Hunting quantum butterflies:
The quantum-classical transition for chaotic systems

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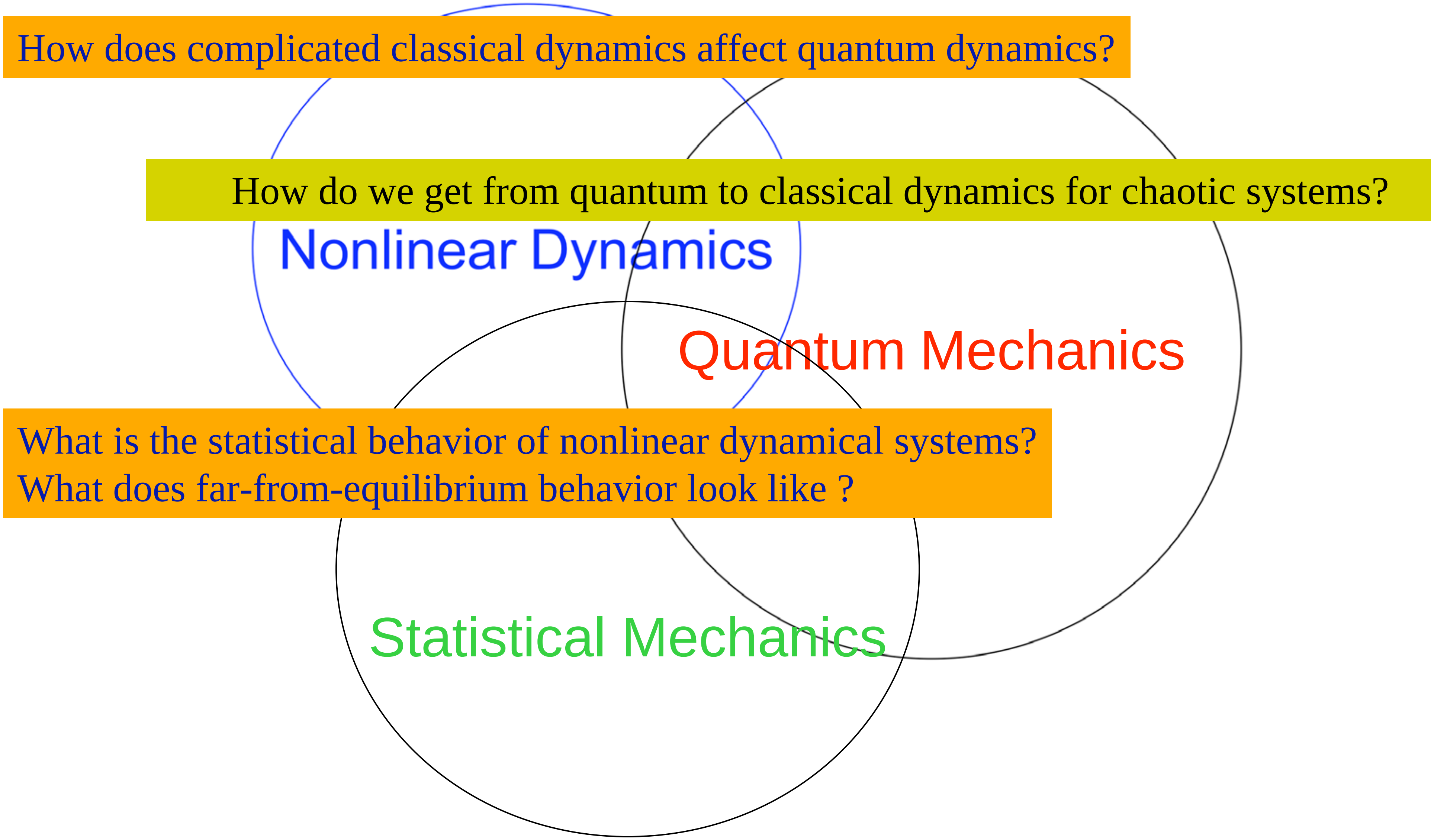
Dynamics: This chancy, chancy, chancy world



