right)^{\frac{N}{2}} \left[1 + \sum_{P\neq I} (-1)^P \frac{e^{-\frac{\omega\alpha_P^2}{2+\omega\beta}}}{\Gamma(N/2)}\right]$

cf Jarzynski equality

$$e^{-\beta\Delta F} = \langle e^{-\beta W} \rangle$$

The Work distribution (attractive interactions, infinite volume)

(ii.) ***Attractive interactions***

For $c<0$ there are bound states of n -bosons, $n=2,\cdots,N$
(momenta k form complex n- strings: $k_j = k + ijc/2, \; j = 1 \cdots n$

- Using our generalization of Yudson representation, we find

$$\mathcal{P}_{c<0}(W) = \mathcal{P}_{\text{unbound}}(W) + \mathcal{P}_{\text{bound}}(W)$$

- $\mathcal{P}_{\text{unbound}}(W)$ same expression as before with $c \to -c$, so $\delta_{\text{eff}} < \delta$ due to attraction.

  Similar to super Tonks-Girardeau experiment, where one prepares system with large $c>0$ and then suddenly quenches to large $c<0$. Note, our expression valid for any c.

- $\mathcal{P}_{\text{bound}}(W)$ is due to the strings, n- string contributing $\mathcal{P}_{n-\text{bound}}(W) \propto |c|^{n-1} e^{-n|c|\delta}$

  Transitions to states containing bound states are highly suppressed and in the true super-TG limit vanish entirely.

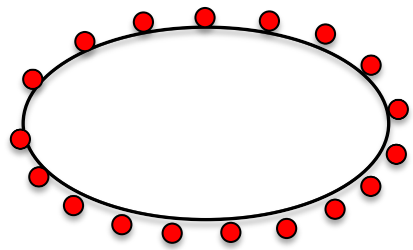
- For finite $c<0$ new effects: bound states lower the energy and work distribution becomes non vanishing at negative values of W.

- Indeed, for a 2- string $\mathcal{P}_{\text{bound}}(W) \approx N \sqrt{\frac{2\pi\omega}{m}} \frac{e^{-|c|\delta - \frac{2W}{\omega}}}{\Gamma\left(\frac{N}{2}-1\right)} \left[\frac{2(W + \frac{|c|^2}{4m})}{\omega}\right]^{\frac{N}{2}-2}$, non vanishing for $-|c|^2/4m < W$

- There is a non zero probability that work can be extracted from the system. This does not violate the 2nd law of thermodynamics since $\langle W \rangle > 0$ *Jarzynski '11*

The Work distribution (repulsive interactions, finite volume)

Start from a **fully occupied lattice** – boundary effects important



As before:

$$P(W) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} e^{iWt} \mathcal{G}(t) = \frac{e^{-\frac{2W}{\omega}}}{W} \left[\frac{2W}{\omega}\right]^{\frac{N}{2}} \sum_P (-1)^P \frac{J_{\frac{N-2}{2}}(2\sqrt{\alpha_P^2 W})}{[\alpha_P^2 W]^{\frac{N-2}{4}}}$$

But boundary conditions enter when calculating $\alpha_P^2 = m\delta_{\text{eff}}^2 \|P\|^2/2$

For instance when $\rho \ll \delta^{-1}$ the permutation $P = (23...N1)$ gives $\alpha_P^2 = m\delta_{\text{eff}}^2 N(N-1)/2$
however with PBC it gives $\alpha_P^2 = m\delta_{\text{eff}}^2 N/2$

- Region $W \sim \langle W \rangle$ not affected by BC, dominated by few exchanges of particles average work is as before.

- Region $W \ll \langle W \rangle$ strongly affected, all permutation contribute

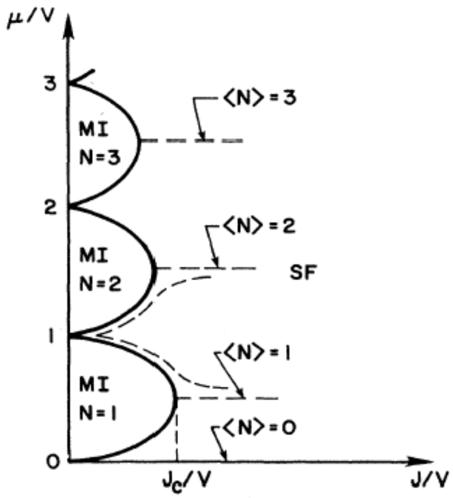
We have $\mathcal{P}(W) \sim W^{\frac{N}{2}}$ corresponding to $|\mathcal{G}(t)|^2 \to 1/t^{N+2}$

As opposed to $\mathcal{P}(W) \sim W^{\frac{N^2}{2}-1}$ and $|\mathcal{G}(t)|^2 \to 1/t^{N^2}$ as before.

