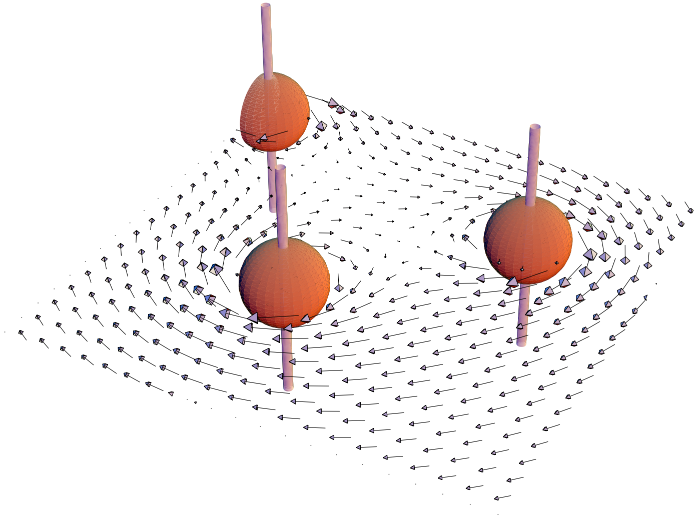
Flatland

Darrell F. Schroeter
Reed College



KITP (07/16/2007)

Flatland : Many-body quantum mechanics in 2D

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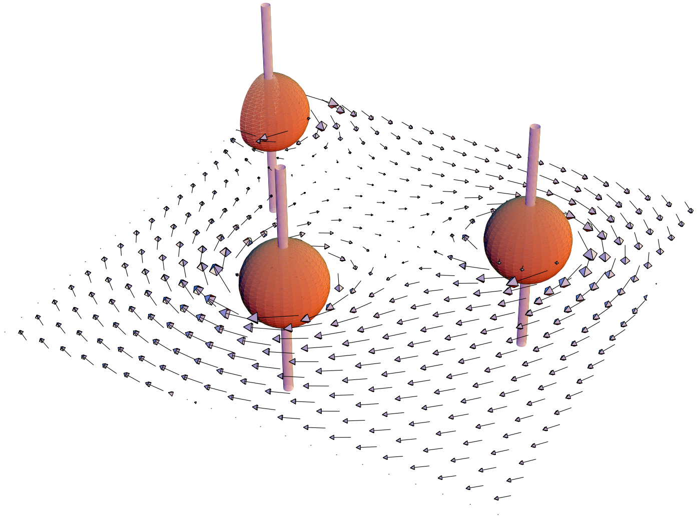


KITP (07/16/2007)

Acknowledgments

- Collaborators
    - ★ Eliot Kapit (Reed College ‘05)
    - ★ Prashant Luitel (Reed College ‘05)
    - ★ Bryan Conner (Occidental College ‘08)
    - ★ Martin Greiter (K.I.T.)
    - ★ Ronny Thomale (K.I.T.)
    - ★ Thomas Mosier (Reed College ‘08)
    - ★ Christopher Knowlton (Reed College ‘08)
- Research Corporation Cottrell College Science Award

Outline

- **Quantum statistics** is the result of the indistinguishability of quantum mechanical particles.
- In **3D** only two types of quantum statistics are possible and the particles that obey them are either **bosons** or **fermions**.
- In **2D** a continuum of quantum statistics are possible and the particles that obey them are **anyons**.
- Anyons do occur in nature.
- Understand how anyons arise in condensed matter systems at a microscopic level.

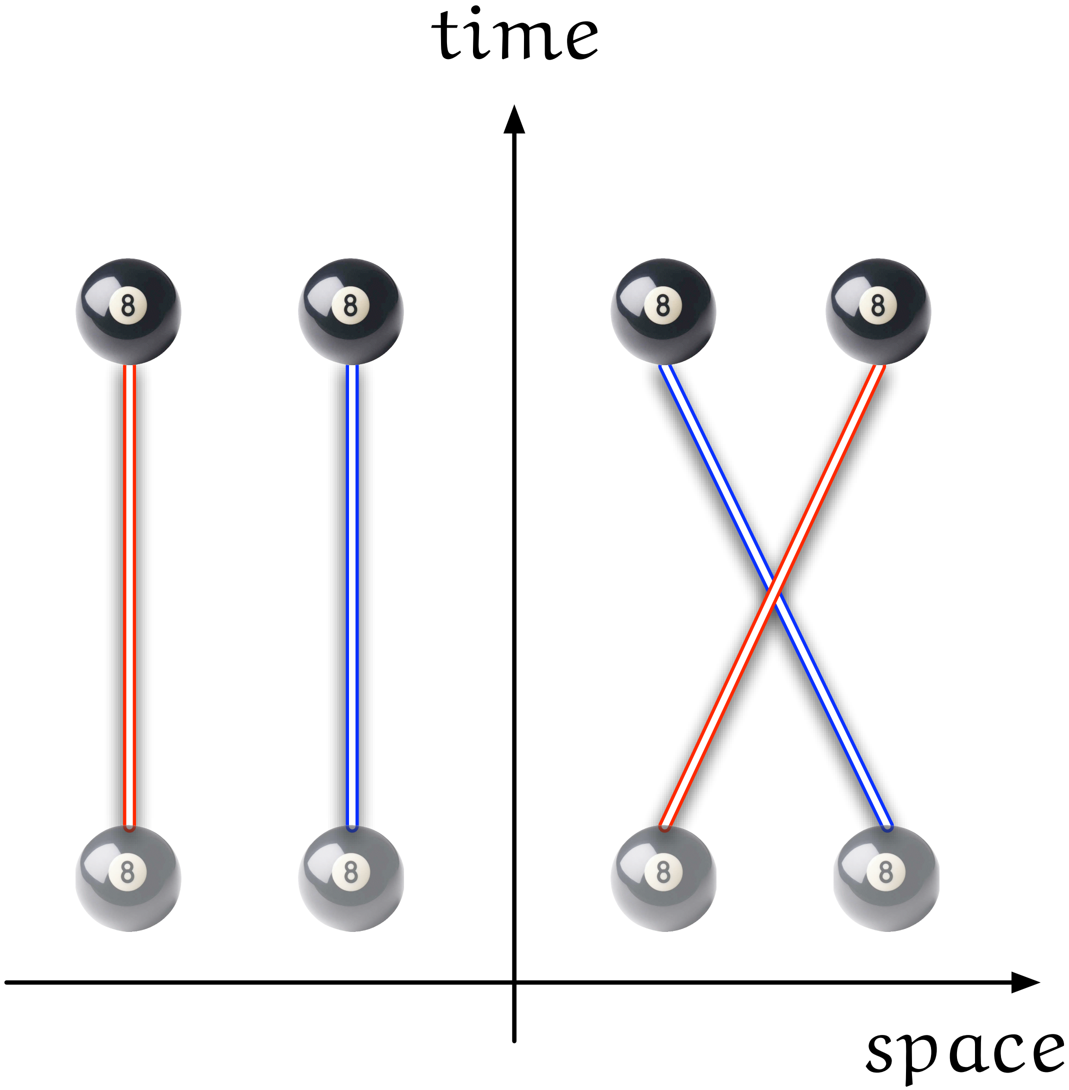
Indistinguishability

Indistinguishable vs. Identical (Classical)

- Classically, it is sensible in principle to have particles that are identical.

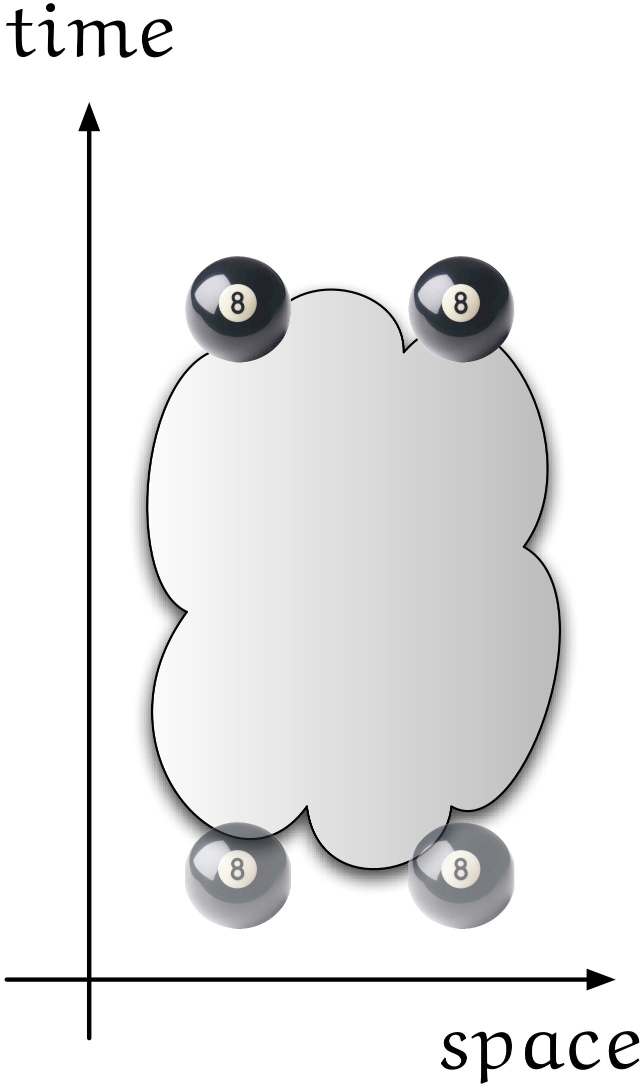
- However, classically these particles remain distinguishable.

- They have trajectories and you may keep them straight by watching them move.



Indistinguishable vs. Identical (QM)

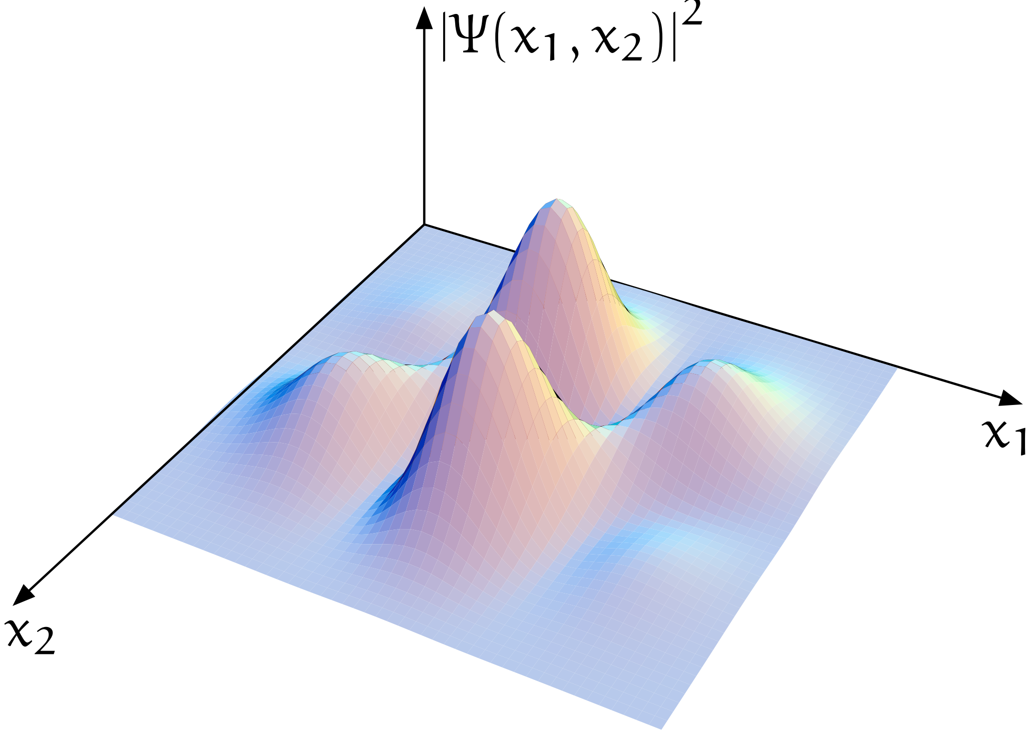
- Quantum mechanically, two particles of the same type (two electrons) are in fact identical.
- In this case they are also indistinguishable.
- Quantum mechanical particles do not have trajectories.
- It doesn't make sense to ask whether the particle on the left is still on the left.



Many-body quantum mechanics

- In the quantum mechanics of many particles (say 2) we represent the state of a system by the wave function $\Psi(\vec{r}_1, \vec{r}_2)$

- The wave function is interpreted probabilistically:
  - $|\Psi(\vec{r}_1, \vec{r}_2)|^2 \; d\vec{r}_1 \; d\vec{r}_2$ is the probability of finding particle 1 within $d\vec{r}_1$ of $\vec{r}_1$ and simultaneously finding particle 2 within $d\vec{r}_2$ of $\vec{r}_2$.



