Electrodynamics of a magnet moving through a pipe:† challenging undergraduates with theoretical projects
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- Introductory remarks
- Formulation
- Results
- Concluding remarks

†Electrodynamics of a Magnet Moving through a Metallic Pipe (with Eliza J. Morris), Canadian Journal of Physics 84, 253 (2006); arXiv:physics/0406085.

1 Introductory Remarks

- A genuine research experience for undergraduates is fast becoming a prerequisite for jobs and graduate school admission. It is also an effective educational strategy.
- Prospects and problems of incorporating undergraduates in theoretical research.
- Pedagogically oriented research as an educational strategy.

2 Formulation

• The popular demonstration involving a permanent magnet falling through a conducting pipe is treated as an axially symmetric boundary value problem.

• Specifically, Maxwell's equations are solved for an axially symmetric magnet moving coaxially inside an infinitely long, conducting cylindrical shell of arbitrary thickness at non-relativistic speeds.

• Previous treatments (Saslow) idealized the problem as a point dipole moving slowly inside a pipe of negligible thickness.

• The results allow a rigorous study of eddy currents and magnetic braking under a broad range of conditions.

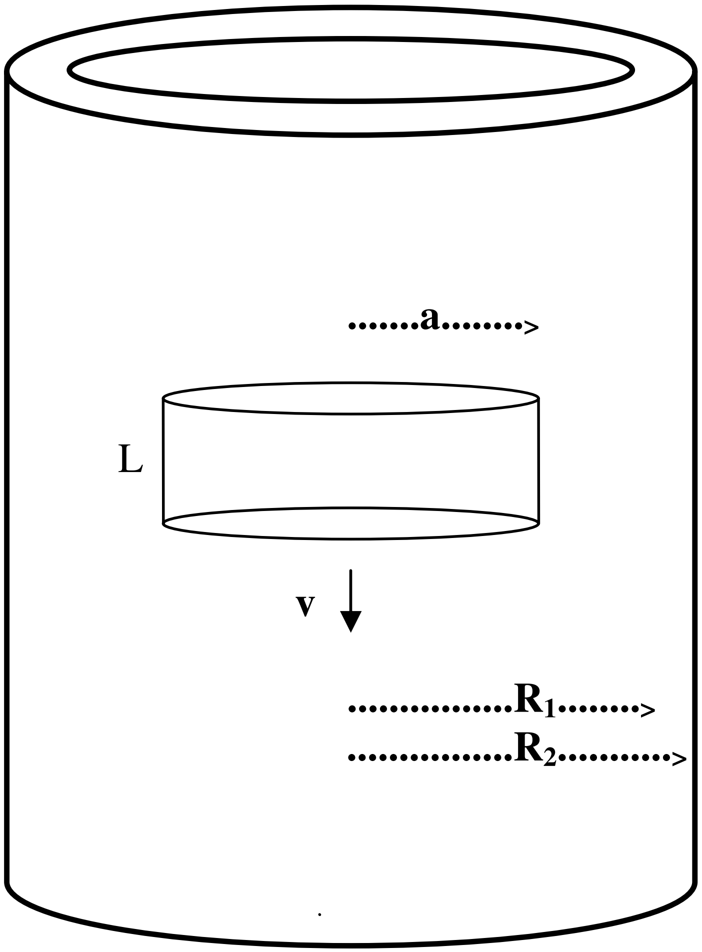

Figure 1: A cylindrically symmetric permanent magnet moving coaxially inside a conducting pipe.

• A permanent (or *hard*) magnet is a ferromagnetic material whose magnetization does not change when immersed in (moderate) external fields, electromagnetic or gravitational.

• Thus, by the equivalence principle, a permanent magnet is not affected by acceleration and can thereby be characterized by equivalent sources in its rest frame $\mathcal{S}'$ (with cylindrical space coordinates ρ', ϕ', z'):

$$\mathcal{M}'(\rho', z') = mP(\rho', z')\hat{\mathbf{z}}', \tag{1}$$

• Equivalently, since $\mathbf{J}_M = \nabla \times \mathcal{M}$,

$$\mathbf{J}'_M(\rho', \phi', z') = -m[\partial P(\rho', z')/\partial \rho']\hat{\boldsymbol{\phi}}' \tag{2}$$

• In the laboratory frame, we find $\mathbf{J}_M(\rho, z, \phi, t) = \mathbf{J}'_M(\rho', \phi', z')$, or

$$\mathbf{J}_M(\rho, \phi, z, t) = -m\frac{\partial P[\rho, z - z_M(t)]}{\partial \rho}]\hat{\boldsymbol{\phi}}, \tag{3}$$

• Using the standard solution for the vector potential in the quasi-static limit, we find after some algebra,

$$\tilde{A}_M(\rho, k) = -\mu_0 m \int_0^{+\infty} d\rho' \frac{\partial \tilde{P}(\rho', k)}{\partial \rho'} \rho' I_1(|k|\rho') K_1(|k|\rho). \tag{4}$$

for the moving magnet in *free* space.