Today

- **Reminders**
  - Quantum-classical difference
  - Chaos

- **Quantum trajectories**
  - Recovering classical chaos -- “standard result”

  - Chaos induced by quantum effects
    (**non-monotonic or anomalous**)

- **Stop me to ask questions!**

Quantum non-classical behavior

Quantum mechanics: matter exists as wave-functions

$$\psi(x) = \psi_1(x) + \psi_2(x)$$

Use wave-functions to construct probabilities

$$P(x) = P_1(x) + P_2(x) + \text{interference between } \psi_1 \text{ and } \psi_2$$

- Interference, Tunneling
- Entanglement: Bizzare, disconcerting, non-local
- Not chaotic: Linearity of Schrödinger's equation+ discrete spectrum

Quantum-classical transition

- The universe is fundamentally quantal
- Classical reality must emerge from quantum properties
- Transitions to and from quantum properties are being probed regularly now (cold atoms/nanomech for example).

- The transition is NOT simple at all
- $\hbar$ → 0 limit of quantum mechanics, but it is non-trivial! [$\hbar$ divided by characteristic action of system → 0]

Need to consider the effect of the environment (all other systems)

- How does Schrödinger → Newton (or Hamilton)?
- How does Hilbert space → Phase-space?

"Space is big. You just won't believe how vastly, hugely, mind-bogglingly big it is." – Douglas Adams, Hitchhiker's guide to the galaxy

- Phase-space = (Number of particles) X (2 X space)
- Hilbert space = $\left(\frac{Phase-space}{\hbar}\right)^{(Number-of-particles)}$

"**Hilbert** Space is **big**$^{\mathbf{N}}$. You just **(won't believe)**$^{\mathbf{N}}$ how vastly, hugely, mind-bogglingly big it is."

Understanding the transition is critical to understanding '**quantum control**', '**quantum information**', '**quantum computing**', '**quantum engineering**' and quantum mechanics itself.

Chaos

Some useful things to know about chaos

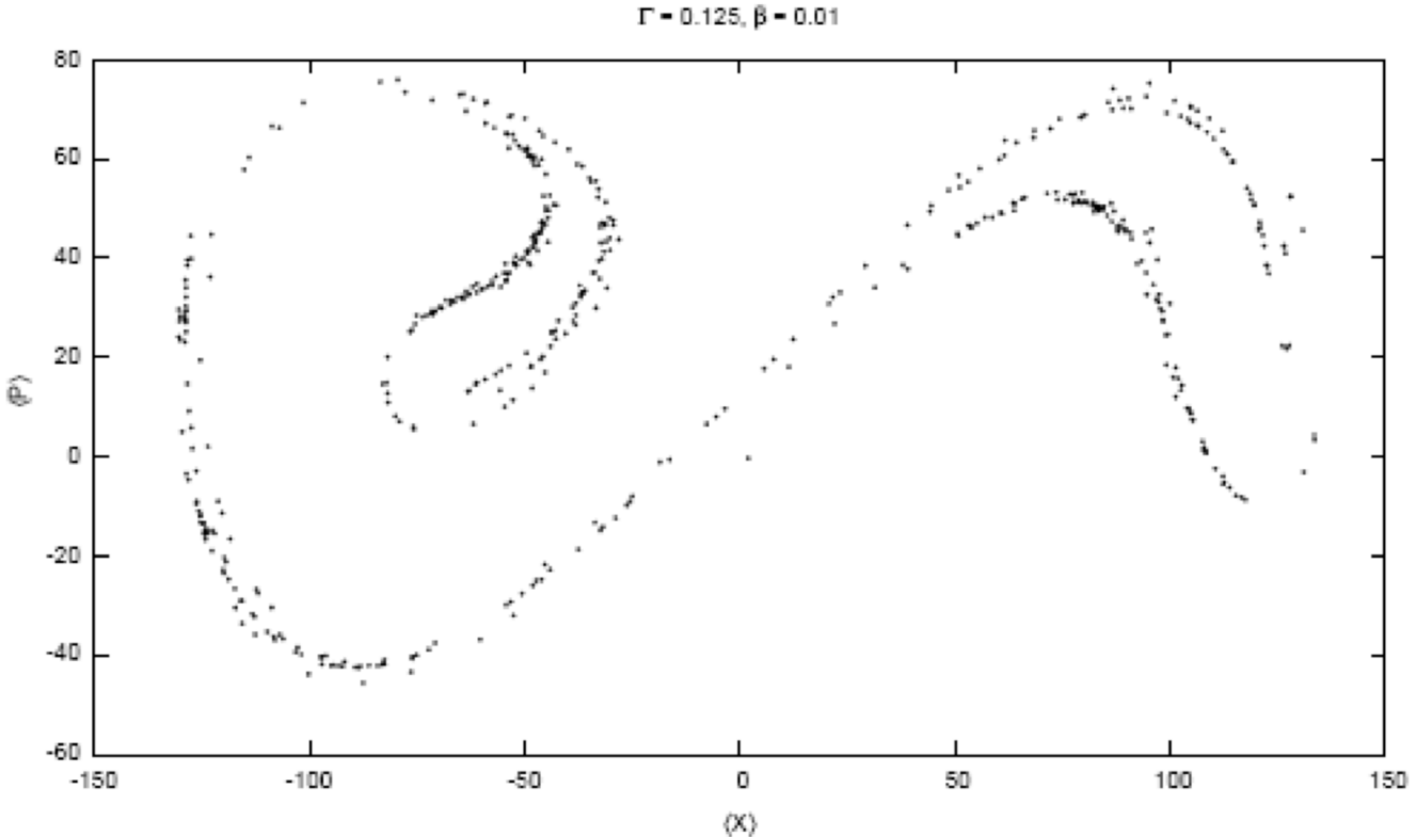
- Chaos = Exponentially rapid divergence of trajectories in phase-space, rate given by λ = Lyapunov exponent

