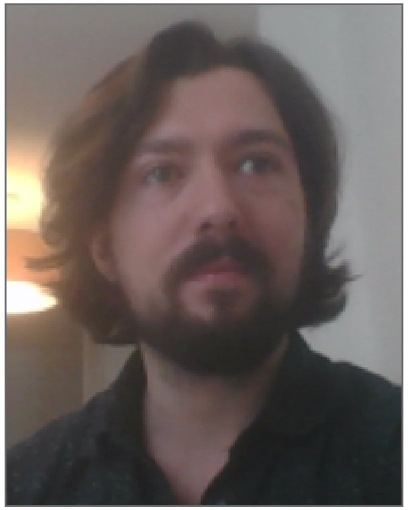
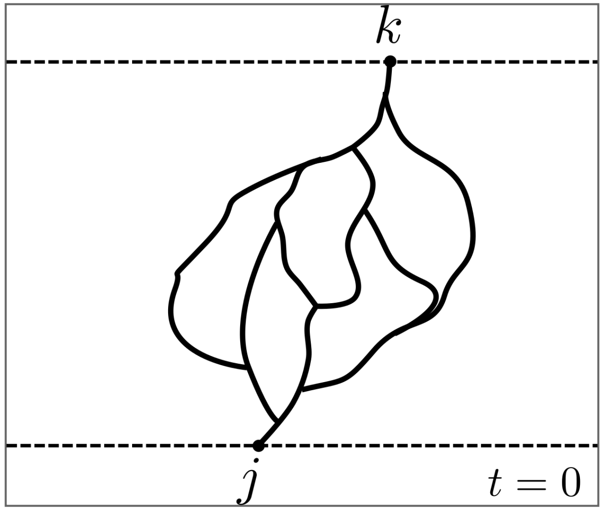
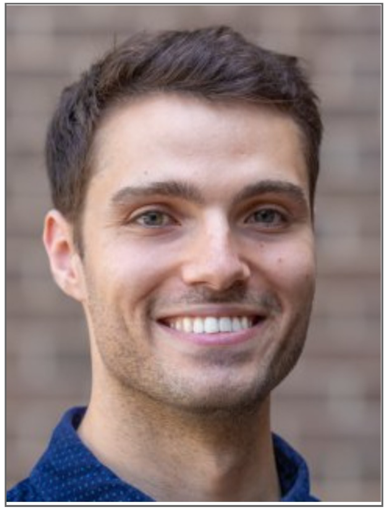
Absence of superdiffusion in certain random spin models

Work with Pieter Claeys and Jonah Herzog-Arbeitman



≡



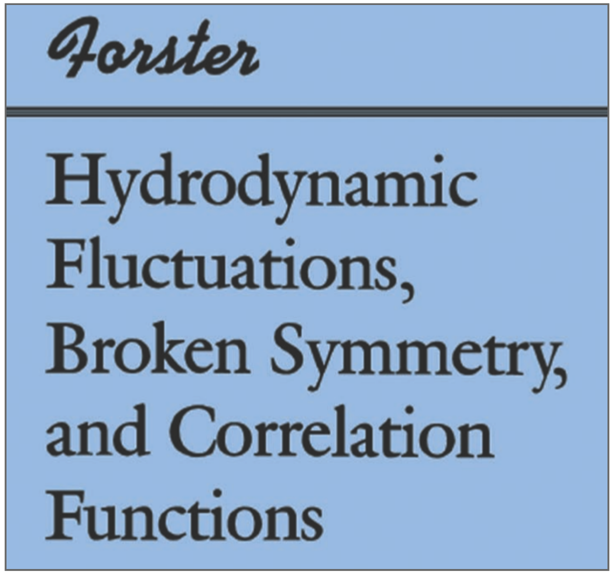


auste.nl/slides/superdiffusion-kitp/

Embarrassingly simple question

- ***What is nature of spin transport in Heisenberg chain?***
  $$H = \sum_j \left[X_j X_{j+1} + Y_j Y_{j+1} + Z_j Z_{j+1}\right]$$
- All 3 components conserved
- (Naive) expectation: diffusion at $T > 0$ (including $T = \infty$)

Simple?



- **Except**: nonabelian, low dimension, integrability, …

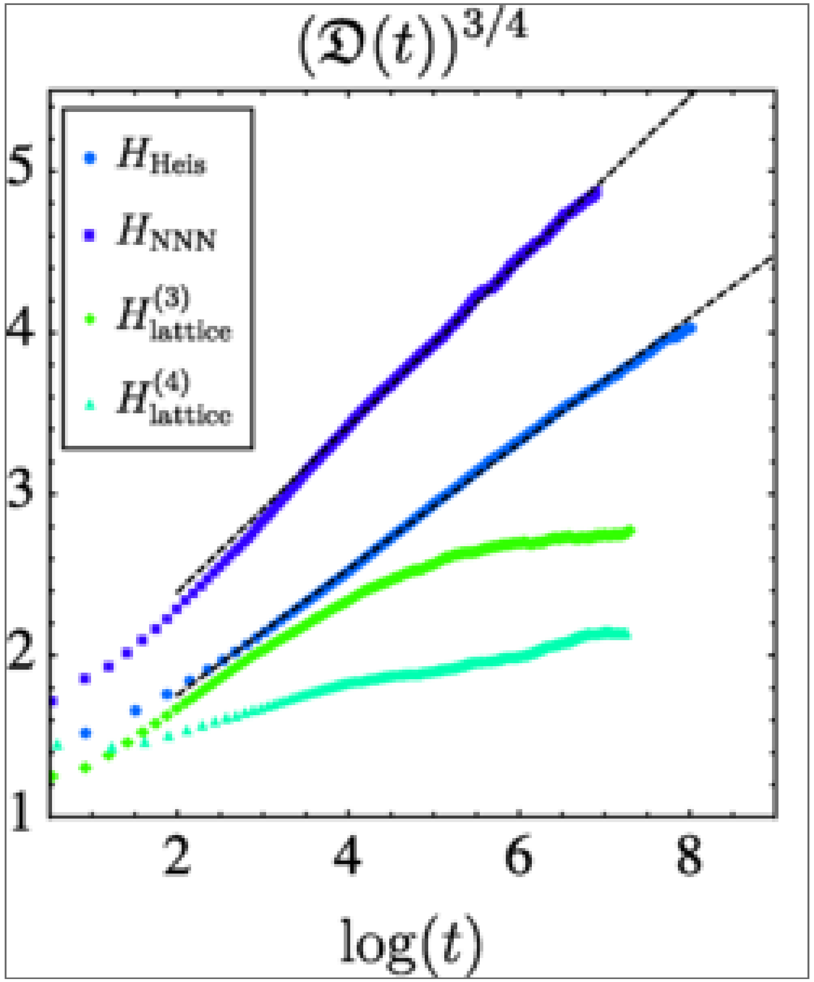
≡

Recent predictions

- (Very) good evidence for KPZ(ish) behavior $\ell \sim t^{2/3}$ in **integrable**, **nonabelian** models, classical and quantum
- Recent review **Bulchandani, Gopalakrishnan, Ilievski (2021)**

