Dynamical Mean-Field Theory in Non-Equilibrium Many-Body Statistical Physics: Aging, Glassy Dynamics, Ecosystems and High-Dimensional Chaos

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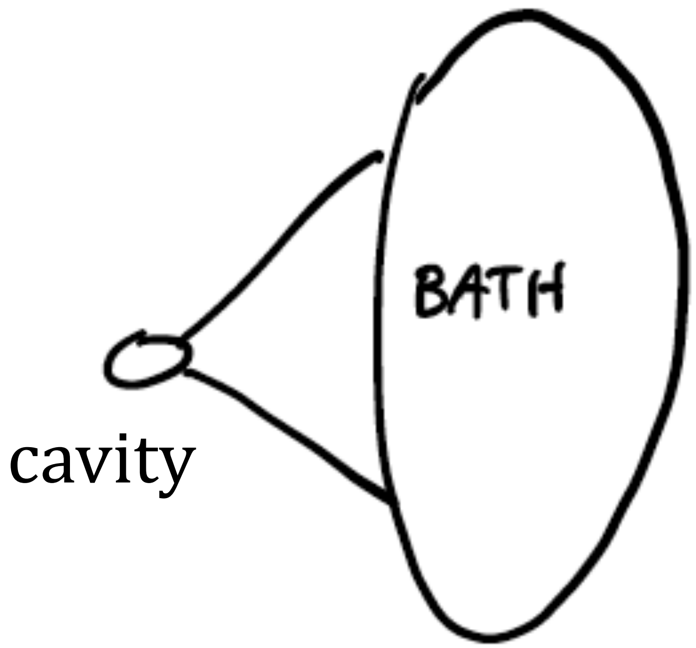
SIMONS FOUNDATION

Crash course on DMFT

The two key ingredients of DMFT

1. Identify the correct degree of freedom and treat the rest of the system as a bath

2. The bath is statistically identical to the singled-out degree of freedom: self-consistency



An example: mean-field spin glasses

$$\dot{S}_i = -\frac{\partial V}{\partial S_i} + \sum_{k(\neq i)} J_{ik} S_k + \xi_i(t)$$

$N \gg 1$

$$\overline{J_{ik}^2} = \frac{1}{N} \;;\; V(s) = (s^2-1)^2 \;;\; \langle \xi_i(t)\, \xi_k(t') \rangle = 2T\delta_{ik}\delta(t-t')$$

$$\dot{s}_k = -\frac{\partial V}{\partial S_k} + \sum_{\ell(\neq k,i)} J_{k\ell} S_\ell + J_{ki} S_i + \xi_k(t)$$

Solve the dynamics of the “bath” as a function of the cavity

An example: mean-field spin glasses

$$\dot{s}_k = -\frac{\partial V}{\partial S_k} + \sum_{\ell(\neq k,i)} J_{k\ell} S_\ell + J_{ki} S_i + \xi_k(t)$$

$N \gg 1$

$$\dot{S}_i = -\frac{\partial V}{\partial S_i} + \underbrace{\sum_{k(\neq i)} J_{ik} S_k^0}_{} + \underbrace{\sum_{k\neq i} J_{ik}\, \delta S_k^0}_{} + \xi_i(t)$$

Noise: Gaussian force   Dissipation: Onsager reaction term

$$\dot{S}_i(t) - \int_0^t ds\, R(t,s) S_i(s) = -\frac{\partial V}{\partial S_i} + \xi_i(t) + \eta_i(t)$$

Self-consistency

$$R(t,s) = \frac{1}{N}\sum_k \left.\frac{\delta S_k(t)}{\delta h_k(s)}\right|_{h_k=0}$$