$$d_1 = d_0 \exp(\lambda t)$$

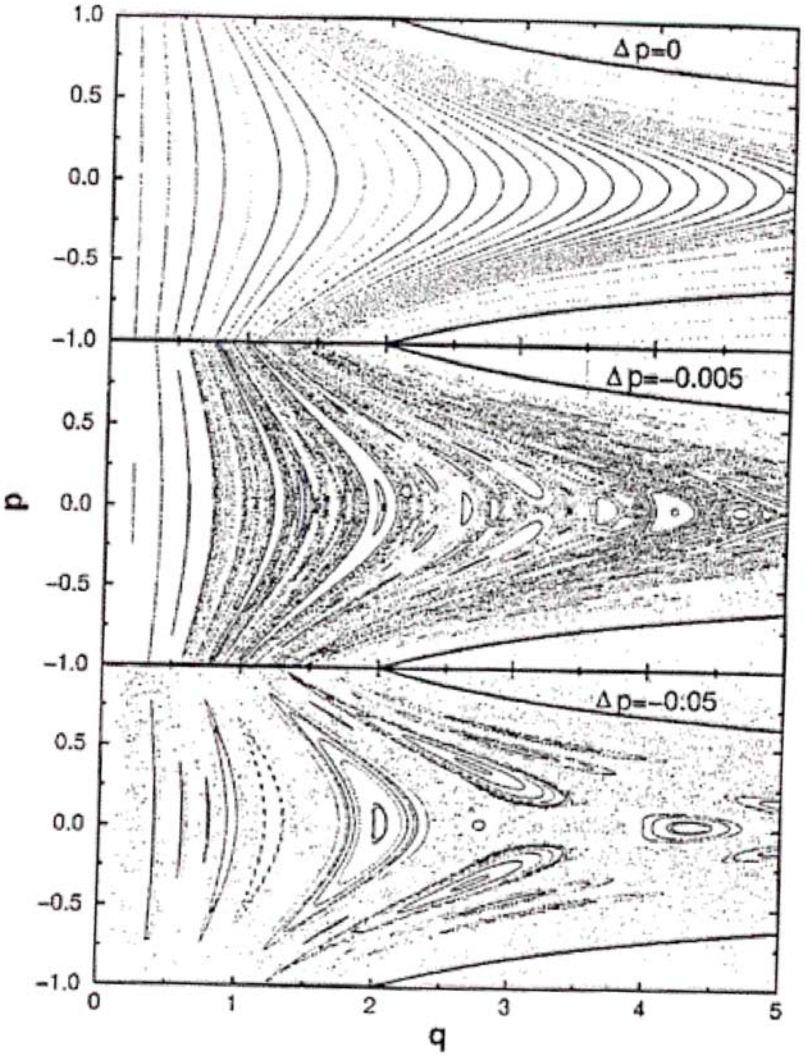
- A generic nonlinear dynamical system is chaotic