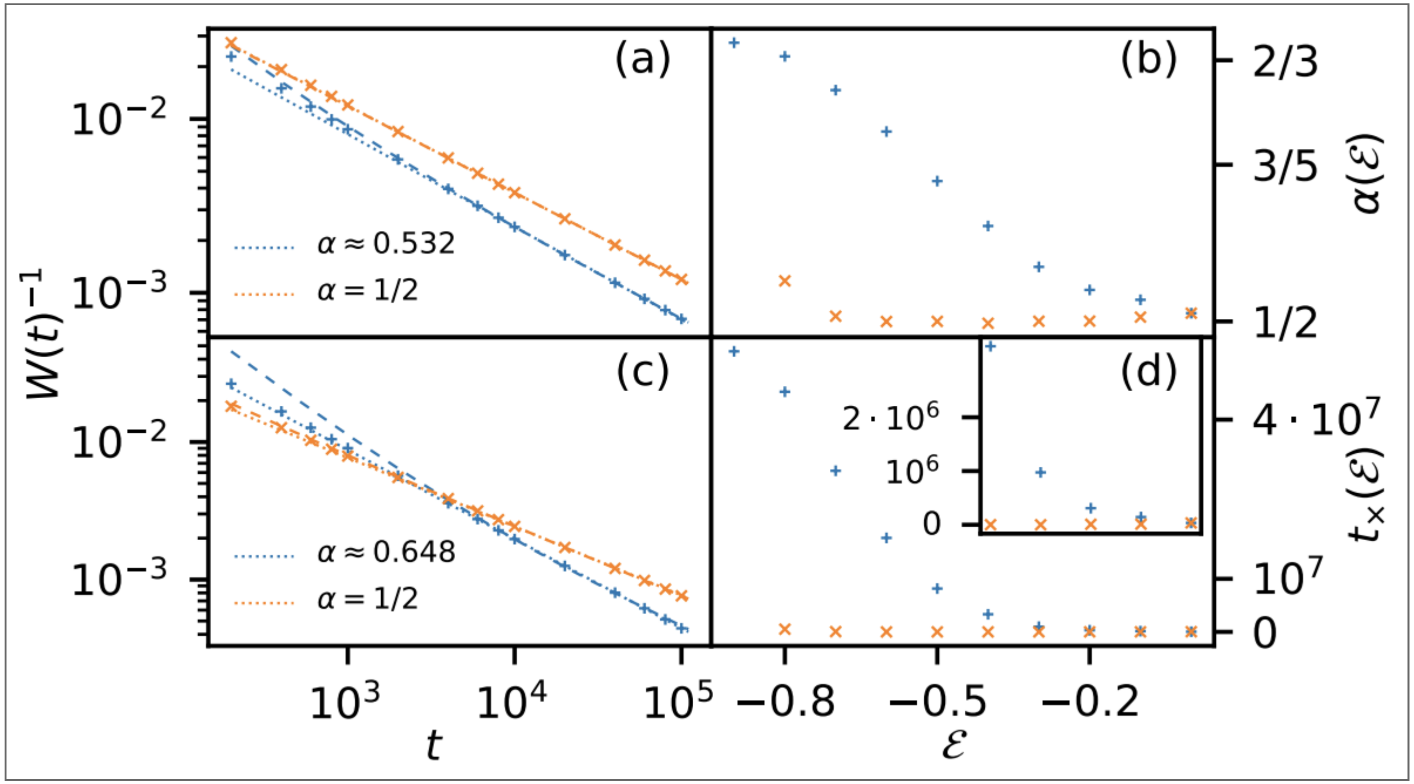
≡

- **De Nardis *et al.* (2020)** $D(t) \sim (\log t)^{4/3}$ in classical Heisenberg chain



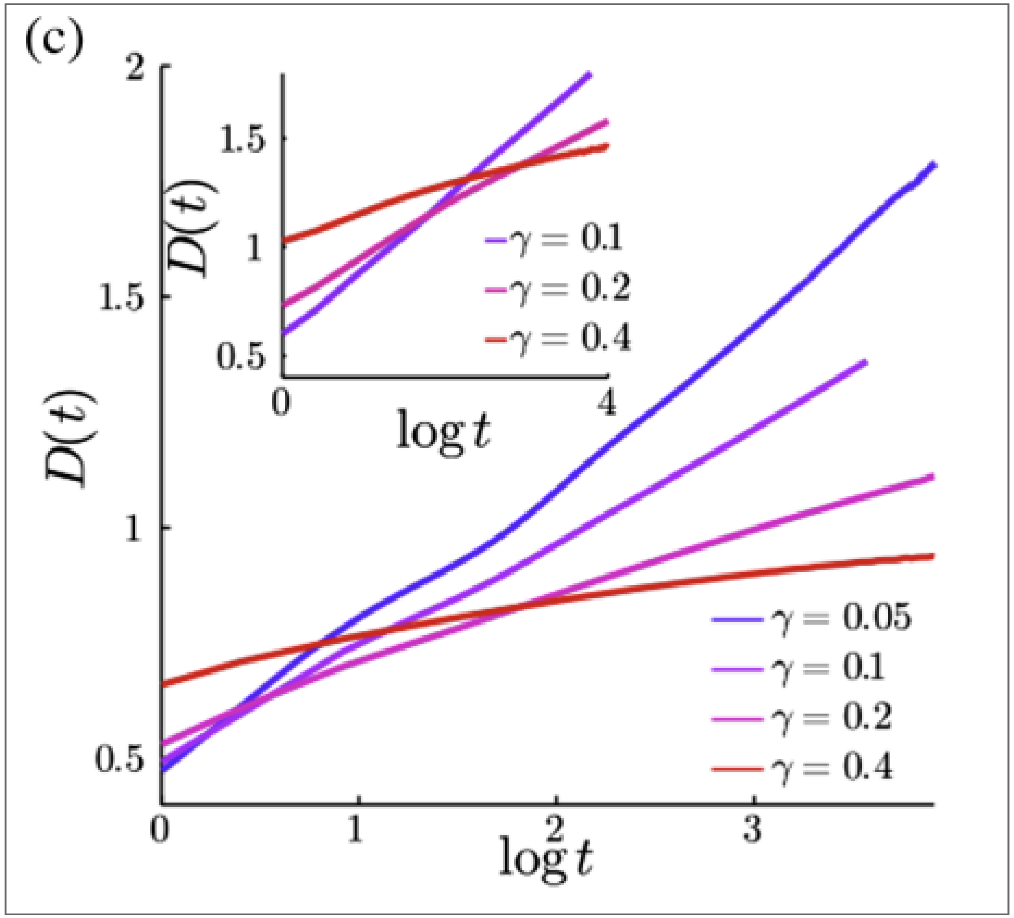
≡

- **McRoberts *et al.* (2021)** on classical FM (blue) and AFM (orange)



- At finite T FM looks anomalous (KPZish); AFM looks normal

- **De Nardis *et al.* (2021)** $D(t) \sim \log t$ with noisy exchange coupling



≡

Lack of theory tools

- No integrability; "weak integrability breaking" in its infancy
- Absence of small parameters: exchange coupling J is only scale