$$\langle \eta_i(t)\eta_i(s)\rangle = \frac{1}{N}\sum_k S_k(t) S_k(s) = C(t,s)$$

DMFT in statistical physics

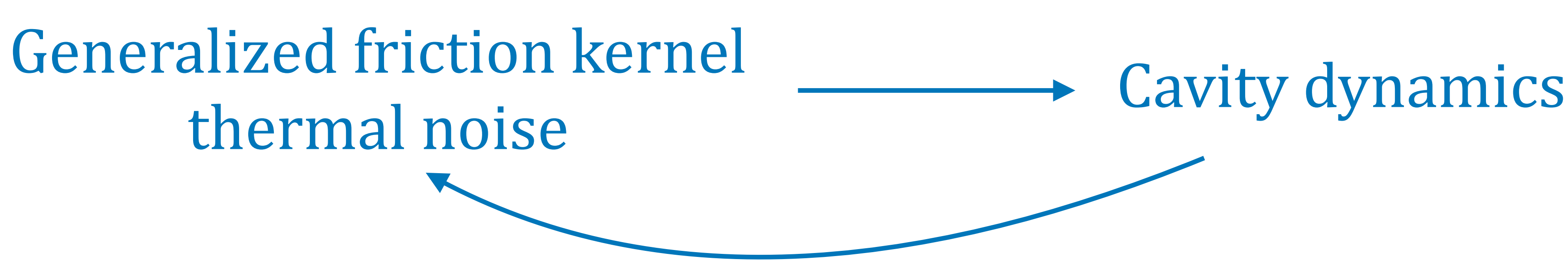
1. Identify the correct degree of freedom and treat the rest of the system as a bath

The effect on the cavity on the rest of the system treated at first order in perturbation theory (exact in the mean-field limit)

Generalized friction kernel and thermal noise

2. The bath is statistically identical to the singled out degree of freedom: self-consistency

Self-consistent stochastic non-markovian dynamics



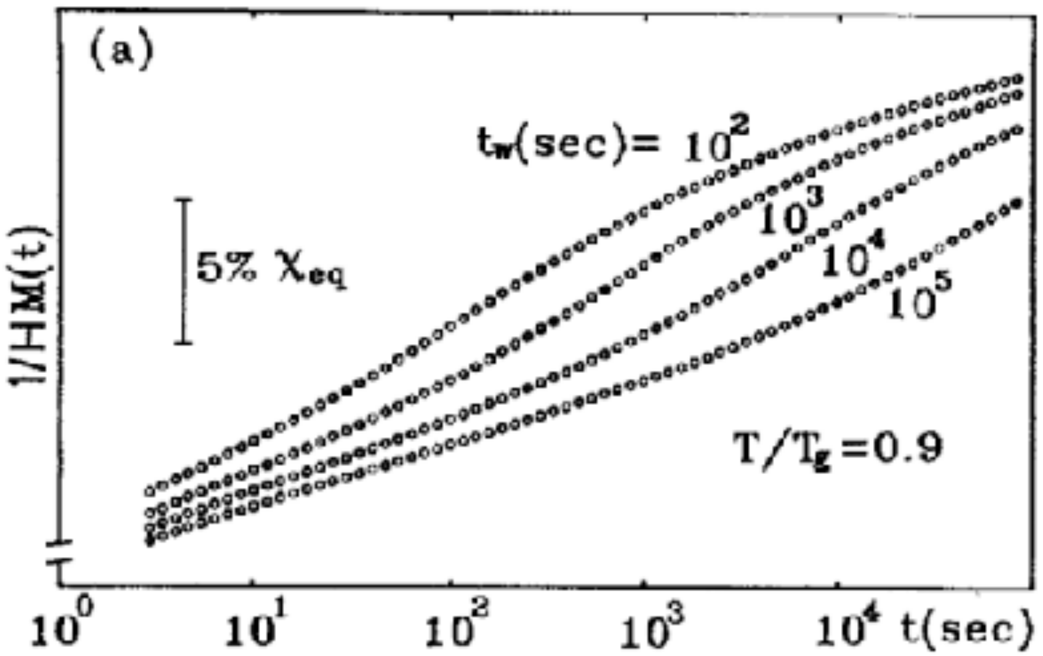
Sompolinsky, Zippelius PRB 1982
Mézard, Parisi, Virasoro, Spin Glass and Beyond