Quantum statistics

The effect of indistinguishability

A mathematical statement of indistinguishability

- But the probability of finding particle 1 at $\vec{r}_1$ and particle 2 at $\vec{r}_2$ doesn’t make any sense if the particles are indistinguishable.

- All one can ask is “what is the probability of finding a particle at $\vec{r}_1$ and a particle at $\vec{r}_2$.”

- The mathematical statement is that the probabilities have to be equal regardless of which particle is where:

$$|\Psi(\vec{r}_1,\vec{r}_2)|^2 = |\Psi(\vec{r}_2,\vec{r}_1)|^2$$

- The wave function differs by at most a phase when the coordinates are interchanged

$$\Psi(\vec{r}_1,\vec{r}_2) = e^{i\theta}\,\Psi(\vec{r}_2,\vec{r}_1)$$

Bosons and fermions

- This phase is constrained through interchanging the particles twice

$$\Psi(\vec{r}_1, \vec{r}_2) = e^{i\theta}\, \Psi(\vec{r}_2, \vec{r}_1) = e^{2i\theta}\, \Psi(\vec{r}_1, \vec{r}_2)$$

- Therefore $e^{2i\theta} = 1$.

Bosons and fermions

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$$\Psi(\vec{r}_1,\vec{r}_2) = e^{i\theta}\,\Psi(\vec{r}_2,\vec{r}_1) = e^{2i\theta}\,\Psi(\vec{r}_1,\vec{r}_2)$$

- Therefore $e^{2i\theta} = 1$. There are only two possibilities:

$\theta = 0$	$\Psi(\vec{r}_1,\vec{r}_2) = \Psi(\vec{r}_2,\vec{r}_1)$	bosons
$\theta = \pi$	$\Psi(\vec{r}_1,\vec{r}_2) = -\Psi(\vec{r}_2,\vec{r}_1)$	fermions

Path integral formulation

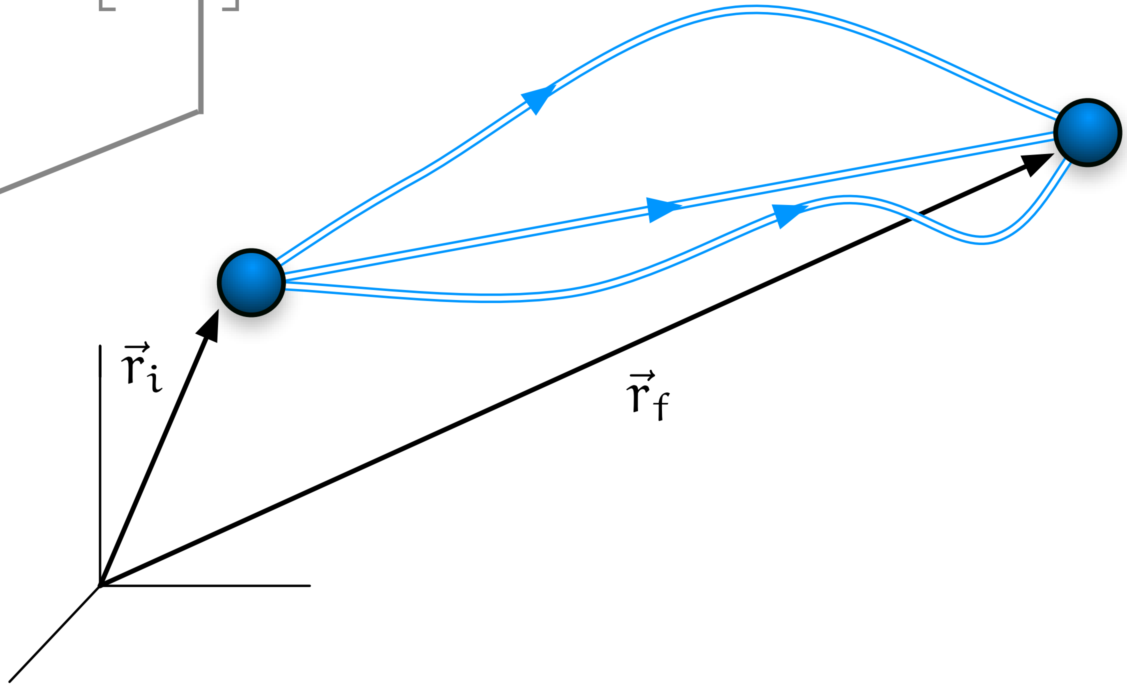
Getting rid of particle labels altogether

Feynman’s path-integral formulation

- Probability for a particle to move from position $\vec{r}_i$ at t_1 to position $\vec{r}_f$ at t_2 can be computed by calculating the classical action along every possible path and computing the sum

$$\langle \vec{r}_i, t_1 | \vec{r}_f, t_2 \rangle \propto \sum_{\text{all paths}} \exp\left[\frac{i}{\hbar} S\right]$$

classical physics



Path-integrals for N-body QM

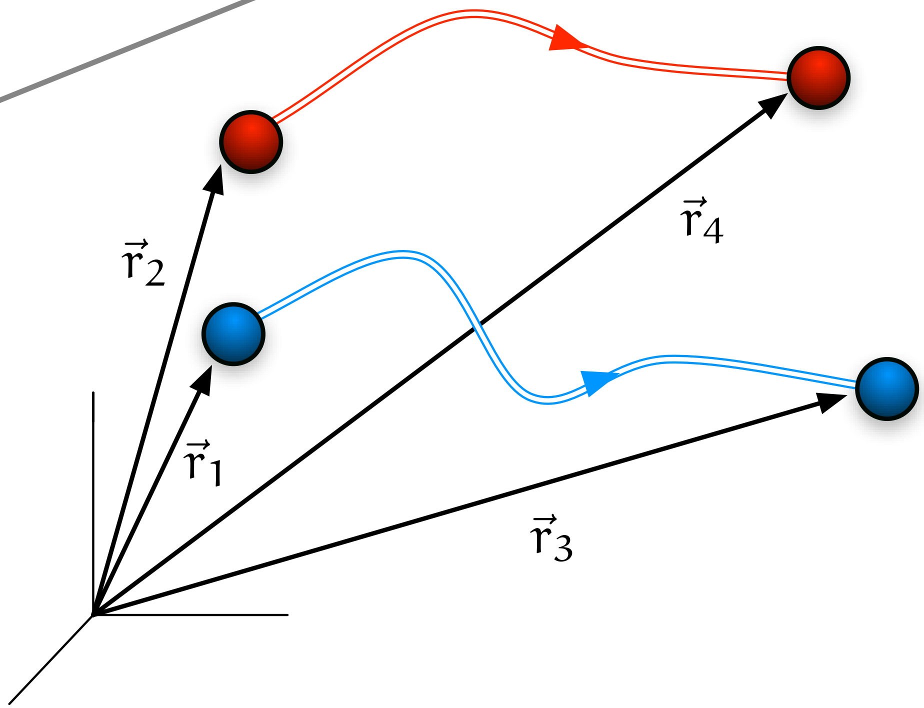
- Formulation is similar for more than one particle[†]

$$\langle \{\vec{r}_i\}, t_1 | \{\vec{r}_f\}, t_2 \rangle \propto \sum_{\mathrm{all\,paths}} e^{i\theta_{\mathrm{path}}} \exp\left[\frac{i}{\hbar} S\right]$$

statistics

classical physics

- θ_{path} must be the same for all paths that can be continuously deformed into one another.



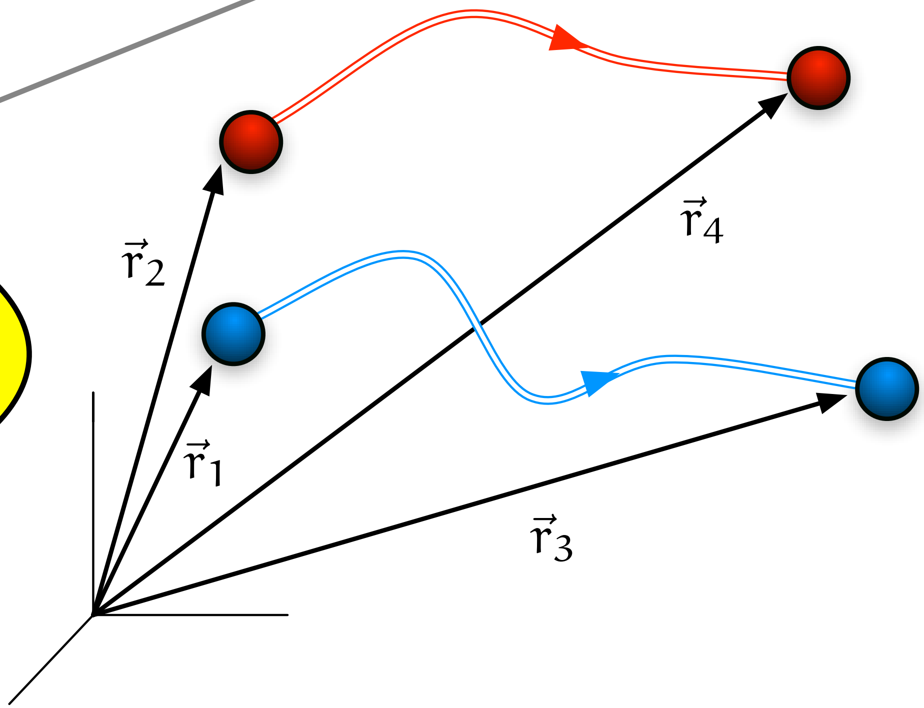
[†] M. G. G. Laidlaw and C. M. Dewitt, Phys Rev D **3**, 1375 (1971).

Path-integrals for N-body QM

- Formulation is similar for more than one particle[†]

$$\langle \{\vec{r}_i\}, t \,|\, \{\vec{r}_f\}, t \rangle \propto \sum_{\text{all paths}} \underbrace{e^{i\theta_{\text{path}}}}_{\text{statistics}} \exp\left[\frac{i}{\hbar} \underbrace{S}_{\text{classical physics}}\right]$$

- θ_{path} must be the same for all paths that can be continuously deformed into one another.



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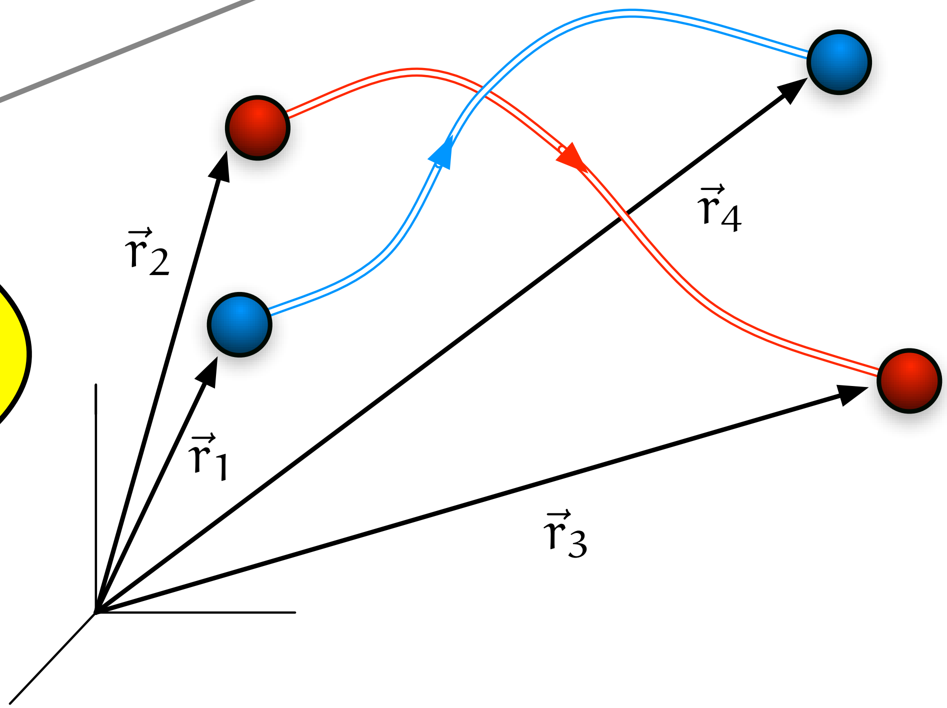


[†] M. G. G. Laidlaw and C. M. Dewitt, Phys Rev D **3**, 1375 (1971).