This work: noisy exchange coupling

$$H = \sum_{j,a} \left[(J + \xi_j(t)) \sigma_j^a \sigma_{j+1}^a \right]$$

- Studied numerically in **De Nardis *et al.* (2021)**
- $SU(2)$ invariance but no energy conservation
- Expect (nonabelian) hydrodynamics of spin modes to play major role
- ***Can develop perturbation theory in*** J

Correlation function

- Spin-1/2 chain of N sites with spin $\boldsymbol{\sigma}_j = (X_j, Y_j, Z_j)$ at site j
- Infinite temperature spin-spin correlator

$$C^{ab}_{jk}(t) \equiv \frac{1}{2^N} \operatorname{tr}\left[\sigma^a_j(0)\sigma^b_k(t)\right] \qquad \sigma^b_k(t) = \mathcal{U}_t^\dagger \sigma^b_k \mathcal{U}_t.$$

- $SU(2)$ invariance: $C^{ab}_{jk}(t) \equiv \delta_{ab} C_{jk}(t)$ with $\sum_{k=1}^N C_{jk}(t) = 1$
- From now on fix $a = b = z$

≡

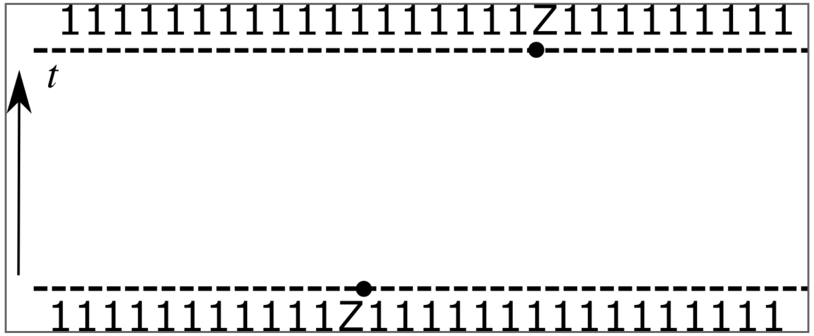
Expansion in Pauli basis

$$Z_j(t) = \sum_{\mu_{1:N}=0,1,2,3^N} \mathcal{C}_{\mu_{1:N}}(t)\sigma_1^{\mu_1} \otimes \cdots \sigma_N^{\mu_N}, \qquad \sigma^\mu = (\mathbb{1}, X, Y, Z)$$

- With initial condition

$$\mathcal{C}_{\mu_{1:N}}(0) = \begin{cases} 1 & \mu_j = z, \mu_k = 0, \forall k \neq j \\ 0 & \text{otherwise,} \end{cases}$$

- Spin correlation function is $C_{jk}(t) = \mathcal{C}_{0\cdots\mu_k=z\cdots 0}(t)$



Model

- Fluctuating exchange coupling gives stochastic Schrödinger equation

$$d|\psi\rangle = \sum_j \left[-i(J\,dt + \sqrt{\eta}dW_j)P_{j,j+1} - \frac{\eta}{2}dt\right]|\psi\rangle.$$

- $P_{j,j+1} = \frac{1}{2}\left[1 + \sum_a \sigma_j^a \sigma_{j+1}^a\right]$ is exchange operator
- W_j independent Brownian motions (white noise $\propto dW_j$)
- **Itô** stochastic differential equation: last term preserves $\langle\psi|\psi\rangle$

Operator dynamics

- Heisenberg equation of motion ($\eta = 1$)

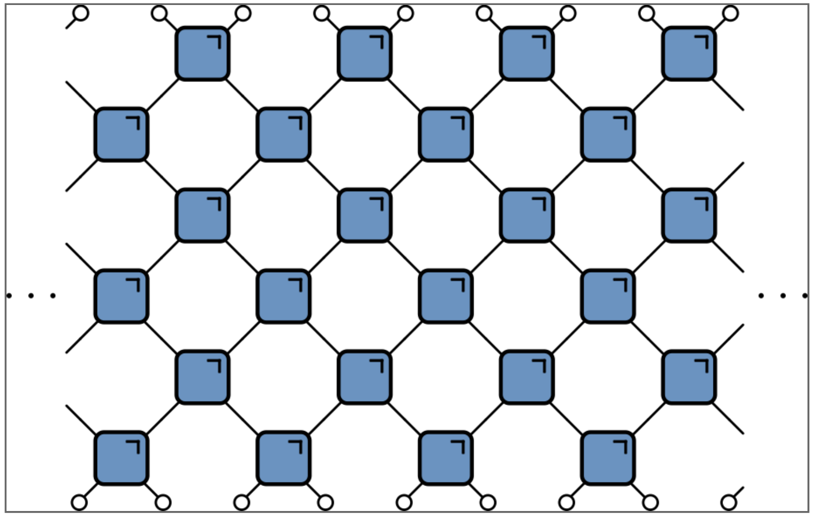
$$d\mathcal{O} = \sum_j \left[i \left(J dt + dW_j \right) \left[P_{j,j+1}, \mathcal{O} \right] + dt \left(P_{j,j+1} \mathcal{O} P_{j,j+1} - \mathcal{O} \right) \right].$$

- $\bar{\mathcal{O}} \equiv \mathbb{E}\, \mathcal{O}$ obeys the (adjoint) Lindblad equation

$$\frac{d\bar{\mathcal{O}}}{dt} = \sum_j \left[iJ \left[P_{j,j+1}, \bar{\mathcal{O}} \right] + \left(P_{j,j+1} \bar{\mathcal{O}} P_{j,j+1} - \bar{\mathcal{O}} \right) \right].$$

≡

Circuit viewpoint



- $SU(2)$ preserving gate

$$U_{j,j+1} = \cos\theta \mathbb{1}_{j,j+1} - i\sin\theta P_{j,j+1}$$

Operator Dynamics

$$U_{j,j+1} = \cos\theta \mathbb{1}_{j,j+1} - i\sin\theta P_{j,j+1}$$

$$\mathcal{O} \longrightarrow U^\dagger_{j,j+1}\mathcal{O}U_{j,j+1} = \cos^2\theta\,\mathcal{O} + \sin^2\theta\,P_{j,j+1}\mathcal{O}P_{j,j+1} + i\sin\theta\cos\theta\left[P_{j,j+1},\mathcal{O}\right]$$

- Take distribution $\theta = \pm\theta_0$ with $p(\theta_0) - p(-\theta_0) \equiv \delta > 0$

≡

Average dynamics

$$\overline{U_{j,j+1}^{\dagger} \mathcal{O} U_{j,j+1}} = \cos^2\theta_0\, \mathcal{O} + \sin^2\theta_0\, P_{j.j+1} \mathcal{O} P_{j.j+1}$$
$$+ i\delta \sin\theta_0 \cos\theta_0 \left[P_{j.j+1}, \mathcal{O}\right]$$

- Interpretation:
  - Operators on sites j and $j+1$ switch with probability $\sin^2\theta_0$
  - Asymmetry δ governs strength of "quantum" dynamics
- Taking $\theta_0 = \sqrt{dt}, \delta = J\sqrt{dt}$ gives continuous time evolution

Back to continuous time

$$\frac{d\bar{\mathcal{O}}}{dt} = \sum_j \left[iJ \left[P_{j,j+1}, \bar{\mathcal{O}} \right] + \left(P_{j,j+1} \bar{\mathcal{O}} P_{j,j+1} - \bar{\mathcal{O}} \right) \right].$$