Quantum: Georges, Kotliar, Krauth, Rozenberg, RMP 1996

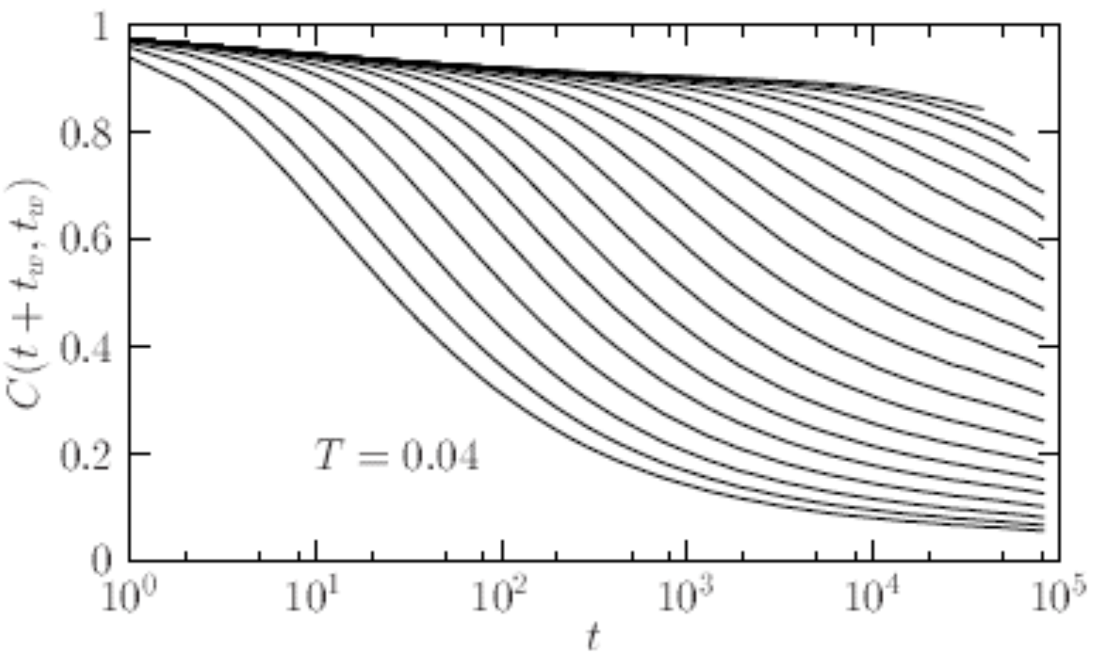
Applications

Aging in spin-glasses

At low temperature spin-glasses display a very slow off-equilibrium dynamics



Zero-Field cooled susceptibility from Zeidlindh et al PRB 87



Heisenberg spin-glass from Berthier Young PRB 04

DMFT for aging in spin-glasses

- Ergodicity breaking transition: the system remains always out of equilibrium below Tc.
  T>Tc *thermalisation at long times*, system and bath verifies FDT
  T<Tc *aging dynamics of the system and the bath*

Cugliandolo, Kurchan, PRL 1993;...
Franz, Mezard 1994,...

- Infinite hierarchy of time-scales and time-sectors: effective temperatures out of equilibrium, and dynamical version of Parisi Full Replica Symmetry Breaking

Cugliandolo, Kurchan PRB 1994; JPSJ 2000,...

Altieri, Cammarota, Biroli J. Phys. A 2020

- Generalisation to quantum systems

Cugliandolo, Lozano PRB 1999
Biroli, Parcollet PRB 2002

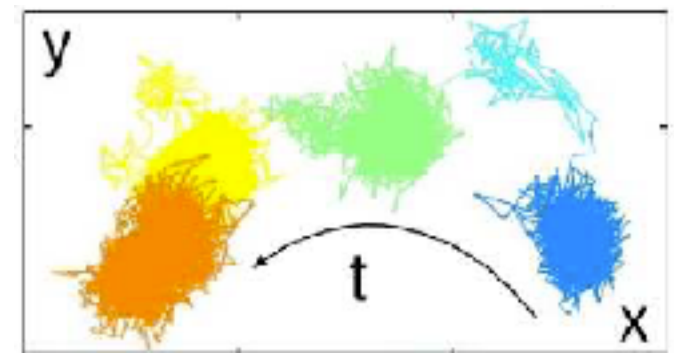
DMFT of the glass transition

Molecular liquids display a dramatic slowing down associated to the glass transition
The relaxation time increases by 14 orders of magnitude