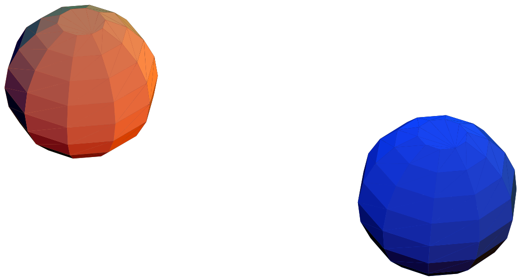
Interchange and dimensionality

The correct way to define statistics ... assignment of θ_{path} to trajectories that interchange particles.

Single interchange of two particles

A path that interchanges two particles can be achieved with

- a rotation of one particle about the other



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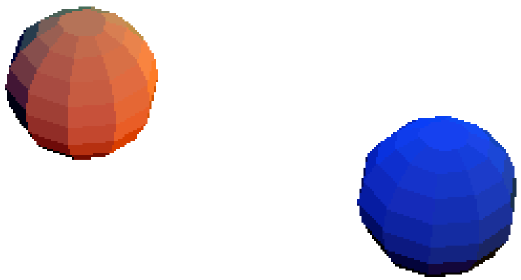
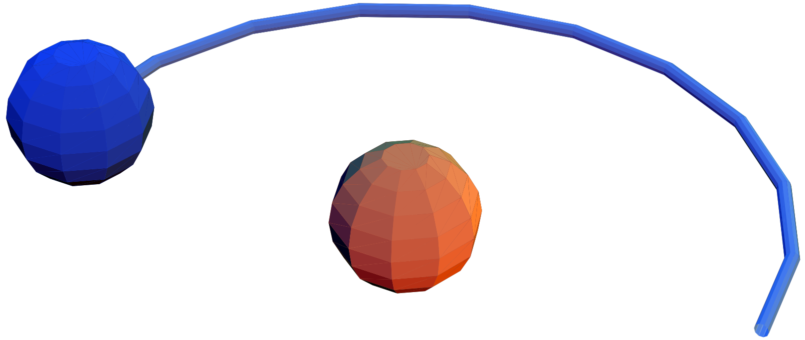


Single interchange of two particles

A path that interchanges two particles can be achieved with

- a rotation of one particle about the other
- and a translation of their center of mass



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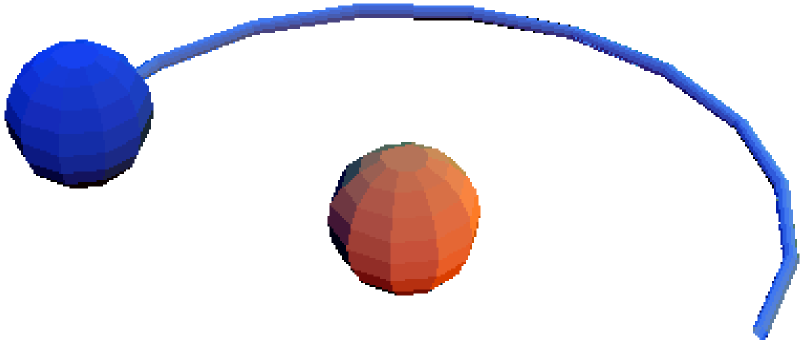
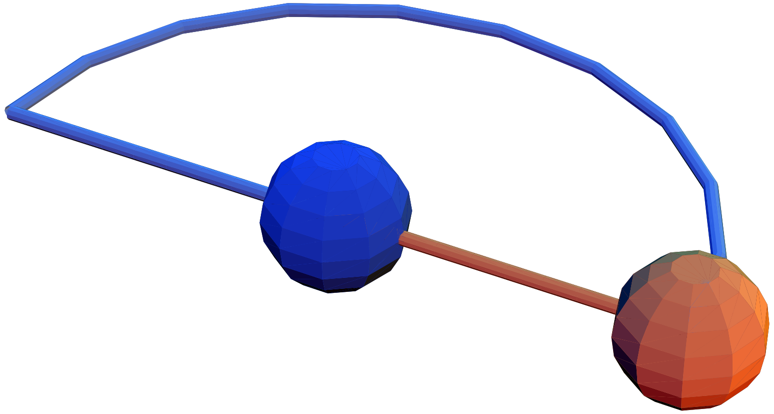


Single interchange of two particles

A path that interchanges two particles can be achieved with

- a rotation of one particle about the other
- and a translation of their center of mass
- the center of mass motion is actually irrelevant

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$$



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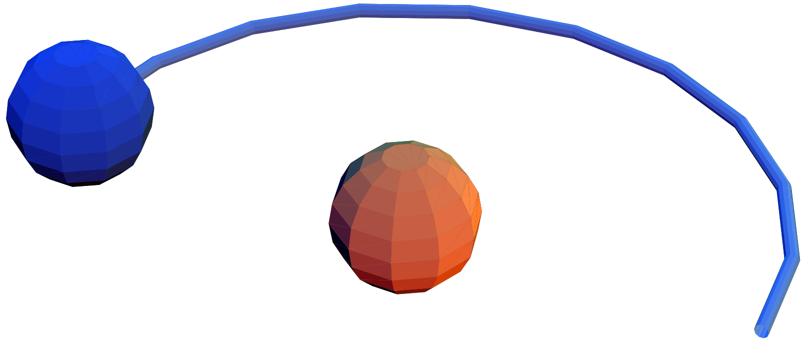
- the center of mass motion is actually irrelevant

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$$

- so these two particles have been interchanged



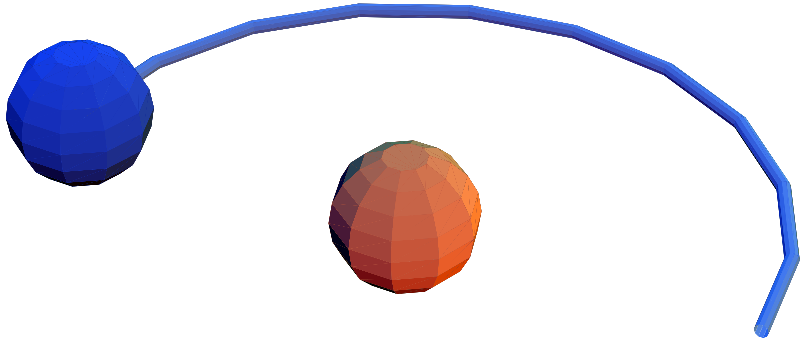
Assign this path

$e^{i\theta}$

Double interchange of two particles

To perform a second interchange

- keep rotating one particle about the other



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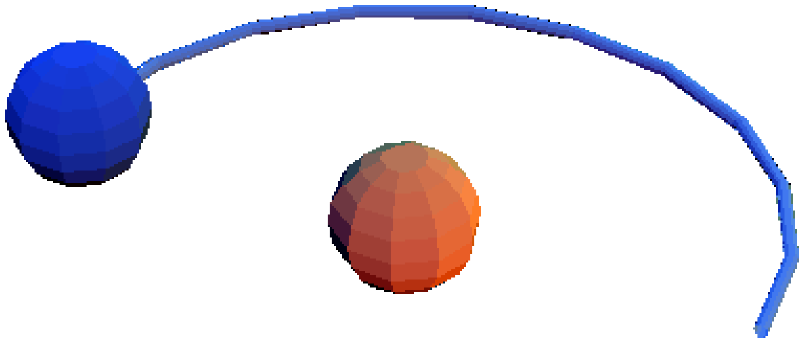
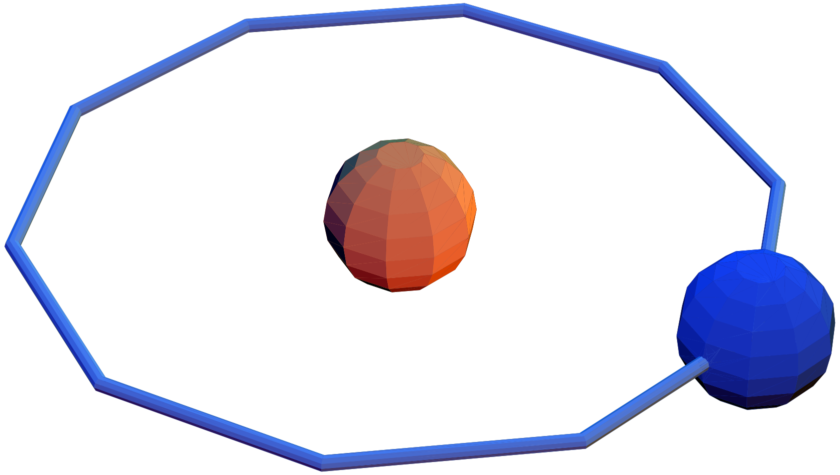


Double interchange of two particles

To perform a second interchange

- keep rotating one particle about the other
- this picks up another factor of the phase $e^{i\theta}$