- $J = 0$: master equation describing **random adjacent transpositions**
- Preserves subspaces corresponding to fixed numbers of each of the σ^μ: 1 operator sector, 2 operator sector, …

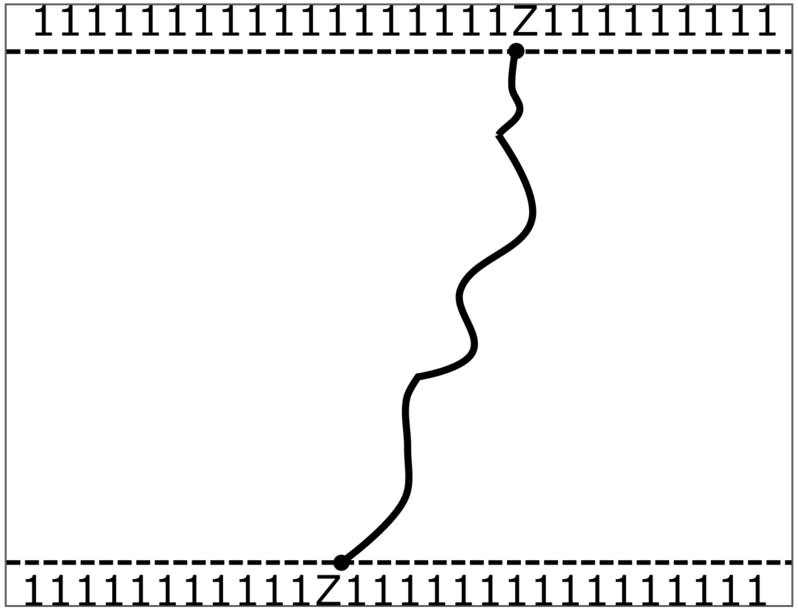
≡

$J = 0$: 1 operator sector

- Writing $C^a_{0\cdots\mu_k=a\cdots0} \equiv C^a_k$ we have equation of motion

$$\partial_t C^a_k = C^a_{k+1} + C^a_{k-1} - 2C^a_k \equiv \Delta_k C^a_k$$

- Diffusion of single σ^a (Δ_k is 1D discrete Laplacian)



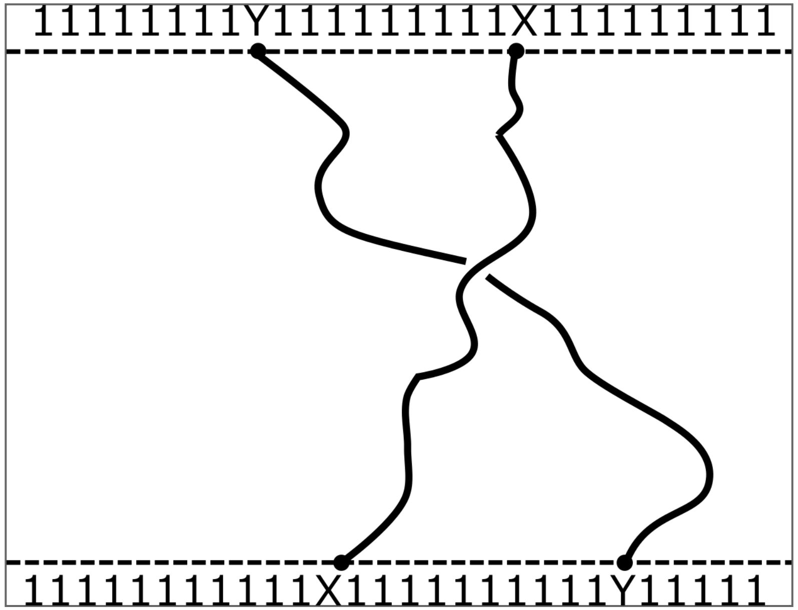
≡

$J = 0$: 2 operator sector

- $C^{bc}_{j,k} \equiv \mathcal{C}_{0\cdots\mu_j=b\cdots\mu_k=c\cdots 0}$

$$\partial_t C^{xy}_{m,n} = \Delta_m C^{xy}_{m,n} + \Delta_n C^{xy}_{m,n} + \delta_{|m-n|-1} C^{xy}_{m,n}$$

- Last term plus condition $C^{xy}_{m,m} = 0$ from hardcore condition



Perturbation theory

$$
\frac{d\bar{\mathcal{O}}}{dt} = \sum_j \left[iJ\left[P_{j,j+1}, \bar{\mathcal{O}}\right] + \left(P_{j,j+1}\bar{\mathcal{O}}P_{j,j+1} - \bar{\mathcal{O}}\right)\right].
$$

$$
\begin{aligned}
i[P, \sigma^a \otimes 1] &= -\epsilon^{abc}\sigma^b \otimes \sigma^c \\
i[P, 1 \otimes \sigma^a] &= \epsilon^{abc}\sigma^b \otimes \sigma^c \\
i[P, \sigma^a \otimes \sigma^b] &= \epsilon^{abc}\left(\sigma^c \otimes 1 - 1 \otimes \sigma^c\right).
\end{aligned}
$$

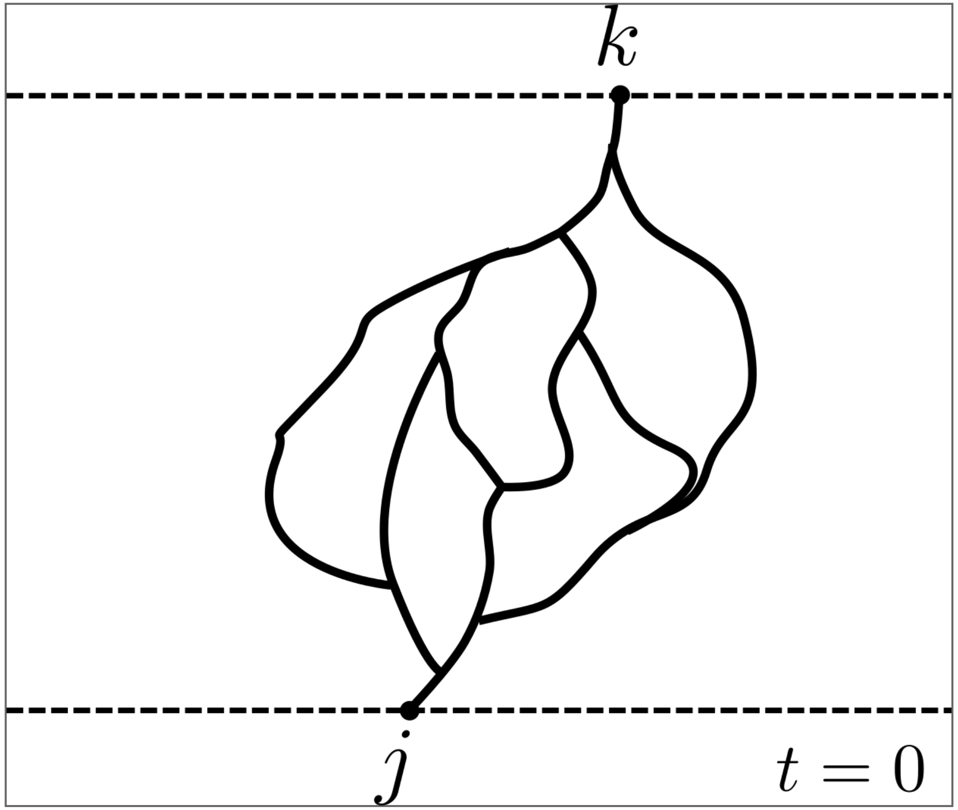
- Sum of first two expressions vanishes by spin conservation
- Describe operator “splitting” ($1 \to 2$) and “merging” ($2 \to 1$).