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{\langle i \neq j \rangle} V(R_i - R_j)$$

$$m \ddot{R}_i = - \sum_{j(\neq i)} \nabla V(R_i - R_j)$$

N interacting particles
<u>in the continuum</u>



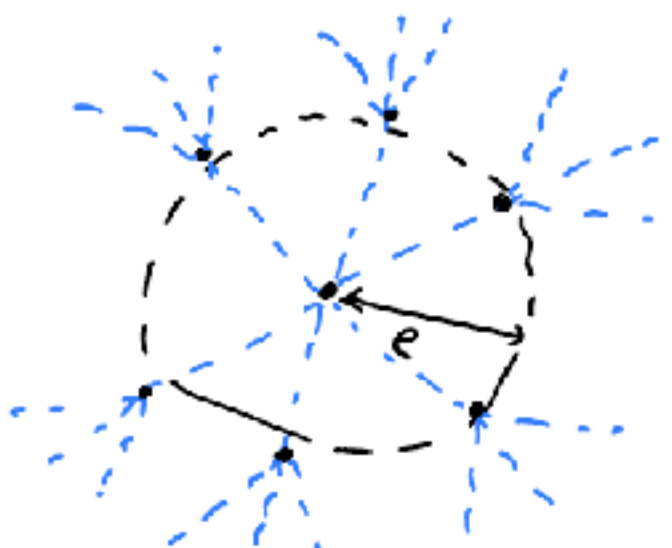
DMFT exact in the large dimensional limit
Challenge: off-lattice system