• Now use the above field as the "incident field":

$$a \leq \rho \leq R_1 : \ \tilde{A}^{(i)}(\rho, k) = \tilde{A}_M(\rho, k) + b_1(k) I_1(|k|\rho), \tag{5}$$

$$R_1 \leq \rho \leq R_2 : \ \tilde{A}^{(ii)}(\rho, k) = b_2(k) K_1(\sqrt{\kappa^2}\rho) + b_3(k) I_1(\sqrt{\kappa^2}\rho), \tag{6}$$

$$R_2 \leq \rho : \ \tilde{A}^{(iii)}(\rho, k) = b_4(k) K_1(|k|\rho). \tag{7}$$

• Continuity conditions are used to find the unknown "b" coefficients. Here b_0 represents the "incident field" of the moving magnet, while the other coefficients correspond to "reflections" and "transmissions."

• Upon imposing the continuity conditions, we find the following set of equations:

$$b_0(k)K_1(|k|R_1) + b_1(k)I_1(|k|R_1) = b_2(k)K_1(\sqrt{\kappa^2}R_1) + b_3(k)I_1(\sqrt{\kappa^2}R_1), \tag{8}$$

$$b_2(k)K_1(\sqrt{\kappa^2}R_2) + b_3(k)I_1(\sqrt{\kappa^2}R_2) = b_4(k)K_1(|k|R_2), \tag{9}$$

$$\frac{|k|}{\mu_0}[b_0(k)K_0(|k|R_1) - b_1(k)I_0(|k|R_1)] = \frac{\sqrt{\kappa^2}}{\mu}[b_2(k)K_0(\sqrt{\kappa^2}R_1) - b_3(k)I_0(\sqrt{\kappa^2}R_1)], \tag{10}$$

$$\frac{\sqrt{\kappa^2}}{\mu}[b_2(k)K_0(\sqrt{\kappa^2}R_2)] - b_3(k)I_0(\sqrt{\kappa^2}R_2)] = \frac{|k|}{\mu_0}[b_4(k)K_0(|k|R_2)]. \tag{11}$$

• This set yields the unknown coefficients which define the solution to our problem.

3 Results

The Drag Force

- The Drag force on the magnet is calculated straightforwardly. The result in terms of the b-coefficients is

∞

$$\mathbf{F} = 2\pi i \mu_0^{-1} \hat{\mathbf{z}} \int_{-\infty}^{+\infty} k dk b_0(-k) b_1(k). \tag{12}$$

- For the uniformly magnetized cylinder we find

$$\mathbf{F}^{uni} = -\hat{\mathbf{v}} \frac{\mu_0 m^2}{2\pi^2} \int_0^{+\infty} dk k^3 \left[\frac{\sin(kL/2)}{(kL/2)} \right]^2 \left[\frac{I_1(ka)}{(ka/2)} \right]^2 \mathrm{Im}[Q(k)], \tag{13}$$

where $Q(k) = b_1(k)/b_0(k)$.

Limiting Cases

(velocity= $\mathbf{v}$, conductivity=σ, relative permeability=μ_{rel})

- **Low magnet speed**(good field penetration into the pipe):

$$\mathbf{F}^{lsp} \cong -\mathcal{C}\sigma v\hat{\mathbf{v}} \;\; (\mu_{rel}\mu_0\sigma v R_1 \ll 1), \tag{14}$$

- **High magnet speed**(skin effect on the inner pipe wall):

$$\mathbf{F}^{hsp} = -\frac{0.274m^2}{R_1^{9/2}}\mathcal{F}_1(a/R_1, L/R_1)\sqrt{\frac{\mu}{\sigma v}}\hat{\mathbf{v}}, (\mu_{rel}\mu_0\sigma v R_1 \gg 1), \tag{15}$$

- **Idealized Model**(*low magnet speed, point dipole, thin-walled pipe*; Saslow):

$$\mathbf{F}^{idl} = -\frac{45\mu_0^2 m^2 s}{1024{R_1}^4}\sigma v\hat{\mathbf{v}}, \tag{16}$$

- Note that $\mathbf{F}$ depends on σ and v through the combination σv.