≡

Equation of motion

- In component form

$$\partial_t \mathcal{C}_{\mu_{1:N}} = \sum_j \left[J\epsilon_{\alpha\beta\mu_j\mu_{j+1}} \mathcal{C}_{\mu_1\cdots\alpha\beta\cdots\mu_N} + \mathcal{C}_{\mu_1\cdots\mu_{j+1}\mu_j\cdots\mu_N} - \mathcal{C}_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_N} \right].$$

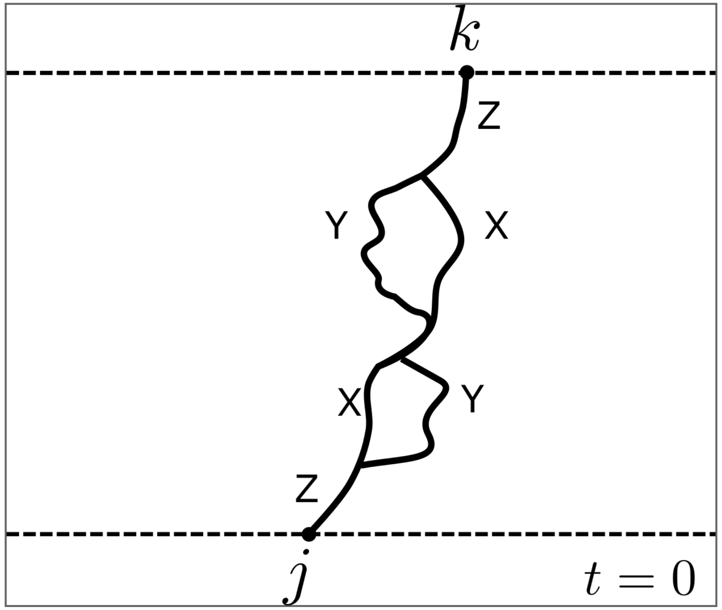


Simple approximation

- 1 and 2 operator sectors, dropping coupling to higher sectors

$$\partial_t C^z_n = J\left[C^{xy}_{n-1,n} - C^{xy}_{n,n-1} - C^{xy}_{n,n+1} + C^{xy}_{n+1,n}\right] + \Delta_n C^z_n,$$

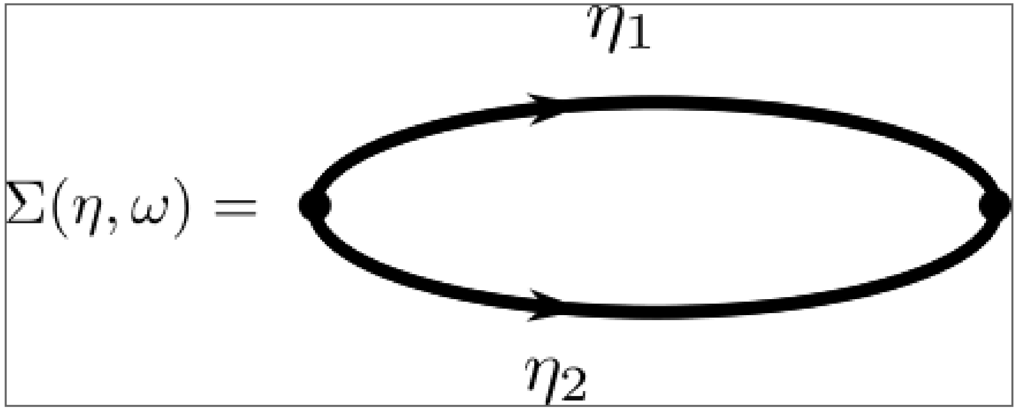
$$\partial_t C^{xy}_{m,n} = J\left[\delta_{m+1,n}\left(C^z_m - C^z_{m+1}\right) + \delta_{m,n+1}\left(C^z_{n+1} - C^z_n\right)\right] + \Delta_m C^{xy}_{m,n} + \Delta_n C^{xy}_{m,n} + \delta_{|m-n|-1} C^{xy}_{m,n}$$



≡

Result for correlator

- $C^z(\eta, \omega) = [i\omega - \Omega(\eta) - \Sigma(\eta, \omega)]^{-1}$ in terms of self-energy



$$\Sigma(\eta, \omega) = \frac{4J^2}{N} \sum_{\eta_1+\eta_2=\eta} \frac{[\cos(\eta_1) - \cos(\eta_2)]^2}{\Omega(\eta_1) + \Omega(\eta_2) - i\omega}, \qquad \Omega(\eta) \equiv 4\sin^2(\eta/2)$$

- Hardcore constraint plays no role due to antisymmetry of vertex

Hydrodynamic limit:

- For $\Omega(\eta) \to \eta^2$ and $\omega = O(\eta^2)$

$$\Sigma(\eta, \omega) = J^2\eta^2 \left[1 + \frac{1}{2}\sqrt{\eta^2 - 2i\omega}\right].$$

- The diffusion pole at $\omega = -i\eta^2$ becomes a pair

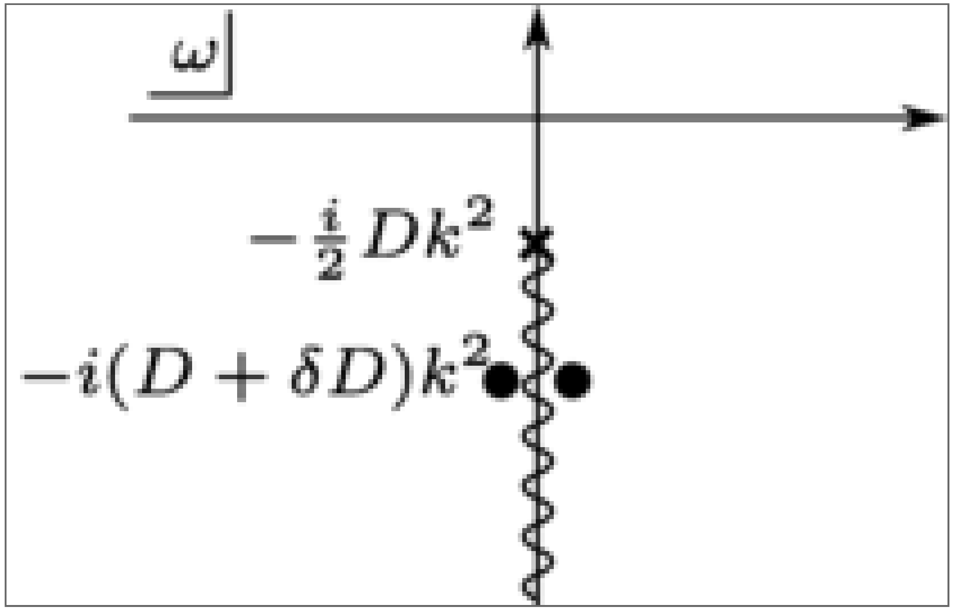
$$\omega_{\pm} = -i(1 + J^2)\eta^2 \pm |\eta|^3 \frac{J^2}{2}\sqrt{1 + 2J^2} + O(\eta^4).$$

- Branch point $\omega = -i\eta^2/2$: min. $\omega(\eta_1) + \omega(\eta_2)$ when $\eta_{1,2} \to \eta/2$.