- **Lorenz butterfly**

Strange attractor in strobe plot of damped driven double-well (Duffing) system



Lorenz butterfly



Chaos in strobe plots of kicked Hydrogen atom

Models and comparisons

- Wave-packets not easily compared to classical trajectory, but can compare with the expectation values <Q> and <P>

  (center of wavepacket)
- Could also compare probabilities to probabilities

- Classically: particles evolving in phase-space + noise (environment)
- Quantum mechanically, wavefunctions + noise (environment)

  (Quantum stochastic dynamics = QSD)

Open quantum system formalism: Lindblad operators

$$d|\psi\rangle = \frac{-i}{\hbar} H |\psi\rangle dt$$

$$+ \left(\langle L^{+} \rangle L - \frac{1}{2} L^{+} L - \frac{1}{2} \langle L^{+} \rangle \langle L \rangle \right) |\psi\rangle dt$$

$$+ (L - \langle L \rangle) |\psi\rangle d\xi$$

Schrödinger dynamics

Environment

Quantum State diffusion/ QSD: Stochastic Schrodinger Equation

$\hat{L}$ is Lindblad operator modeling interaction with environment

Environment = all other systems in the world, that we typically ignore or are trying to isolate our system from

Example: Duffing problem

$$\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \beta^2 x^3 - x = \frac{g}{\beta}\cos(\Omega t)$$

- Classical damped driven double-well, with chaotic attractor: dynamics unchanged, but size changes with β
- Quantized version:

$$\hat{H}_\beta = \hat{H}_D + \hat{H}_R + \hat{H}_{ex}$$

$$\hat{H}_{ext} = -\frac{g}{\beta} Q Cos(\Omega t)$$

$$\hat{H}_D = \frac{1}{2}\hat{P}^2 + \frac{\beta^2}{4}\hat{Q}^4 - \frac{1}{2}\hat{Q}^2$$

$$\hat{H}_R = \frac{\Gamma}{2}(\hat{Q}\hat{P} + \hat{Q}\hat{P})$$

$$\hat{L} = \sqrt{\Gamma}(\hat{Q} + i\hat{P})$$

Γ is 'damping' or friction parameter
$\beta^2 \sim \hbar$, degree of 'quantumness'

- Classical equations obtain for x=<Q>,p= <P> from those equations as β to 0

- But what are the details of the transition from classical mechanics to quantum mechanics?

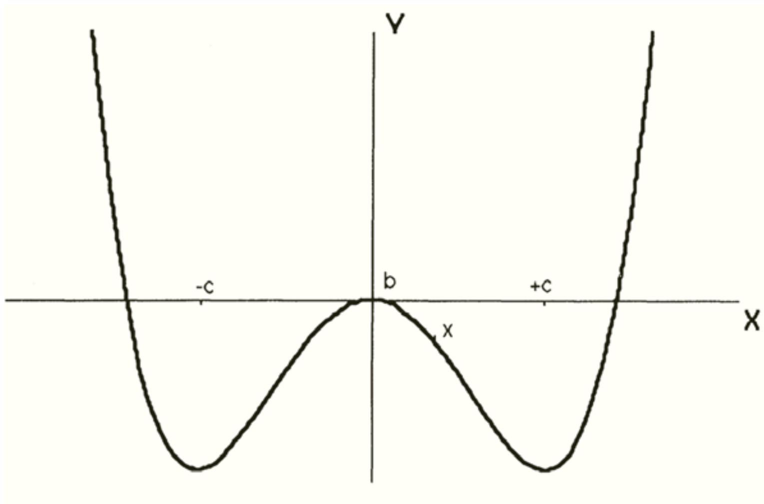
- Two ‘knobs’ for quantum problem: Γ and β