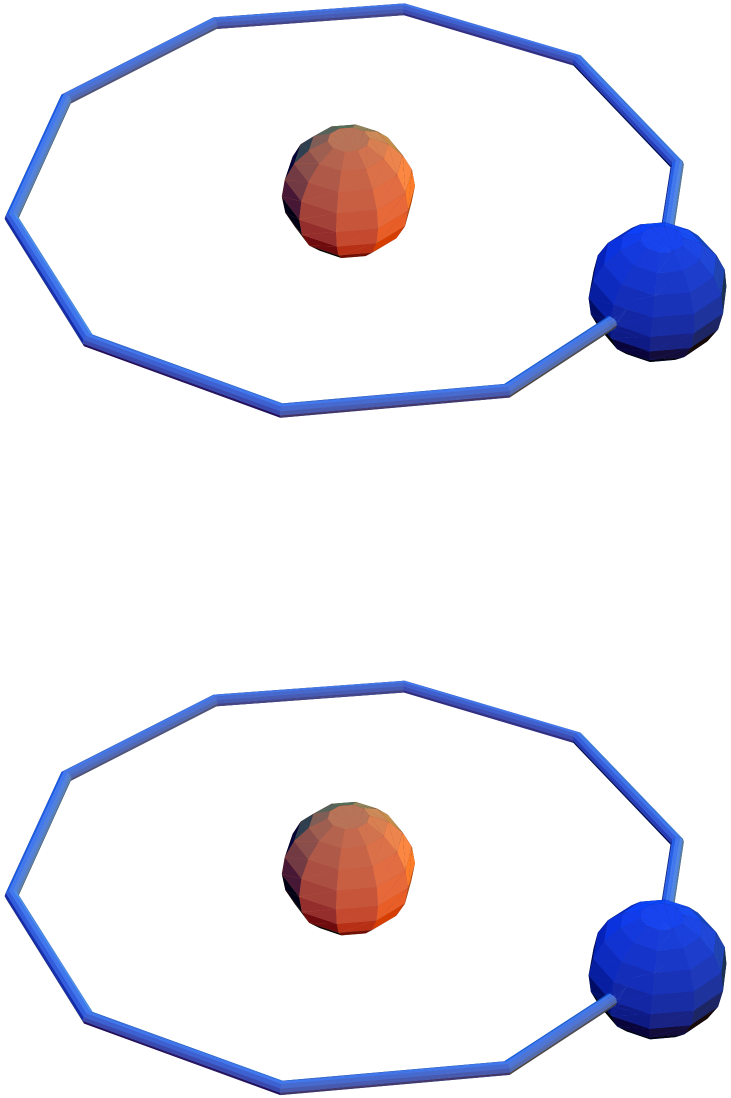


The phase associated with this path is

$$e^{i\theta}\,e^{i\theta} = e^{2i\theta}$$

The role of dimensionality

In three dimensions this loop may be deformed into a point



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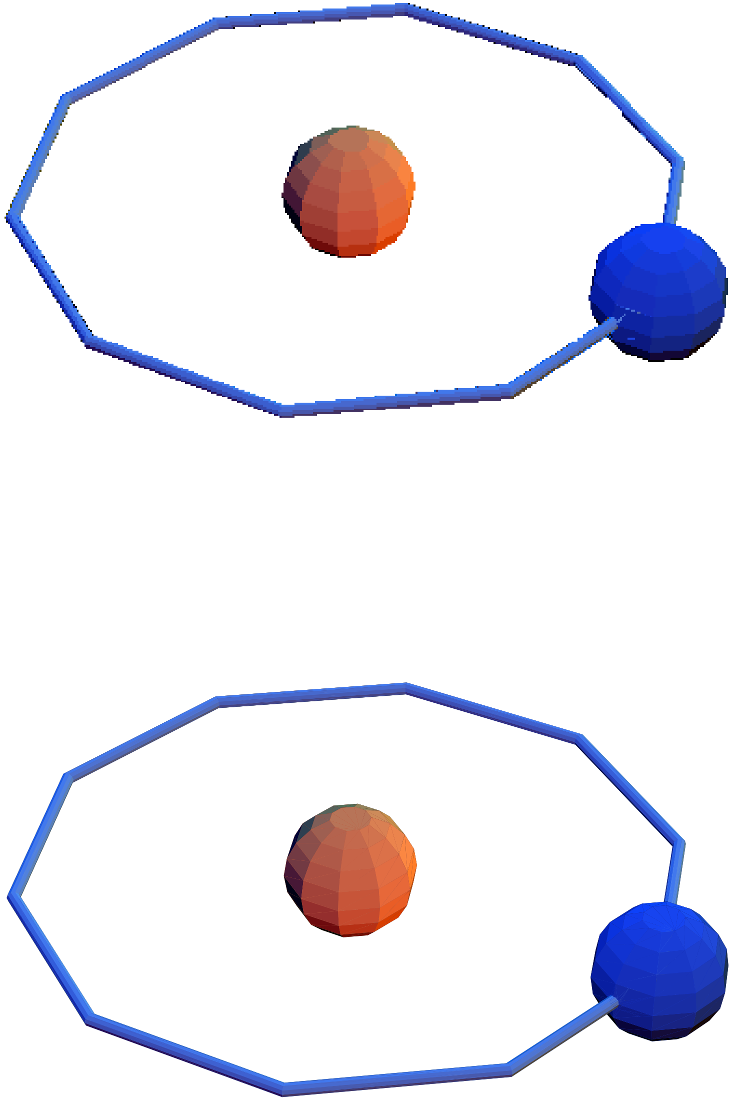
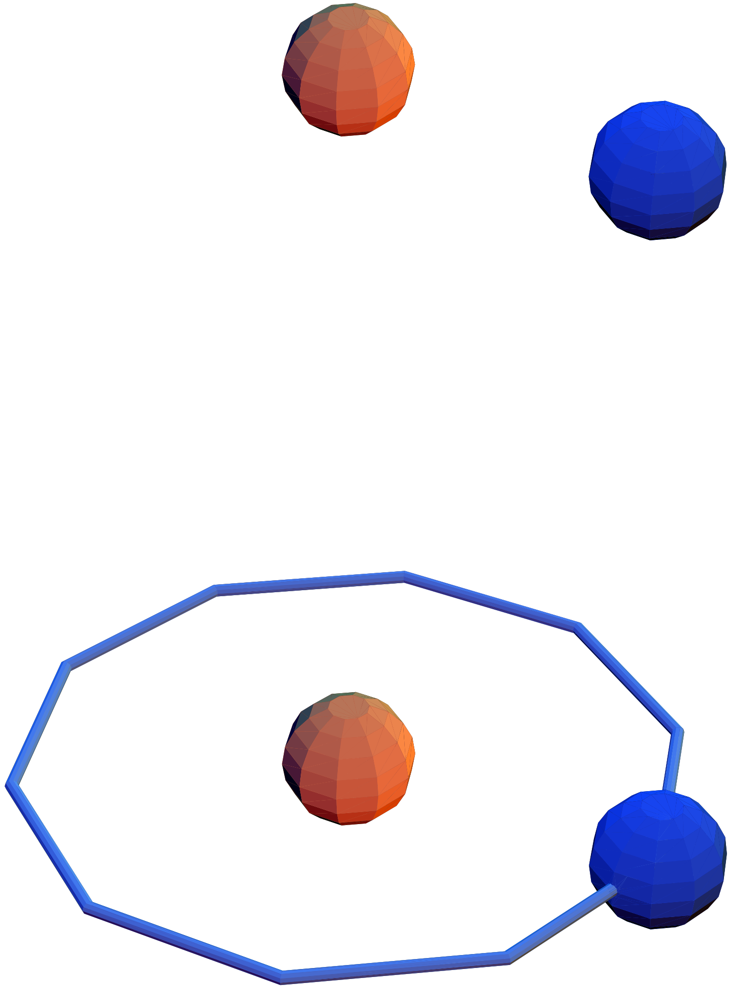


The role of dimensionality

In three dimensions this loop may be deformed into a point

- $e^{2i\theta} = 1$
  - $\theta = 0$ (bosons)
  - $\theta = \pi$ (fermions)



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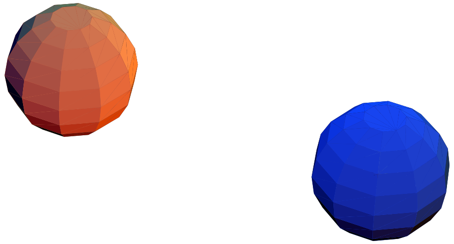

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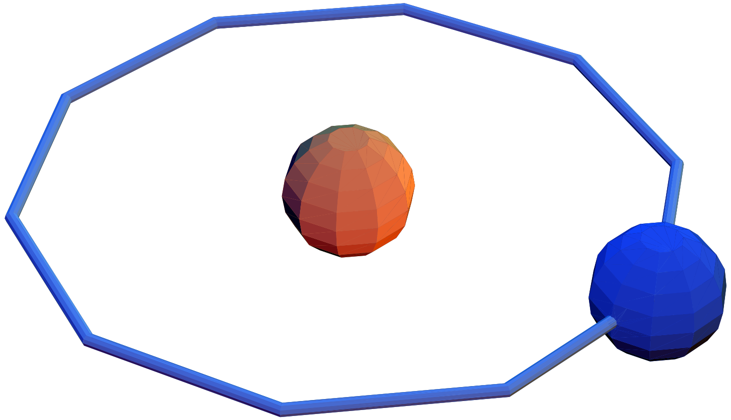
But in two dimensions this loop may not be deformed to a point



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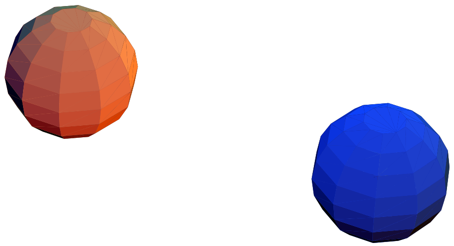

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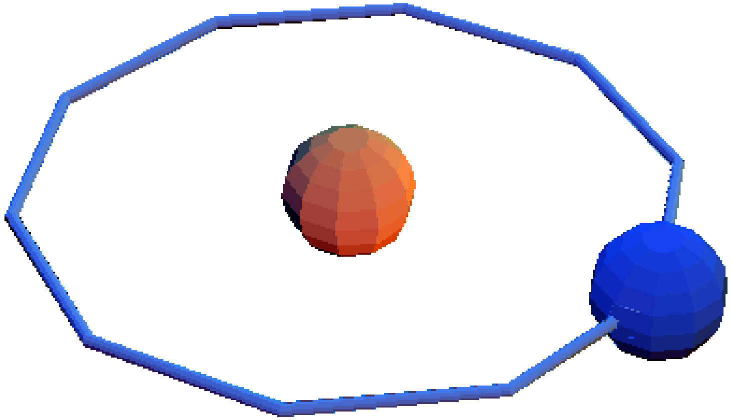


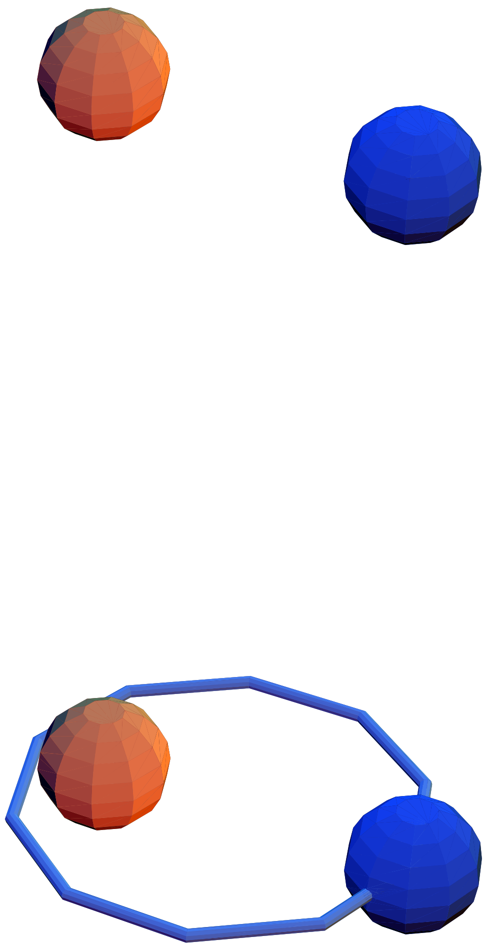
The role of dimensionality

In three dimensions this loop may be deformed into a point

- $e^{2i\theta} = 1$
    - $\theta = 0$ (bosons)
    - $\theta = \pi$ (fermions)

But in two dimensions this loop may not be deformed to a point[1]

- $e^{2i\theta} =$ anything
    - $\theta =$ anything (anyons[2])



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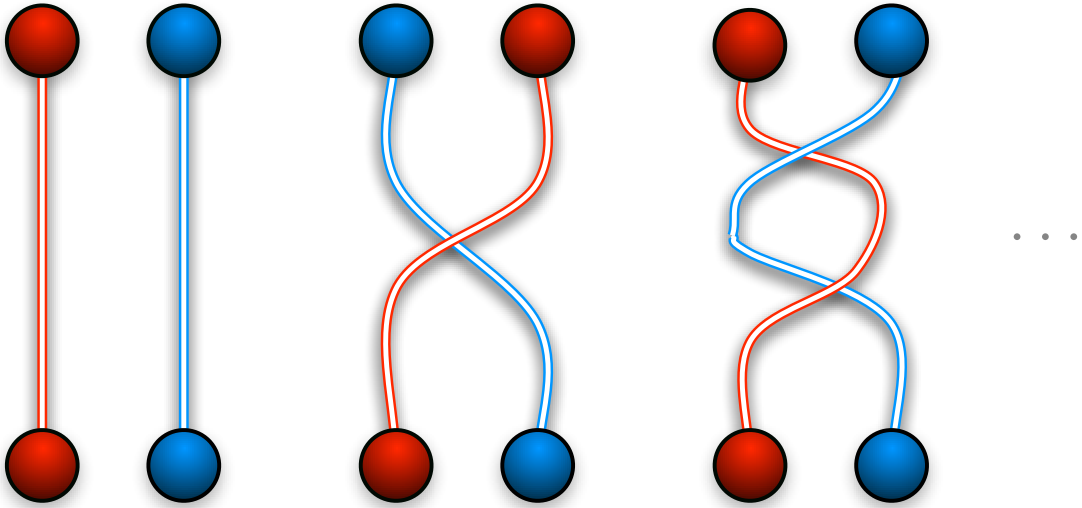
[1] J. M. Leinaas and J. Myrheim, Il Nuovo Cimento **37**, 132 (1977).

[2] F. Wilczek, Phys Rev Lett **49** (1982) 957.

Anyons

- In two dimensions, there is a continuous range of statistics.
- These particles extrapolate between bosons and fermions and are richer than either.