≡

Analytic structure

- c.f. **Xinyi Chen-Lin, Luca V. Delacrétaz, and Sean A. Hartnoll (2019)**



≡

Enhanced diffusion

$$\omega = -i(1+J^2)\eta^2 \pm |\eta|^3 \frac{J^2}{2}\sqrt{1+2J^2} + O(\eta^4).$$

- J enhances **ordinary** diffusion
- Find transient diffusion constant using $D(t) = -\frac{1}{2}\partial_t \partial_\eta^2 C^z(\eta;t)\Big|_{\eta=0}$

$$D(t) = 1 + J^2 - J^2 e^{-4t}\left[I_0(4t) + I_1(4t)\right] \xrightarrow[t\to\infty]{} 1 + J^2 - \frac{J^2}{\sqrt{2\pi t}}$$

≡

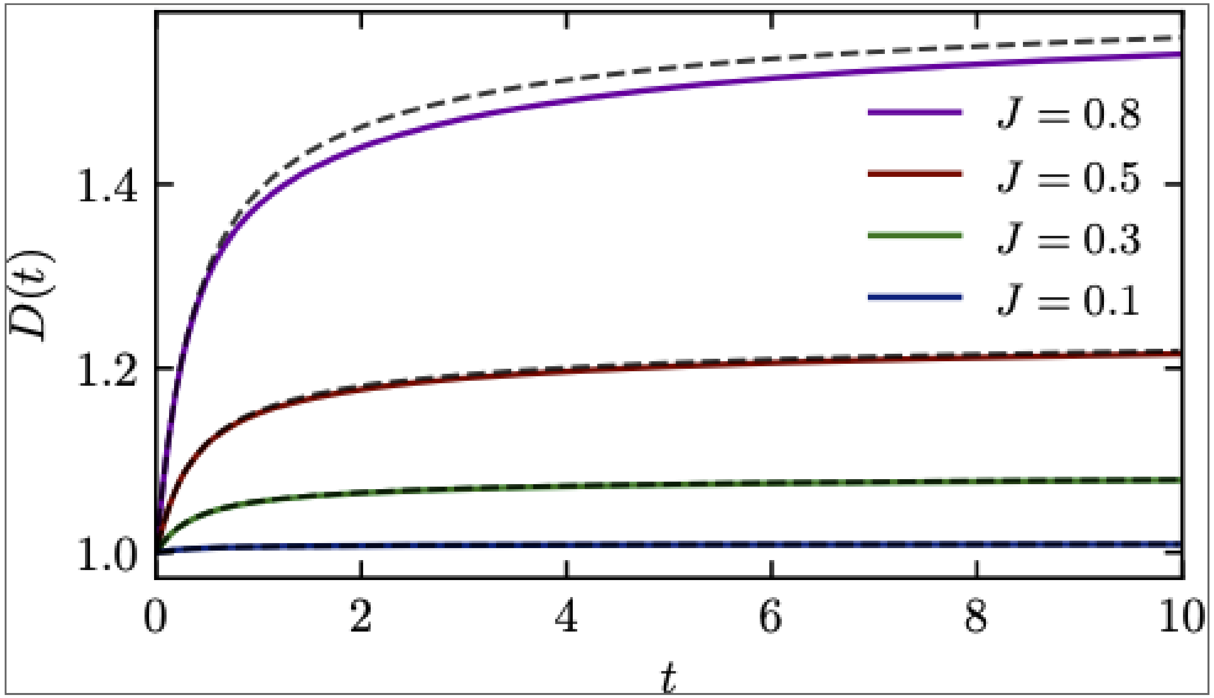
Numerics

- Represent $Z_j(t)$ using MPO and evolve using TEBD (based on **TeNPy**)
- $\chi = 400$, truncation error $\epsilon = 10^{-12}$, $\delta t = 10^{-2}$
- Exact for $J = 0$ ($\chi = 2$)