The strongly interacting particles have no space to expand into unlike previously, resulting in slower decay of the echo.

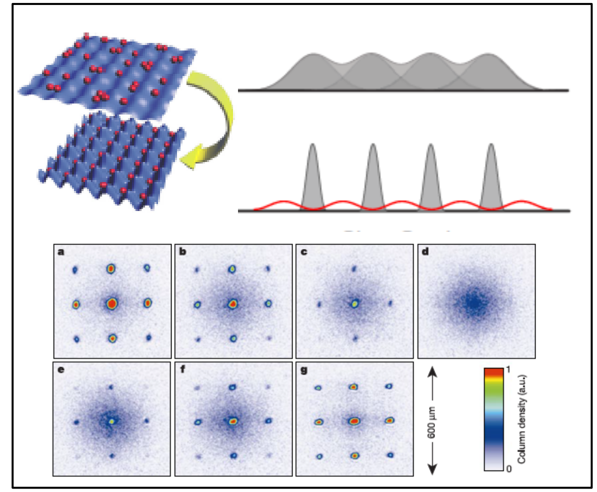
Quench of an optical lattice: *gapped scenario*

- Quench system from an initial Mott state, the ground state of very high barriers (BH model)



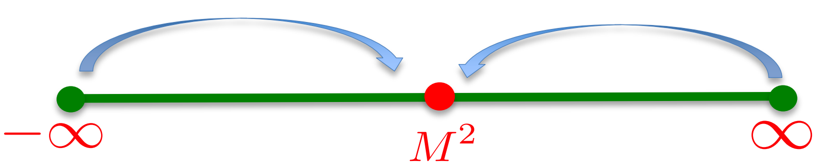
Fisher et al, 1989

Greiner et al '02 Bloch et al '08

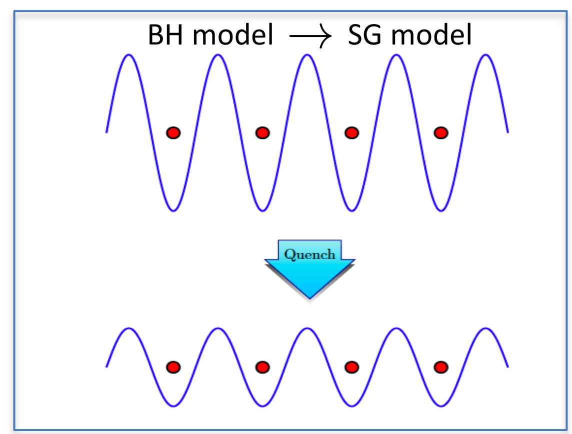


- Quench (*e.g.* SG as effective low-E theory of BHub)

$$H_{SG}(M^2 = \pm\infty, \beta) \rightarrow H_{SG}(M^2, \beta)$$



$$H_{SG}(M^2,\beta) = \frac{1}{2}\int \mathrm{d}x \left\{ \Pi^2(x) + [\partial\phi(x)]^2 - M^2 \cos[\beta\phi(x)] \right\}$$



Mott

SF

- Another realization: pair of coupled one-dimensional condensates of interacting atoms *(Gritsev et al '07)*

Conclusions

- Quenches realizable in cold atoms experiments, work distribution measurable

- Calculated effect of quench on local correlation functions:

- RG flow in time
- Dynamical fermionization

- Calculated work distribution of a quench in a strongly interacting, gapless system.

- Studied bound state contributions to the work distribution.
  Showed they dramatically change the distribution and allow for negative values of work

- Showed that interactions strongly affect the universal edge exponents of the work distribution and also the long time decay of the Loschmidt echo

- Calculated for attractive and repulsive interactions, determined critical exponents at threshold

To do

- Connect to nonequilibrium thermodynamics: entropy production, fluctuation theorems
- Small and large systems - increase role of fluctuations
- Quench across critical points, defect production, Kibble-Zurek dynamics, scaling and universality
- Time dependent quenches: slow drives, Floquet