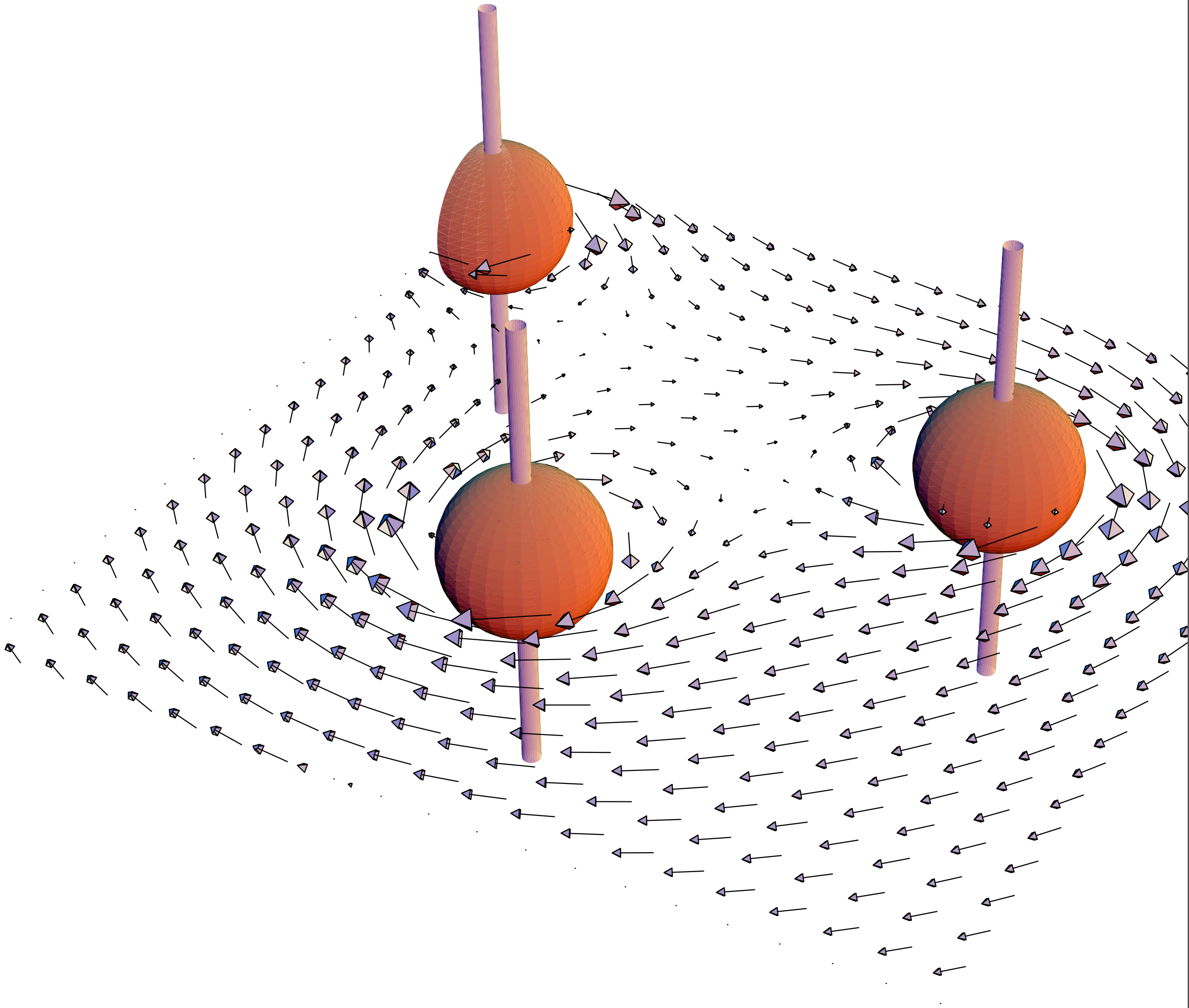
bosons	$+1$	$+1$	$+1$	$\cdots$
fermions	$+1$	-1	$+1$	$\cdots$
anyons	$+1$	$e^{i\theta}$	$e^{2i\theta}$	$\cdots$

Wave functions for anyons?

Can be constructed in three ways

- Wave functions $\Psi(\vec{r}_1, \vec{r}_2)$ are multi-valued in anyon coordinates
- Transmutation of statistics. Anyons can be treated as bosons carrying fictitious charge and a fictitious solenoid.

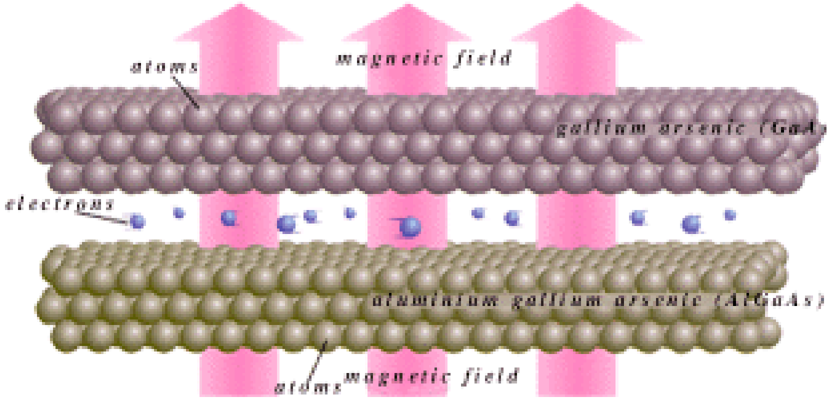
Anyons are composite particles. Can describe them in terms of the wave functions of their constituents (electrons).

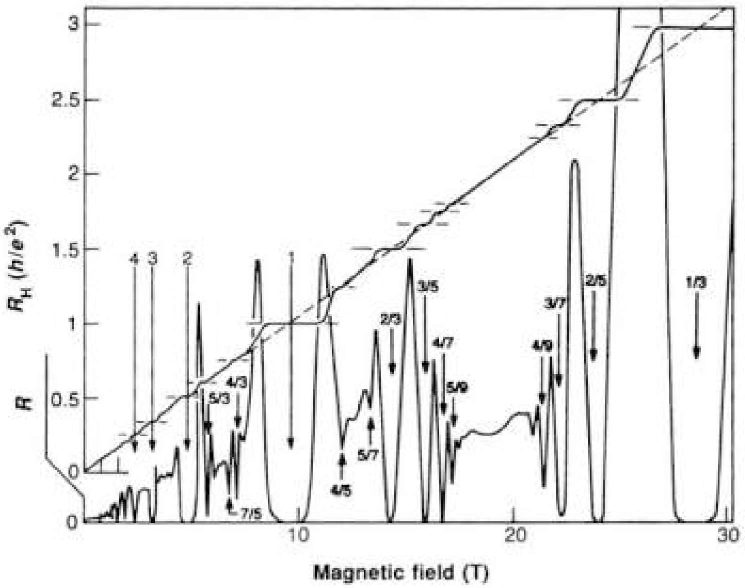


whatever is not forbidden is compulsory...

Fractional Quantum Hall Effect

- The interface between two semiconductors creates a **2D** electron gas.
- At low temperatures and in a magnetic field, the collective excitations of this electron gas behave like **fractions of electrons**.
- These composite particles are **anyons**.
- 1998 Nobel prize to Tsui, Störmer and Laughlin.





Images from the Nobel Foundation, Illustrated Presentation of the 1998 Nobel Prize in Physics.

Anyon wave functions in the FQHE

- The wave function for an anyon at position z_0 is

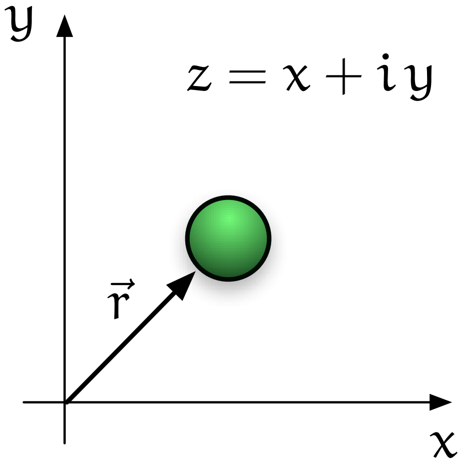
$$\Psi_{z_0}(z_1 \cdots z_N) = \prod_i (z_i - z_0) \prod_{i<j} (z_i - z_j)^m \prod_i e^{-\frac{1}{4}|z_k|^2}$$

where electron coordinates are in red.

- These “particles” have statistics

$$\theta = \frac{\pi}{m}$$

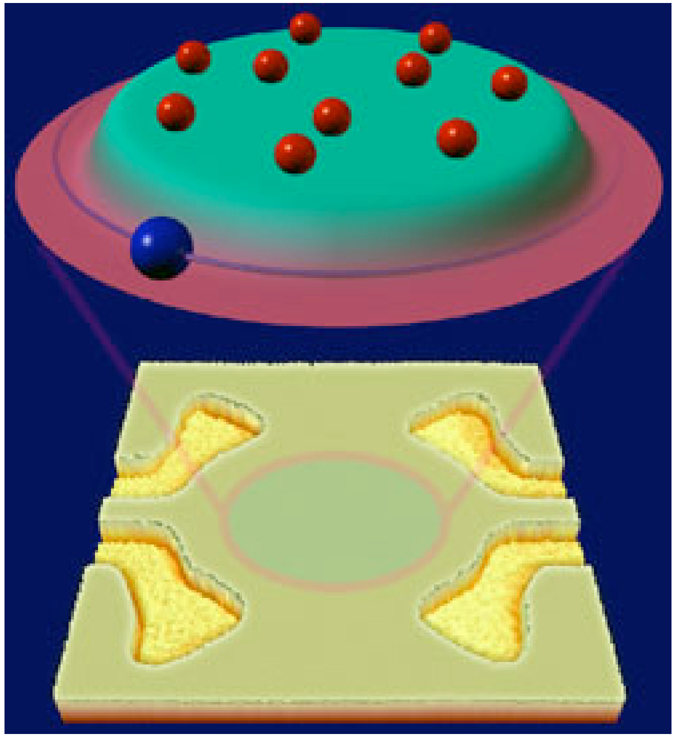
with m is an odd integer.



Direct observation of anyons (2005)

- An interferometer was created where anyons circled an island of anyons with the phase shifts producing interference fringes.

- Similar to the Aharanov-Bohm effect except that the interference is due to statistics not a real magnetic field.



F. E. Camino, Wei Zhou, and V. J. Goldman, PRB **72**, 075342 (2005).
D. Lindey, Phys Rev. Focus, 2 November 2005.

Current interest in anyons (besides FQHE)

- Quantum computing. An anyon quantum computer would be insensitive to interactions with its environment.

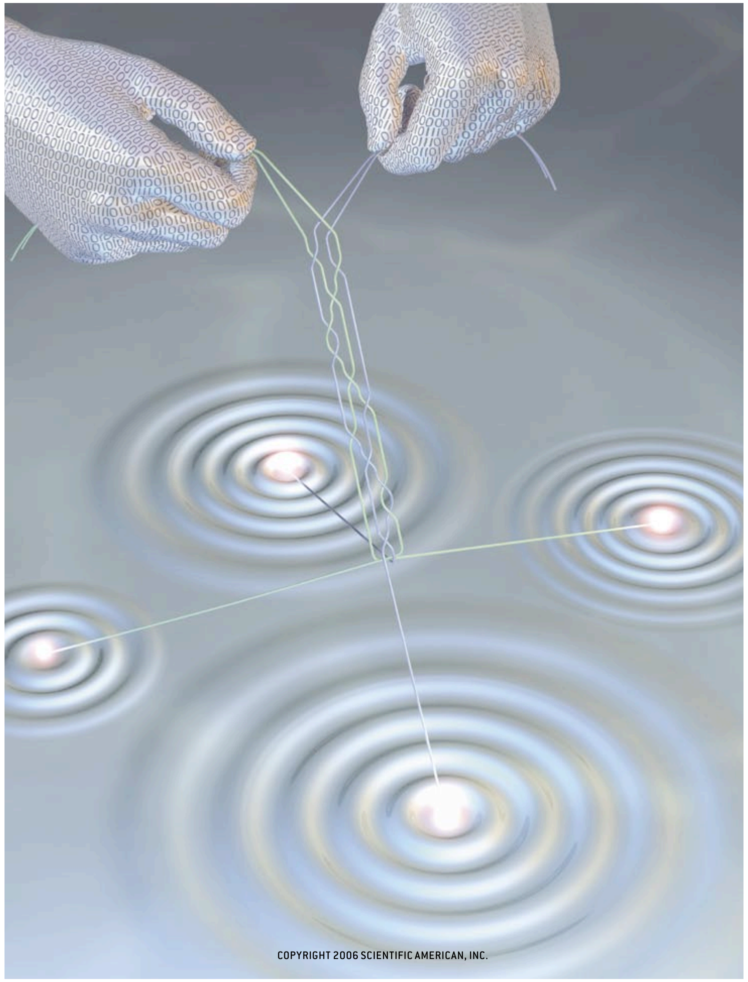


Image from G. P. Collins, Scientific American, April 2006, p. 57.

Current interest in anyons (besides FQHE)

- Quantum computing. An anyon quantum computer would be insensitive to interactions with its environment.

- Superconductivity.
  - Anyons superconduct.
  - The high T_C materials are effectively 2D.

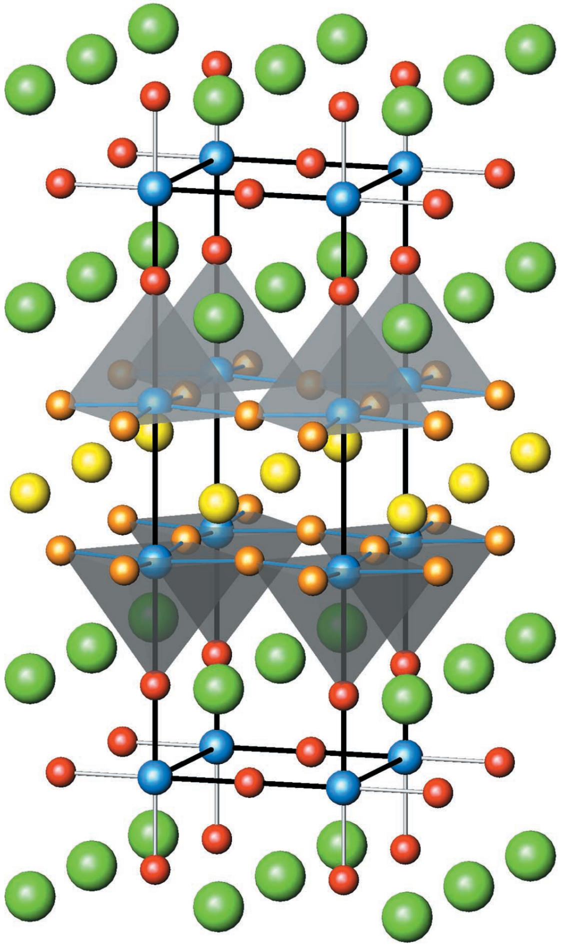


Image from C. Homes, Rad. Synch. News, **18** (3), 2005, p. 9.