≡

Diffusion constant

- 100 spins



- Analytic calculation (dashed) upper bounds $D(t)$

≡

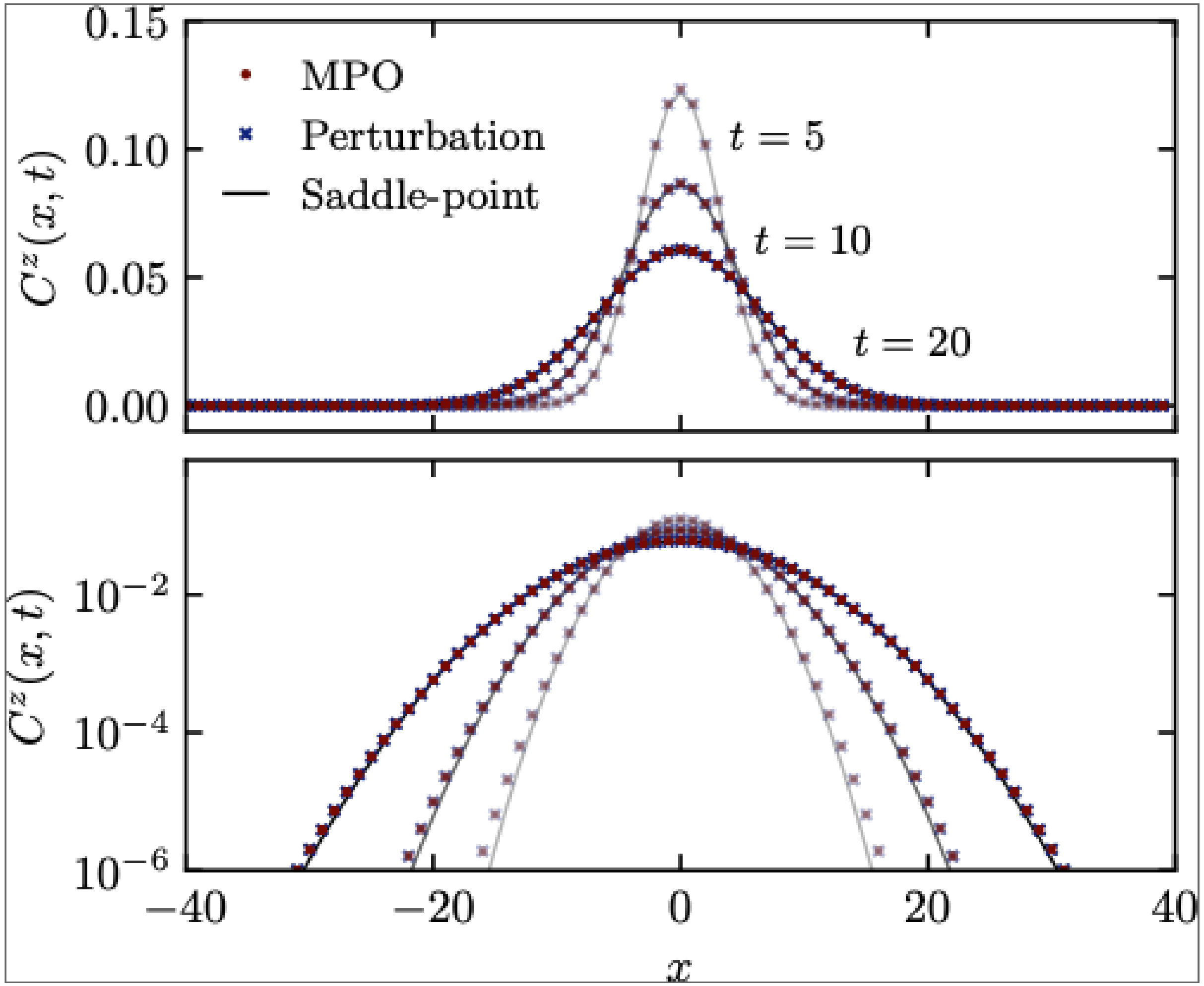
Profile

$$\omega = -i(1+J^2)\eta^2 \pm |\eta|^3 \frac{J^2}{2}\sqrt{1+2J^2} + O(\eta^4).$$

- Assuming poles dominate profile saddle point analysis yields

$$C(x;t) \propto \exp\left(-\frac{x^2}{2Dt}\right)\exp\left(-\frac{J^2\sqrt{2J^2+1}}{2D^3}\frac{|x|^3}{t^2}\right)$$

- 2nd factor hints at $\ell \sim t^{2/3}$ for KPZ!



$$C(x;t) \propto \exp\left(-\frac{x^2}{2Dt}\right)\exp\left(-\frac{J^2\sqrt{2J^2+1}}{2D^3}\frac{|x|^3}{t^2}\right)$$

≡

Nonabelian hydrodynamics

- **Glorioso *et al.* (2020)**: corrections to current

$$J^a = -D\nabla s^a + \lambda\epsilon_{abc}s^b s^c$$

- Implies $\sim \lambda^2/\sqrt{t}$ corrections to diffusion constant
- Consistent with $D(t) \underset{t\to\infty}{\longrightarrow} 1 + J^2 - \frac{J^2}{\sqrt{2\pi t}}$

≡

Higher orders?

- Recall 2nd order result

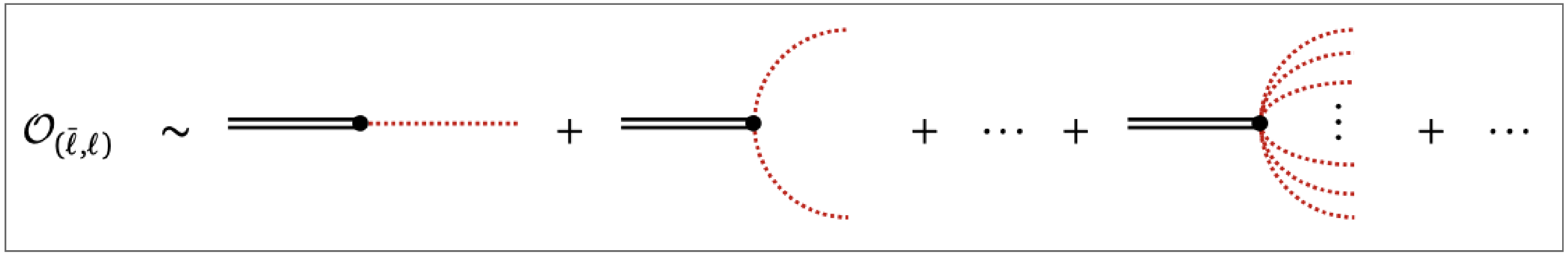
$$\Sigma(\eta, \omega) = J^2\eta^2 \left[1 + \frac{1}{2}\sqrt{\eta^2 - 2i\omega}\right]$$

- Branch point $\omega = -i\eta^2/2$: min. $\omega(\eta_1) + \omega(\eta_2)$ when $\eta_{1,2} \to \eta/2$.
- On kinematic grounds: at order J^{2n} branch point at $\omega = -i\eta^2/n$: minimum $\omega(\eta_1) + \omega(\eta_2) + \cdots + \omega(\eta_n)$ for $\eta_{1,2,\ldots n} \to \eta/n$.

≡

Diffuson cascade?

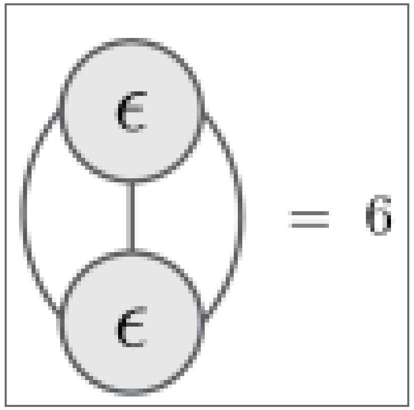
- **Luca V. Delacrétaz (2020)**

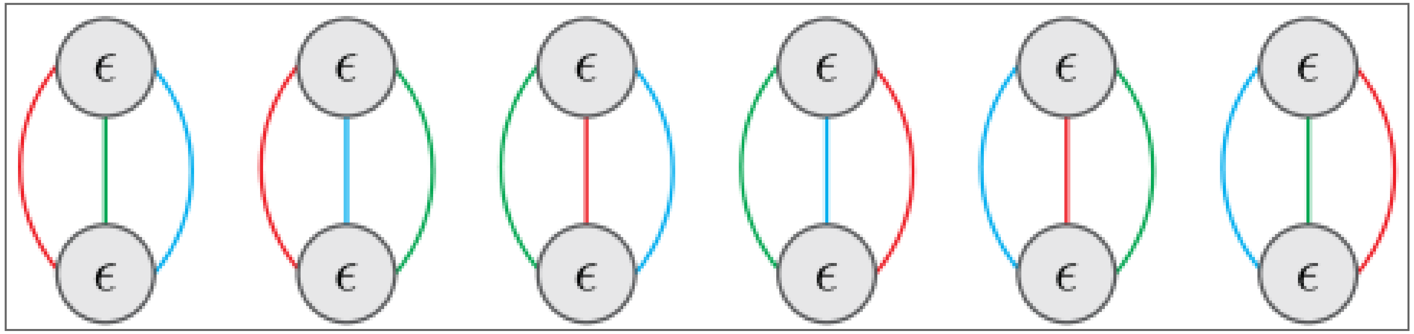


- Contribution of n-diffusons is $\sim n!(k\ell_{\text{th}})^{nd}\exp\left(-\frac{Dk^2t}{n}\right)$
- Optimal n gives contribution $\sim \exp\left(-\alpha\sqrt{Dk^2|t|}\right)$
- Would be interesting to see this in a microscopic model!