$d \to \infty$

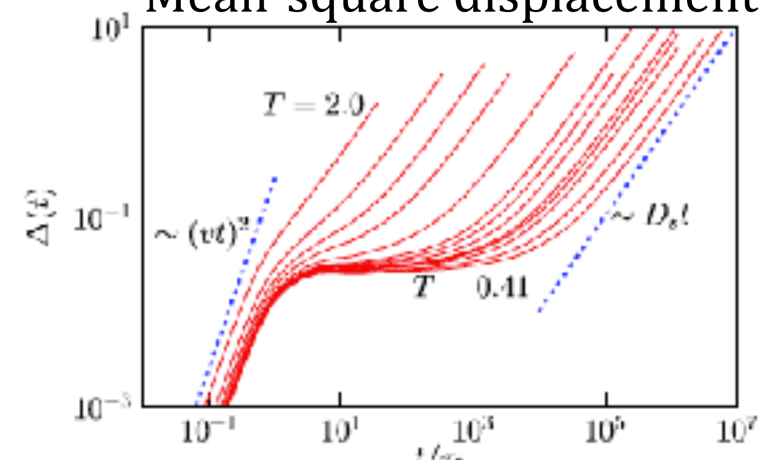
$$u_i(t) = R_i(t) - R_i(0)$$

$$V(R) = \bar{v}\left(d\left(\frac{R}{\ell} - 1\right)\right)$$

$u_i^2 \sim \frac{1}{d}$



Mean-square displacement



Berthier, Biroli RMP 2011

Cavity degree of freedom

$$U_{i,\alpha}(t)$$

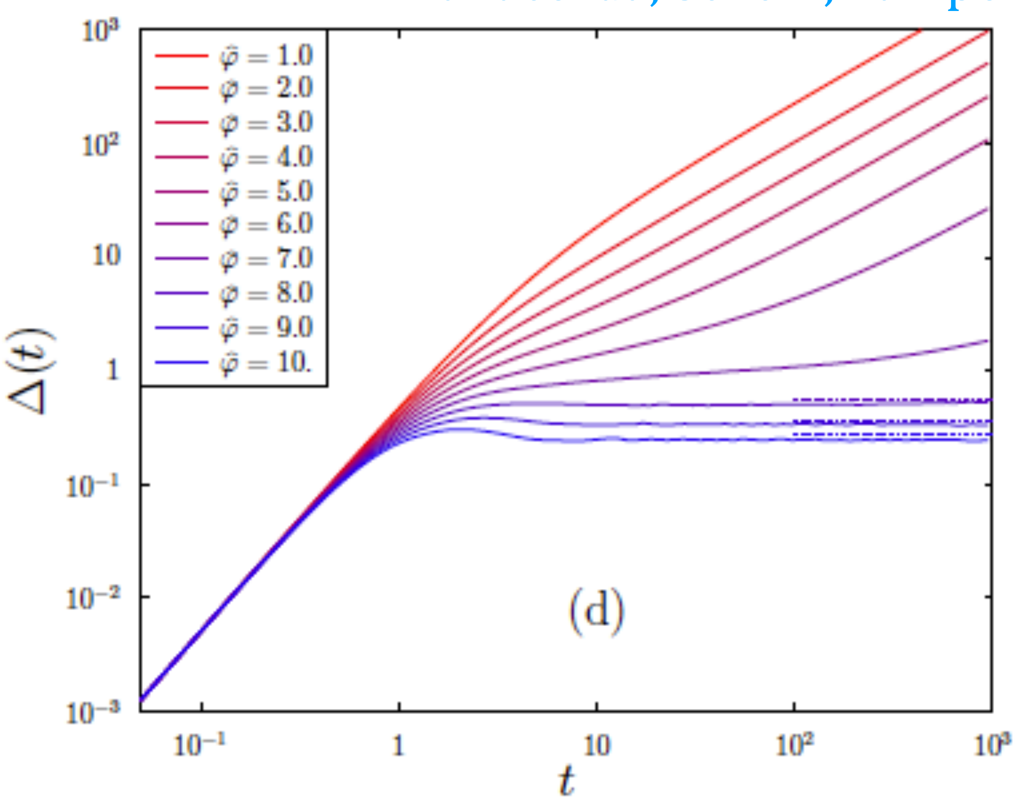
DMFT of the glass transition

- Exact solution of infinite-d many particle dynamics

  Maimbourg, Kurchan, Zamponi PRL 2016
  Agoritsas, Maimbourg, Zamponi J Phys A 2019
  Chen, Biroli, Reichman, Szamel 2021

- Glass transition at Td with the same properties of Mode-Coupling theory

Manacorda, Schehr, Zamponi JCP 2020



- Detailed description of glassy dynamics but fails to obtain super-Arrhenius behavior (non-perturbative in 1/d)

Perspective: cluster extension to obtain a full description of glassy dynamics

Chen, Biroli, Reichman, Szamel 2021

DMFT for many-species ecosystems

“Traditional” ecosystems

“Modern” ecosystems

MANY INTERACTING SPECIES -> STAT PHYS FOR NON CONSERVATIVE FORCES

Generalized Lotka-Volterra

$$\frac{dN_i}{dt} = N_i \left[r_i (K_i - N_i) - \sum_{j(\neq i)} \alpha_{ij} N_j \right] + \lambda_i$$

$$i = 1, \dots, N \qquad (N \gg 1) \; ; \; N_i \geq 0$$

DMFT exact in the Mean-Field Disordered Version

Roy, Bunin, Biroli, Cammarota, JPA 2019

$$\overline{\alpha_{ij}^2}^c = \frac{\sigma^2}{N} \; ; \quad \overline{\alpha_{ij}\alpha_{ji}}^c = \gamma \frac{\sigma^2}{N} \; ; \quad \overline{\alpha_{ij}} = \frac{\mu}{S}$$

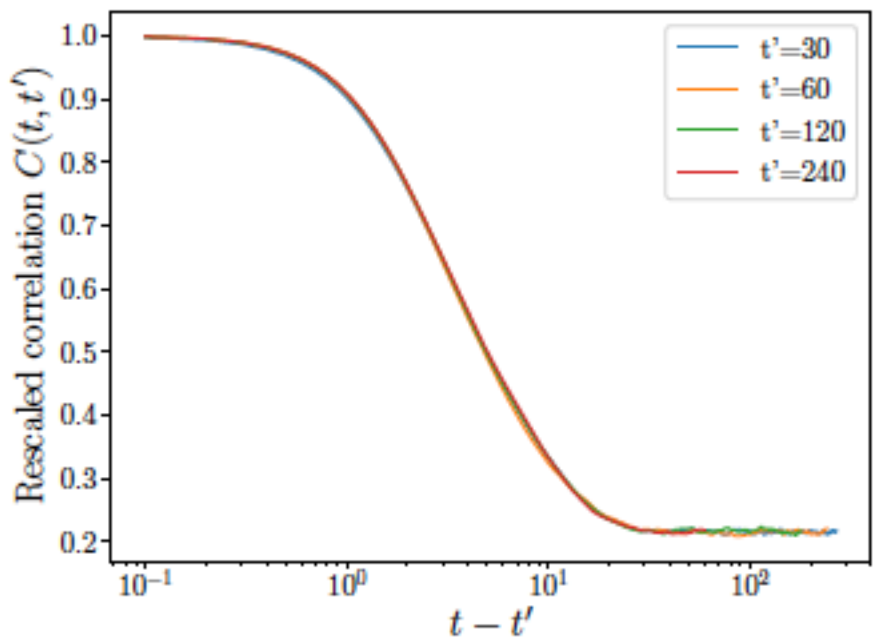
R. May 1972
Bunin, PRE 2017
Barbier et al PNAS 2018