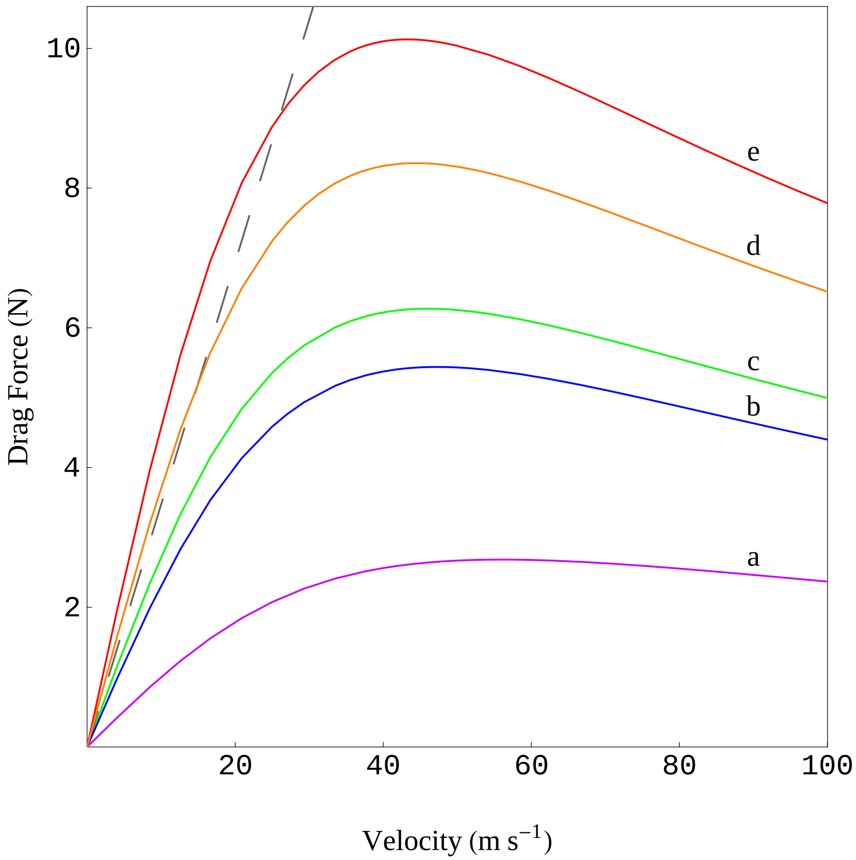

Figure 2: Plot of the drag force versus the magnet speed for a fixed value of the dipole moment and four different shape parameters $(L/2a, a/R_1)$: (a) typical cylinder, $(\frac{2}{1}, 0.60)$, (b) "square" cylinder, $(\frac{1}{1}, 0.60)$, (c) "point-like cylinder" $(\frac{1}{1}, \simeq\ 0)$, (d) short cylinder $(\frac{5}{8}, 0.96)$, and (e) circular wafer, $(\simeq\ 0, \frac{3}{5})$. The dashed line represents the idealized limit.

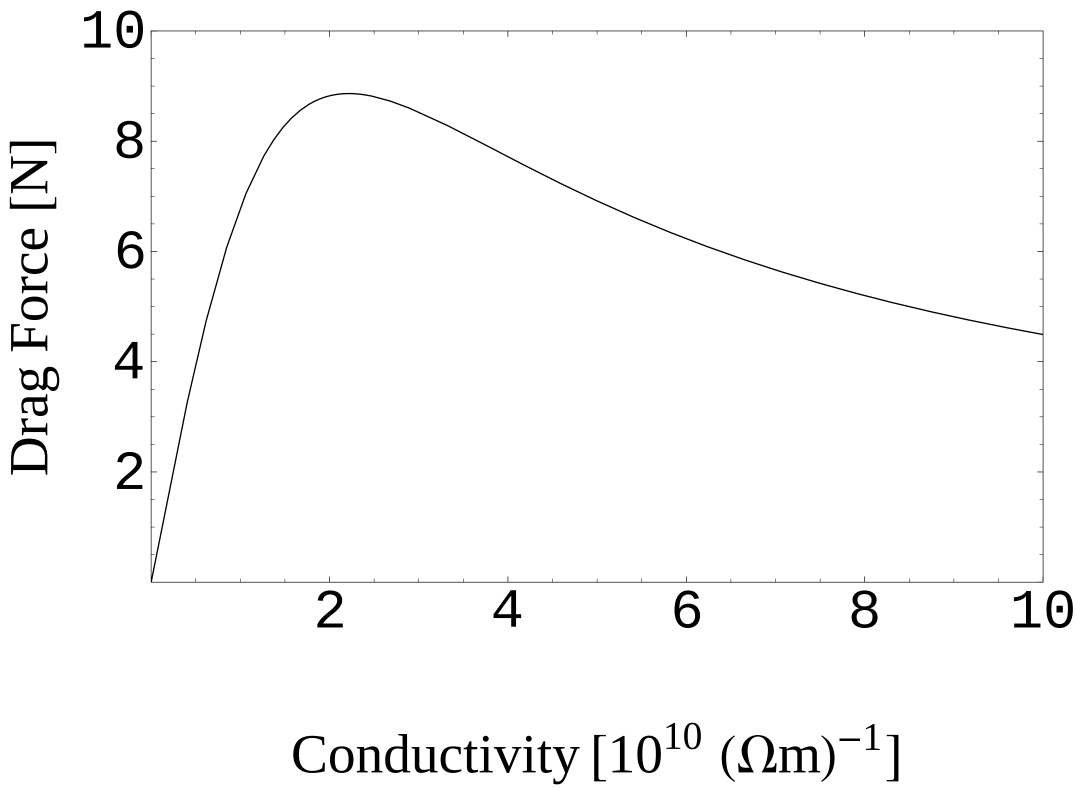

Figure 3: Plot of the drag force versus the conductivity of the pipe for case (d) with $v = 0.10$ m s^{-1}.