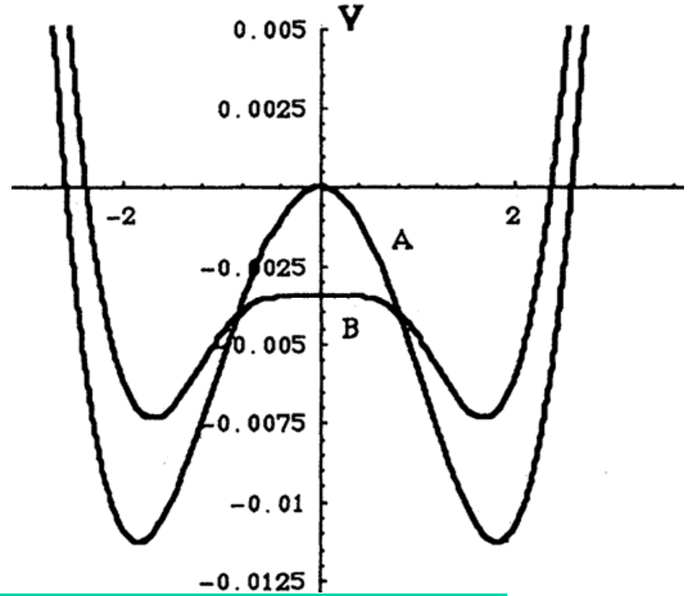
- Γ increases damping, and β increases ‘quantum-ness’

- Remember, classical mechanics is unchanged by β

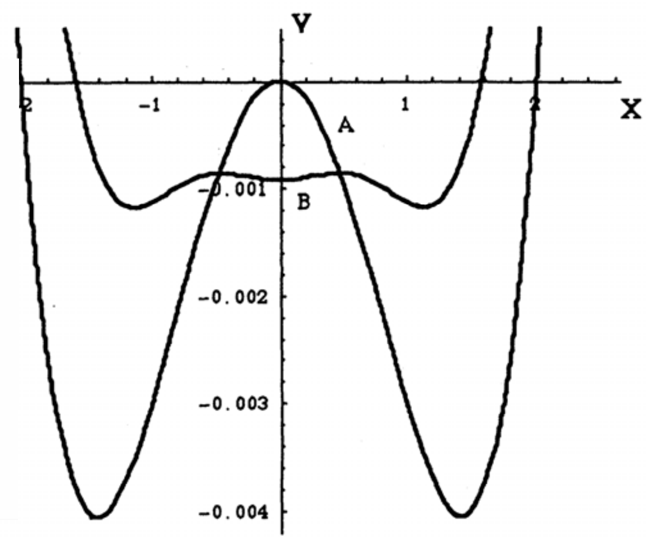
- Results shown for
  - Γ =0.125, β = 0.01, 0.3, 1
  - Γ =0.3, β = 0.01, 0.3, 1

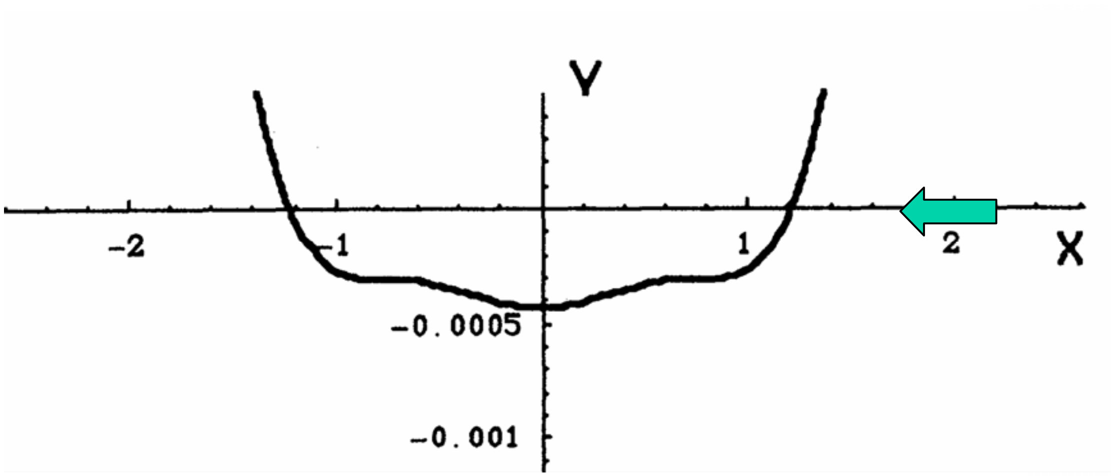
QM approximated by effective potential, shown as β increases