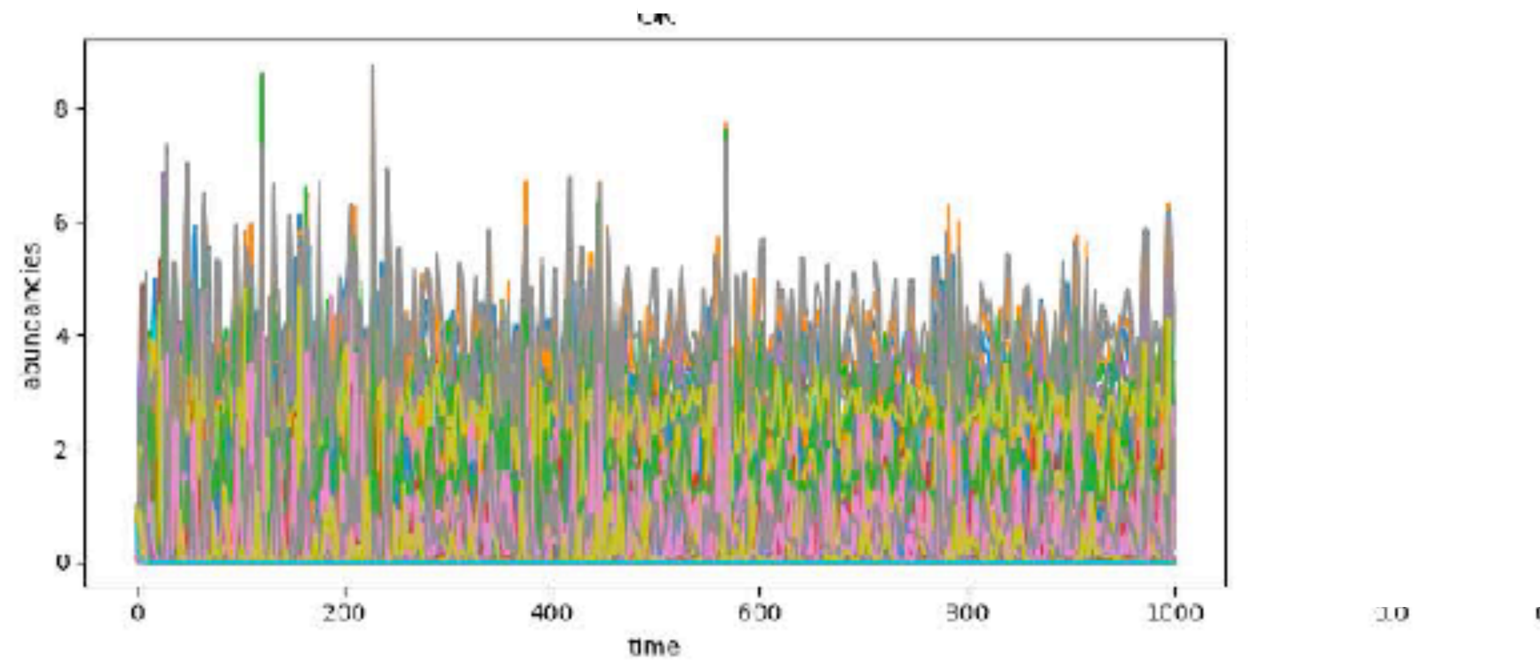
Chaos & Endogeneous fluctuations

Can endogenous fluctuations survive in a large interacting ecosystem?