Limiting Cases . . .

• **Highly Diamagnetic Pipe**($\mu_{rel} \to 0$, as in magnetic flux expulsion, e.g., the Meissner effect):

$$\mathbf{F}^{hdm} = -\frac{\mu_0^2 m^2 \sigma v}{2\pi\sqrt{2}R_1^3}[\ln(R_2/R_1)]^{-\frac{1}{2}} \mu_{rel}^{3/2}\, \hat{\mathbf{v}}. \tag{17}$$

• **Highly Paramagnetic Pipe**($\mu_{rel} \gg 1$, as for "soft" ferromagnetic materials):

$$\mathbf{F}^{hpm} = -\frac{0.0536\mu_0^2 m^2}{R_1^{7/2}}\mathcal{F}_0(a/R_1, L/R_1)\sqrt{\frac{\sigma v}{\mu}}\hat{\mathbf{v}}, \tag{18}$$

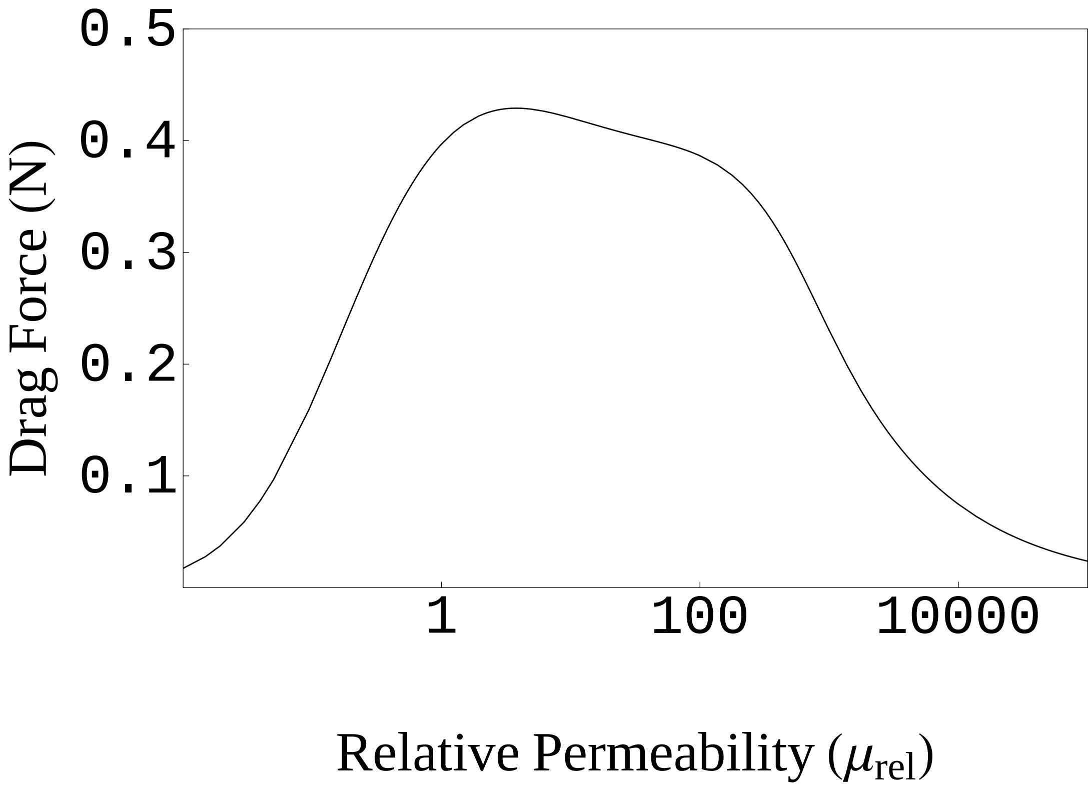

Figure 4: Plot of the drag force versus the relative permeability of the pipe for case (d) with $v = 1.0$ m s^{-1}.

4 Concluding Remarks

• In retrospect, the magnet-pipe system offers a rich landscape of concepts and methods, demonstrating the interplay of physical reasoning with mathematical analysis.

• The published paper is supplemented with a computer program posted on the web which can be used to compute the drag force.