Penrose colouring

- Recall each vertex has ϵ_{abc}. For planar graph this counts **3-colorings**

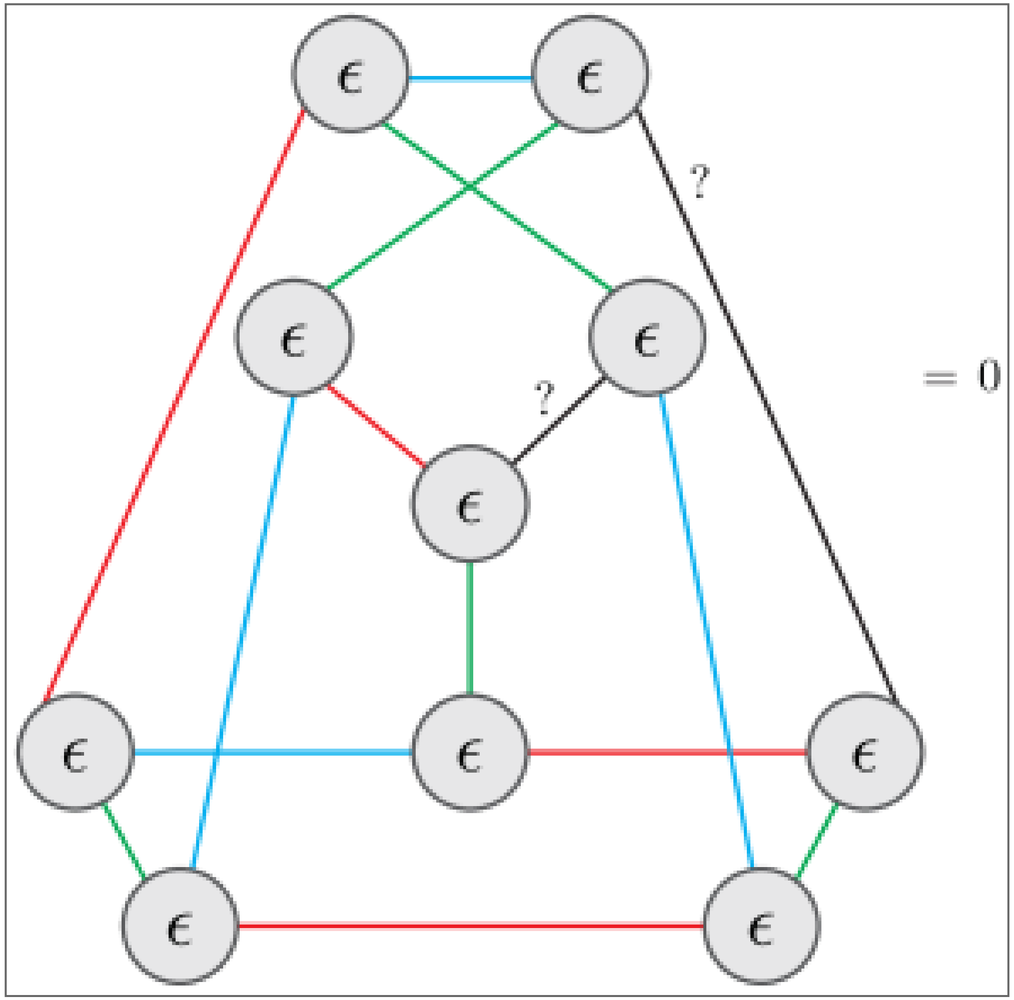




Biamonte and Bergholm (2017)

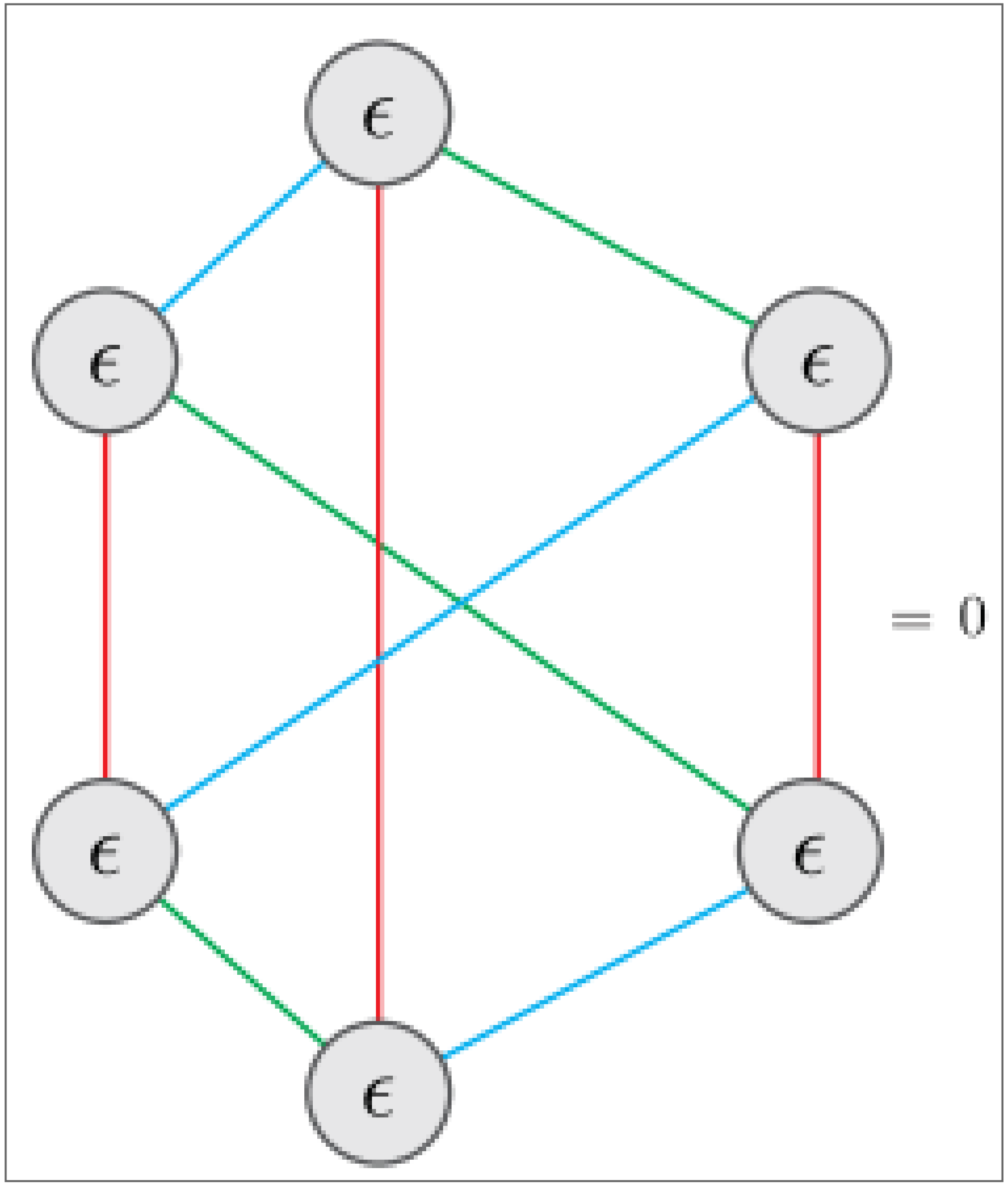
≡

- Some graphs can’t be 3-colored



≡

- Some non-planar graphs give zero



≡