Anyon superconductivity?

Avinash Kare writes in Fractional Statistics and Quantum Theory (2005):

“Two basic issues are involved when discussing anyon superconductivity.

i. One must really start from the microscopic condensed matter physics and get some kind of effective field theory in which the excitations turn out to be semions.

ii. One then has to show that a gas of semions exhibits superfluidity and also superconductivity in case the semions are charged.

I may add here right away that the first issue is very hard and much less established than the second one.”

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[2] E. Kapit, Reed College Senior Thesis (2005).

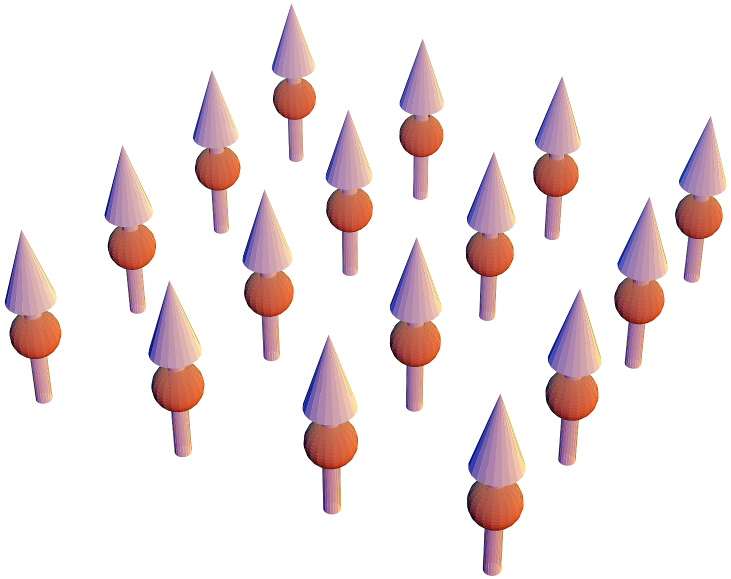
INCLUDE CITATIONS HERE

Frustrated magnets and spin liquids

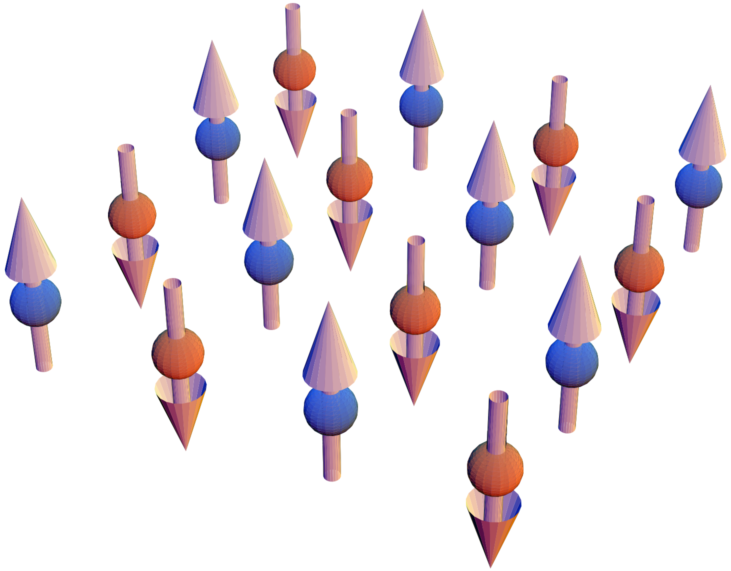
A place to look for anyons

Magnets

- Ferromagnets
  - Neighboring electron spins align
  - Stick it to your fridge



- Anti-ferromagnets (QM magnets)
  - Neighboring electron spins anti-align
  - Won’t stick to your fridge

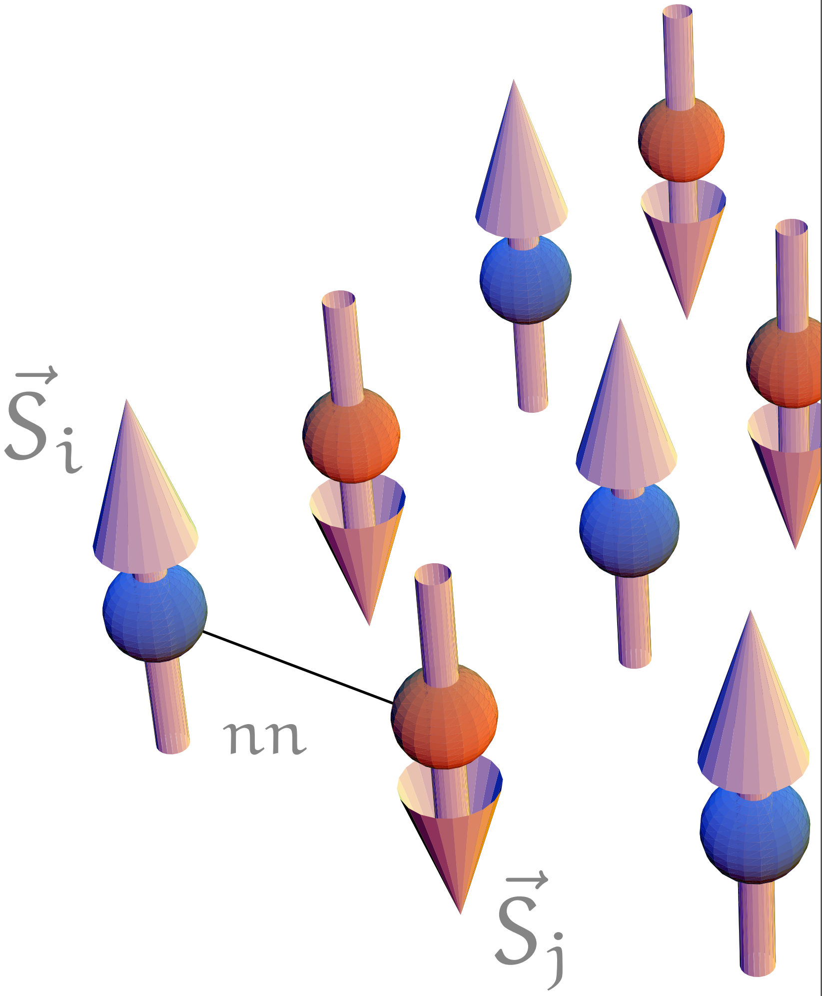


Heisenberg Model

- The “simplest” model of magnetism is the Heisenberg model

$$H = J \sum_{nn} \vec{S}_i \cdot \vec{S}_j$$

  - If $J < 0$ neighboring moments align ... ferromagnet
  - If $J > 0$ neighboring moments anti-align ... anti-ferromagnetism
- For a two-dimensional system, the latter case has not been solved



Spin liquids

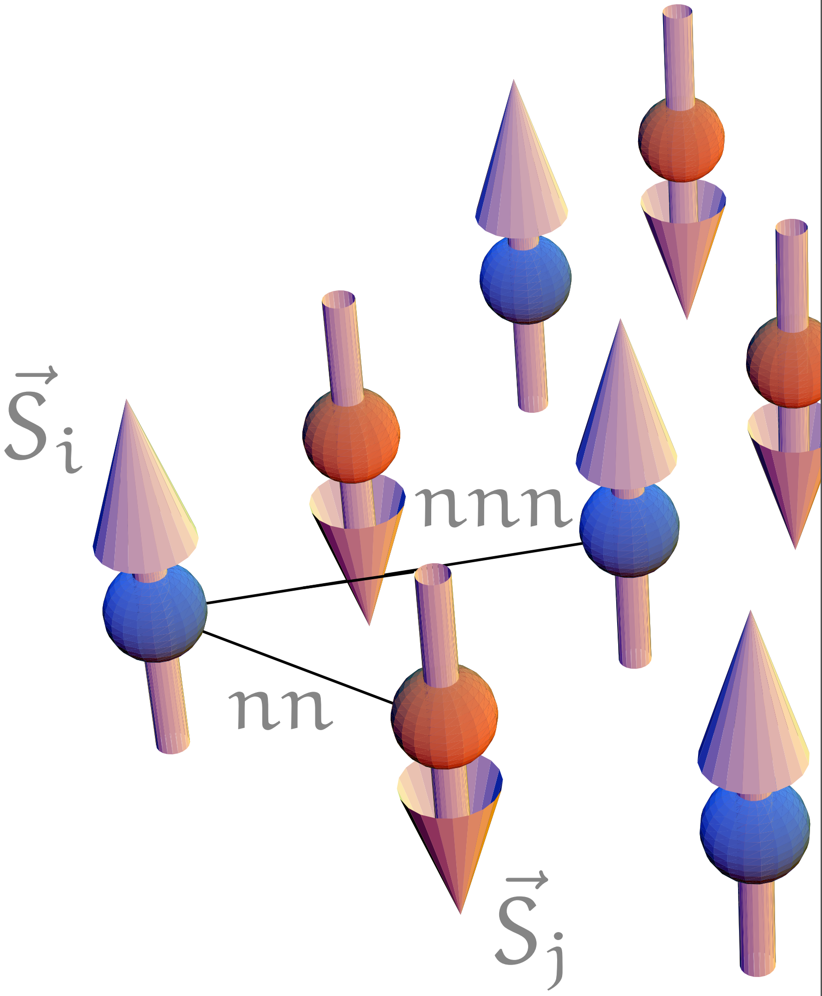
- Starting with an anti-ferromagnetic state, add frustrating interactions

$$H = J_1 \sum_{nn} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{nnn} \vec{S}_i \cdot \vec{S}_j$$

or more generally

$$H = \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

- Correlations between spins remain anti-ferromagnetic but become extremely short-ranged[†]



[†] E. Kapit, P. Luitel, and D. F. Schroeter, Phys Rev B **73** (7), p. 75310 (2006).

INCLUDE CITATIONS HERE

An exact solution for the spin liquid

with anyons

The idea ... work backwards

If you were interested in wave functions that looked like this

$$\psi(x) = \left(\frac{m\,\omega_0}{\pi\,\hbar}\right)^{1/4} \exp\left[-\frac{m\,\omega_0}{2\,\hbar}\,x^2\right]$$

and wanted to know what the potential was for which it was a stationary state, you could do the following

$$-\frac{\hbar^2}{2\,m}\,\frac{d^2\psi}{dx^2} + V(x)\;\psi = E\,\psi$$

$$V(x) = E + \frac{\hbar^2}{2\,m}\,\frac{1}{\psi}\,\frac{d^2\psi}{dx^2} = \left(E - \frac{\hbar\,\omega_0}{2}\right) + \frac{1}{2}\,m\,\omega_0^2\,x^2$$

and you would have determined the model that gave the behavior you were interested in.

Exactly-solvable model

- Start with a wave function that is known to support anyons, the quantum Hall wave function.

$$\psi_{CSL}\underbrace{(z_1 \cdots z_{N/2})}_{\text{locations of up spins on lattice}} = \psi_{FQH}\underbrace{(z_1 \cdots z_N)}_{\text{locations of electrons in FQH state}}$$

- Obtain the coefficients J_{ij} such that the chiral spin liquid is the exact ground state[†]

$$H = \sum_{i<j} J_{ij}\, \vec{S}_i \cdot \vec{S}_j + \ldots$$

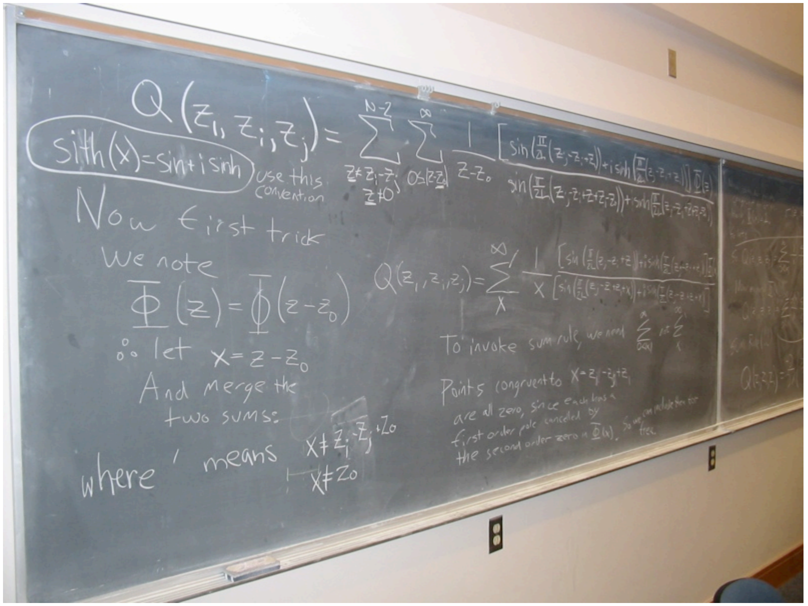
for any number of particles N.