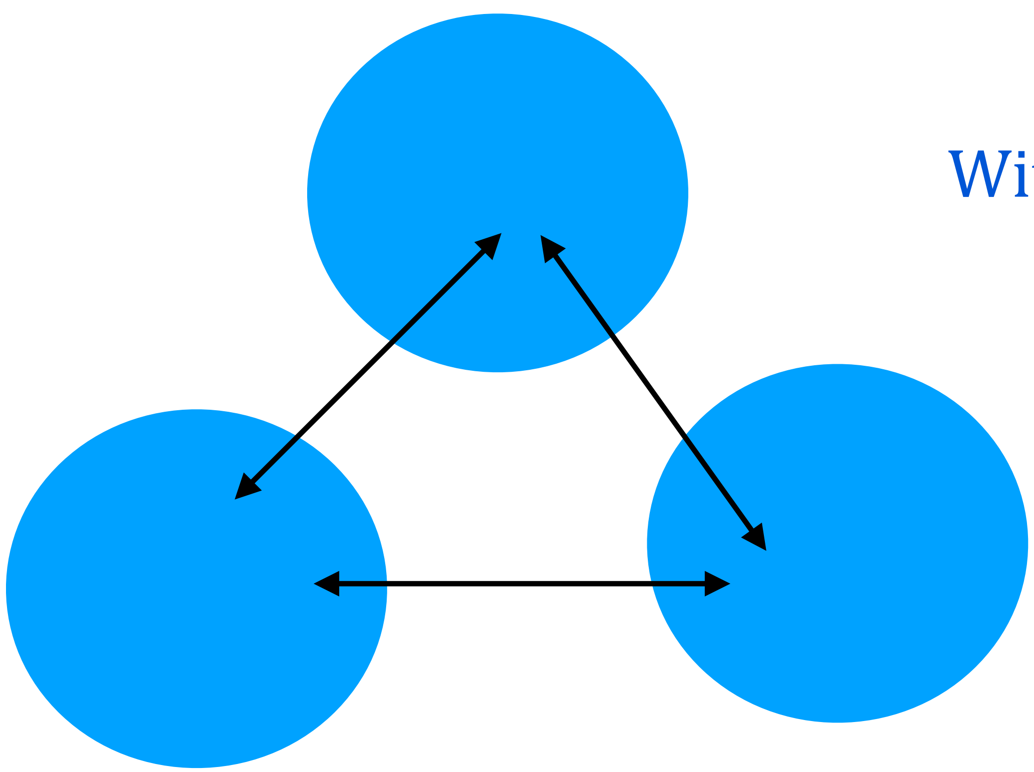
“Are ecological systems chaotic—and if not, why not?” Berryman, Millstein 1989

Transition from one equilibrium phase to stable high-dimensional chaos with immigration





Roy, Bunin, Biroli, Cammarota, JPA 2019



Without immigration chaos is metastable for many communities and with diffusion

Roy, Barbier, Biroli, Bunin PLOS 2020

D. S. Fisher, A. Agarwala, M. Pearce PNAS 2019

Conclusion

DMFT in classical statistical physics: a powerful method to analyze and unveil complex dynamical phenomena

Many different applications

- *Aging dynamics in spin-glasses*
- *Very slow relaxation and glass transition*
- *Chaotic dynamics in large interacting ecosystems*
- Chaos and learning in neural networks
- Optimization algorithms in computer science
- Strongly correlated electrons in equilibrium and driven

DMFT for many-species ecosystems

"Traditional" ecosystems

"Modern" ecosystems

MANY INTERACTING SPECIES

- Communities formed by individuals belonging to different species.
- Interactions between individuals intra and inter species.
- Competition for resources--Cooperation.
- Abundances of species vary dynamically due to the births and deaths.

Many interesting open questions

- Can endogenous fluctuations survive in a large interacting ecosystem?

Endogenous versus exogenous fluctuations

"Are ecological systems chaotic—and if not, why not?" Berryman, Millstein 1989