[†] D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, accepted to Phys. Rev. Lett.

Verification

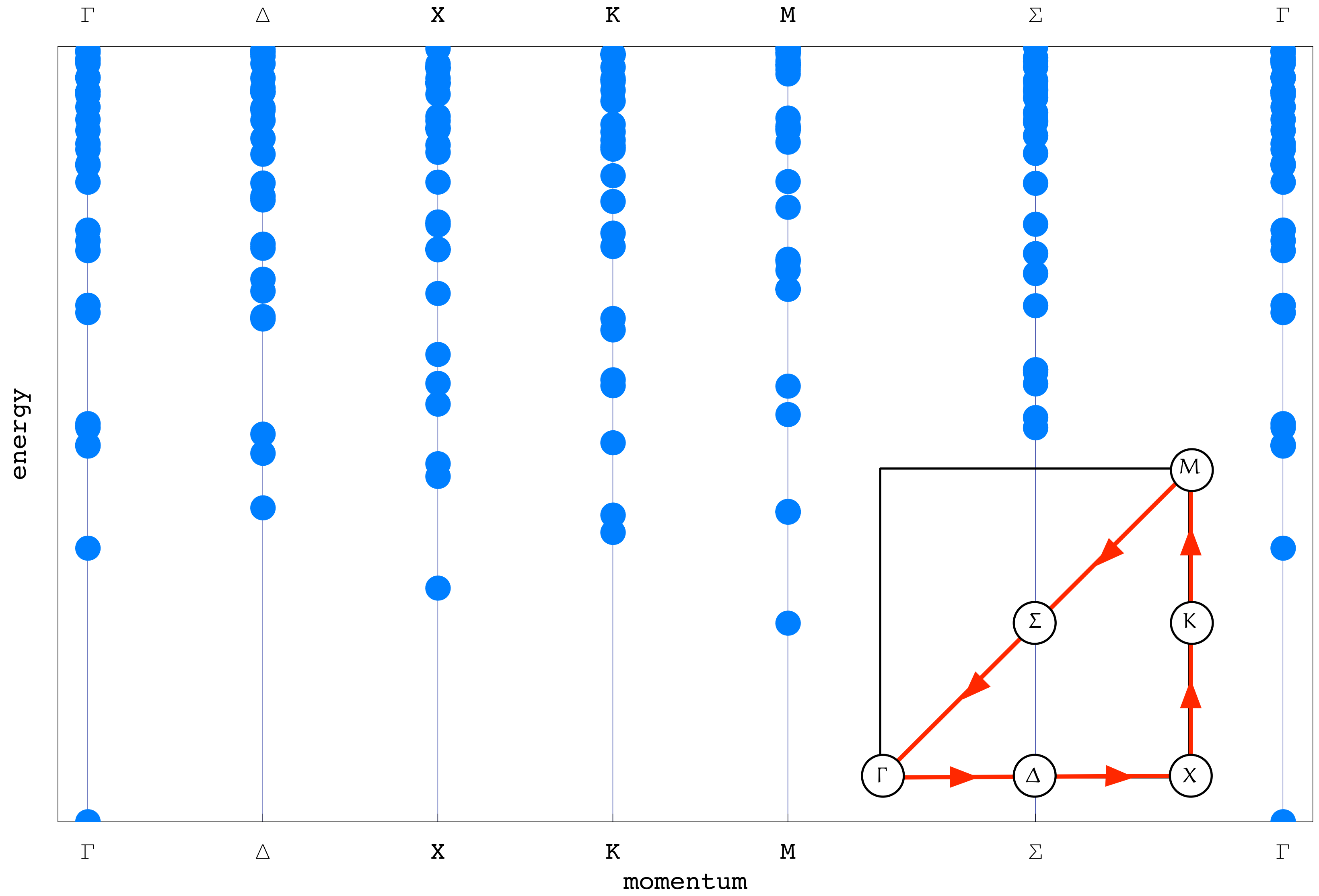
- This calculation was previously attempted by Laughlin[1] and later shown to be incorrect.[2]

- Laughlin is a very smart man.
  - How do we know it is correct this time?
  - Numerics. The model is exact for any N and we have now computed all 65536 states on a lattice of $\mathrm{N} = 16$ sites.[3]

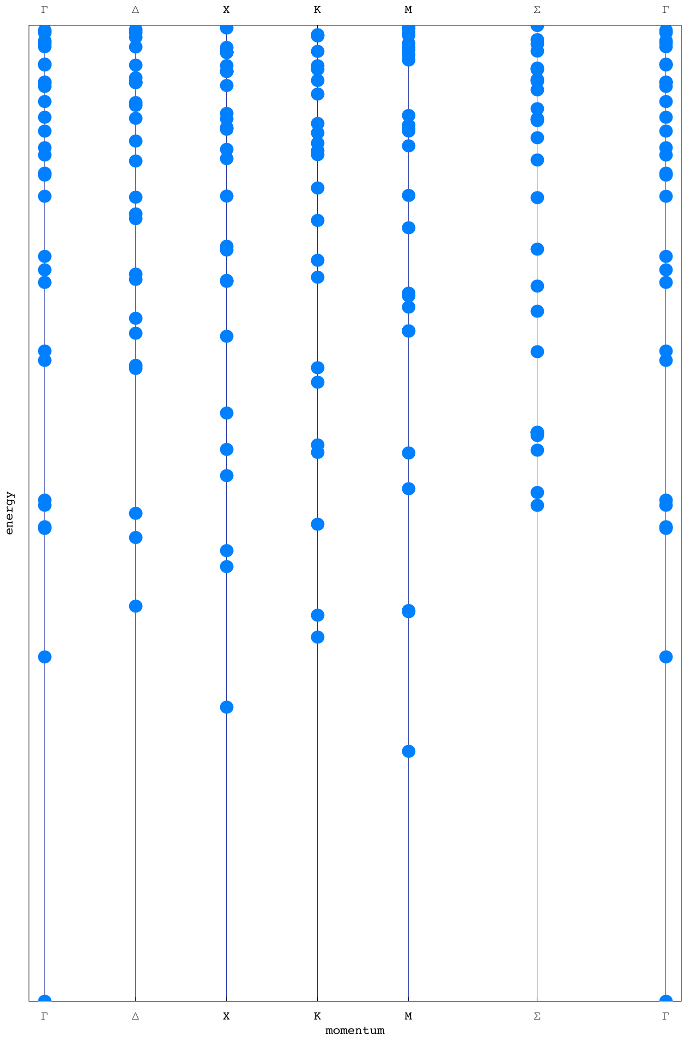


[1] R. B. Laughlin, Ann Phys **191**, p. 163.
[2] D. F. Schroeter, Ann Phys **310**, p. 155.
[3] R. Thomale, D. F. Schroeter, and M. Greiter, to be submitted to Phys. Rev. E.

Energy spectrum of the chiral spin liquid



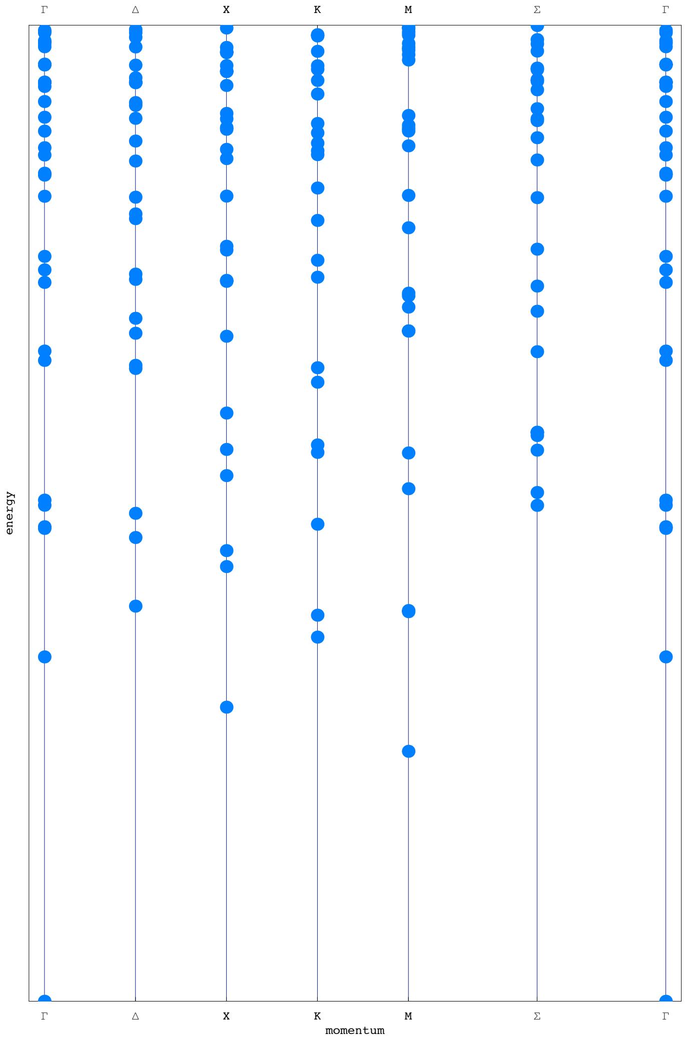
Where are the anyons?



Where are the anyons?

It can be shown[†] that

- if the spectrum of a system has an energy gap E_g between the ground state and all excitations



[†] Chetan Nayak, Quantum Condensed Matter Physics, §19.3

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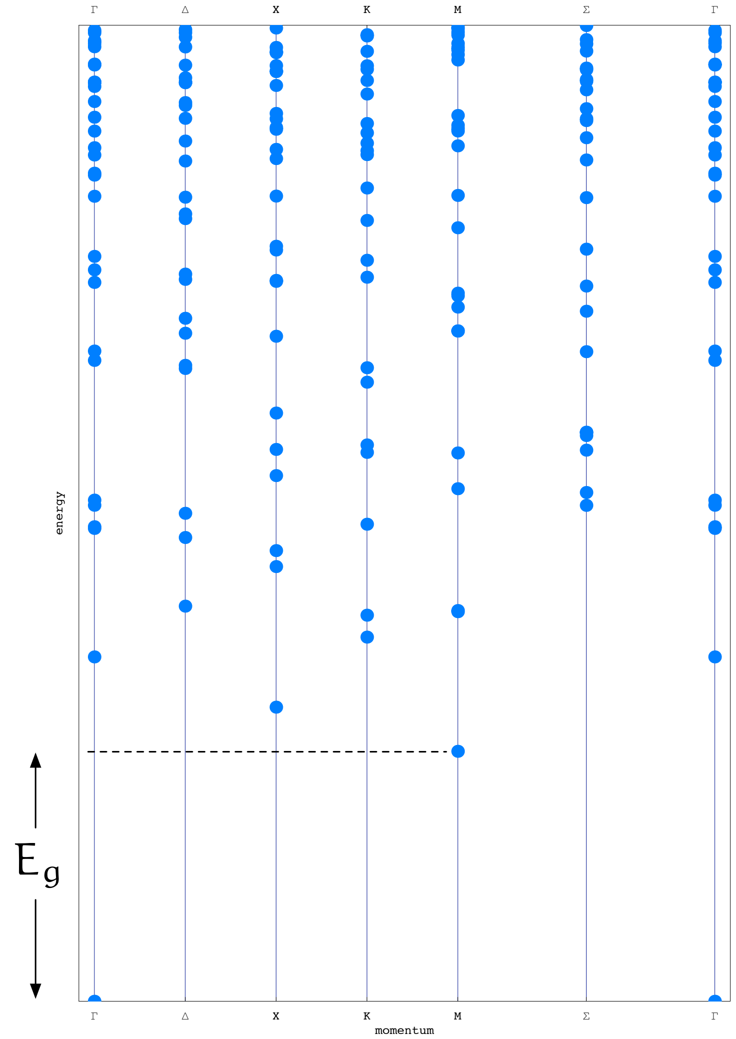

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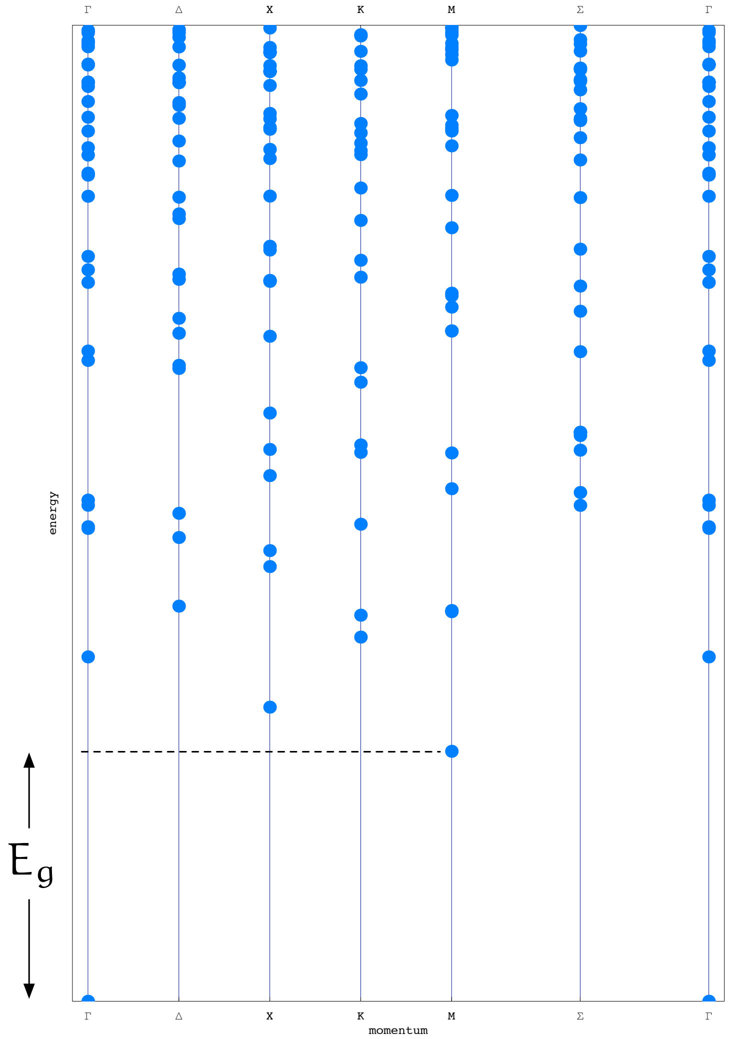
- if the spectrum of a system has an energy gap E_g between the ground state and all excitations
- and the system has a ground state degeneracy of m



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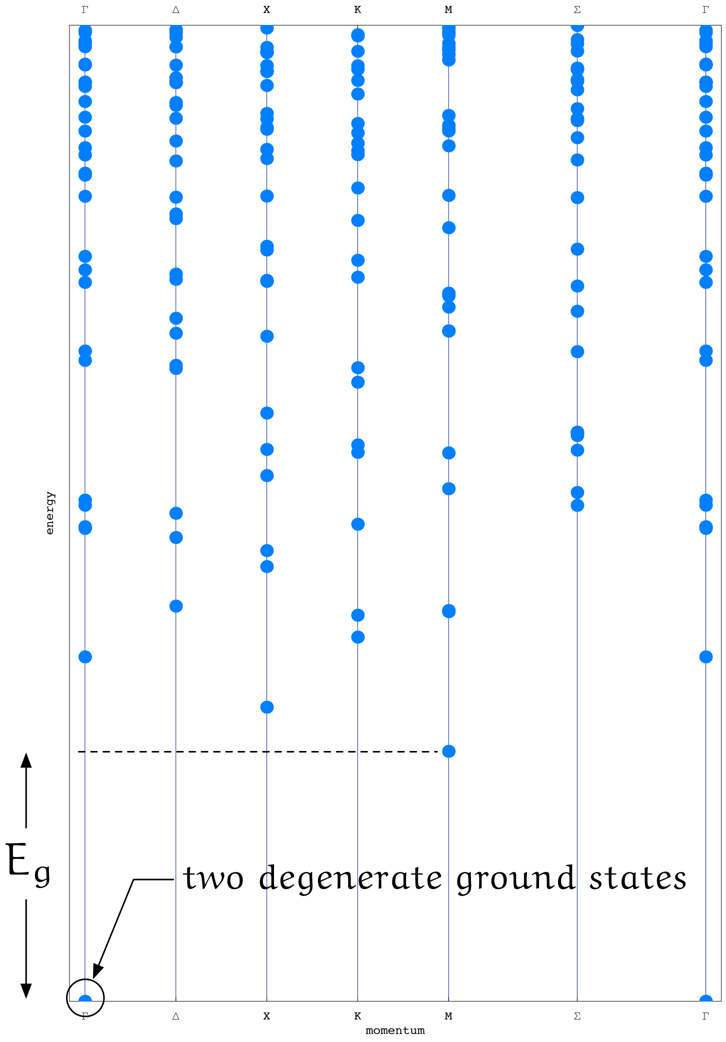

two degenerate ground states

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Where are the anyons?

It can be shown[†] that

- if the spectrum of a system has an energy gap E_g between the ground state and all excitations
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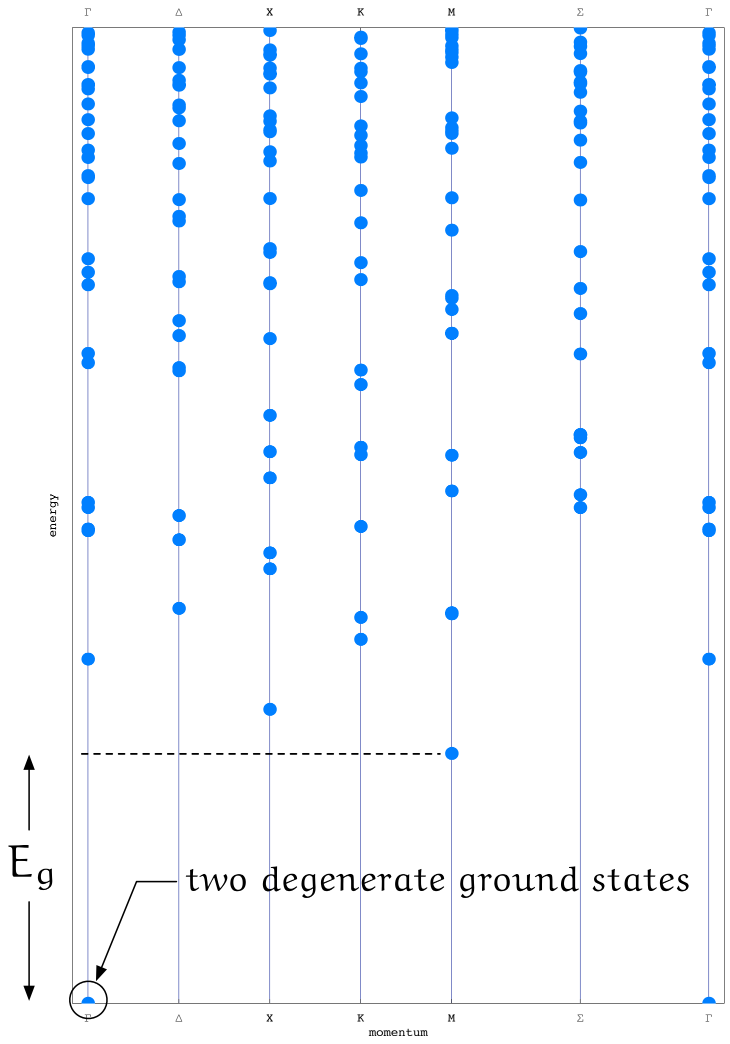
- then the degeneracy is explained if the excitations have fractional statistics

$$\theta = \pi/m$$



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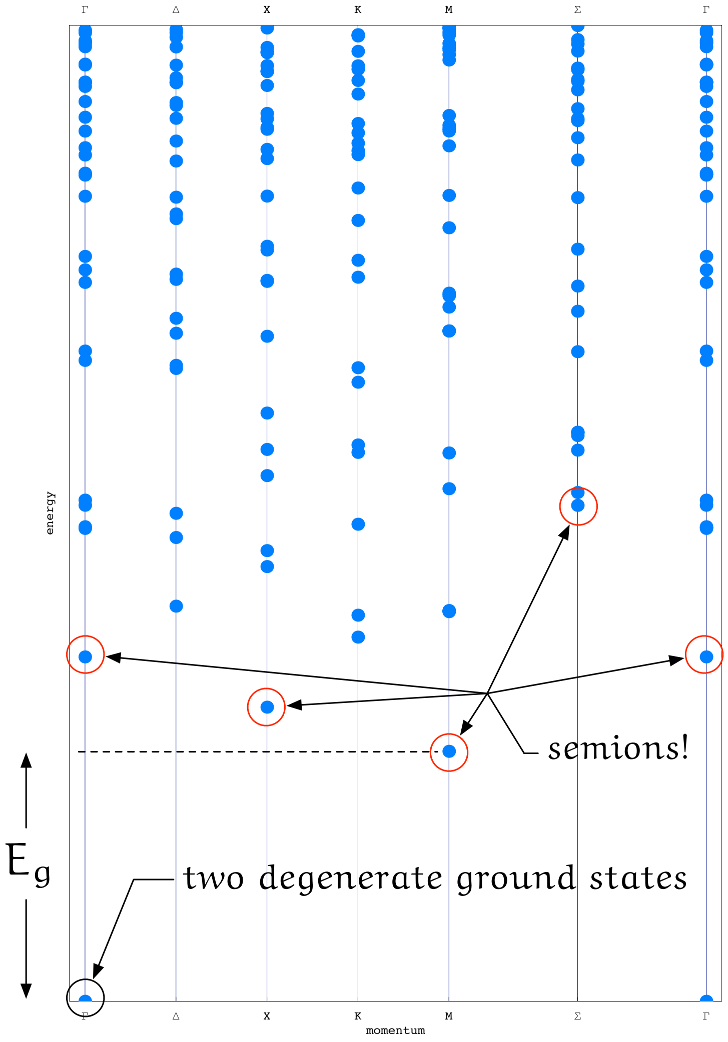

So we expect half-fermions (semions) with $\theta = \pi/2$!

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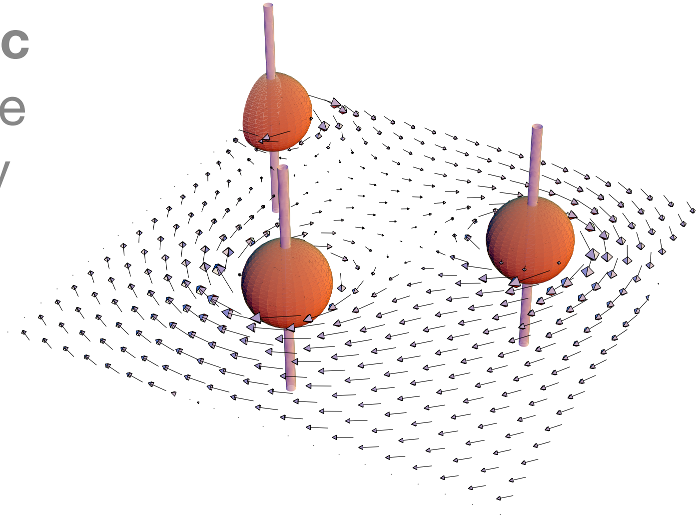
†Chetan Nayak, Quantum Condensed Matter Physics, §19.3

Conclusions

- The spin liquid presented here describes a new state of matter that has fractional excitations.

- This is the only exact model for such a state.

- For the first time, we have the opportunity to study **how real anyons behave in a frustrated magnetic material**. In addition to their statistics, do they have interactions mediated by the liquid from which they arise and through which they travel?



  - What will this study tell us about the potential of anyon superconductivity, of using anyons to do quantum computation?

Future Work

- Direct demonstration that the excitations are anyons. Obtain the wave functions for the excitations and adiabatically transport two particles around each other.
- Interactions between anyons. Study the two-anyon wave functions to determine the effective interaction between these particles.
- Explore the full class of models. The proof generates an entire family of model systems with the CSL ground state. Only the simplest was shown here.
  - What are their common features?
  - Are any of them fully integrable?