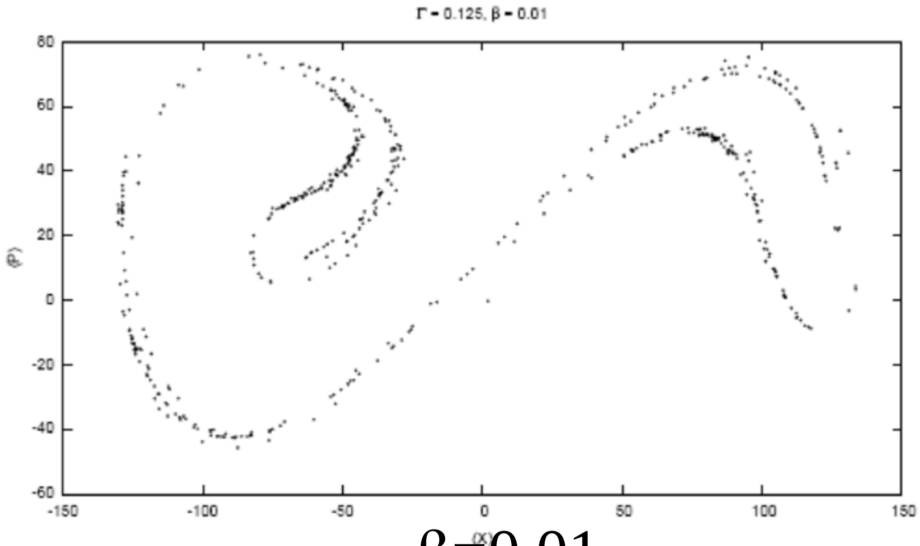




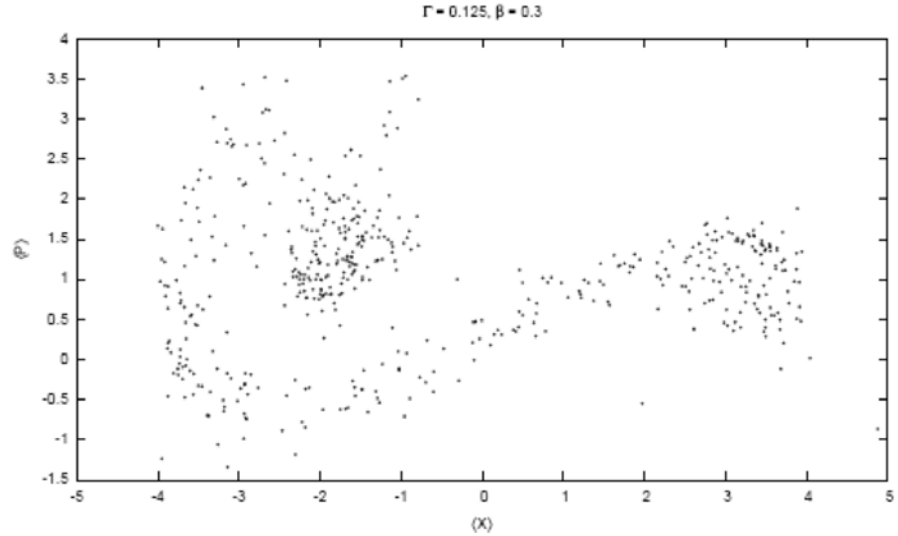
Roughly, quantum effects: Tunneling + zero-point energy



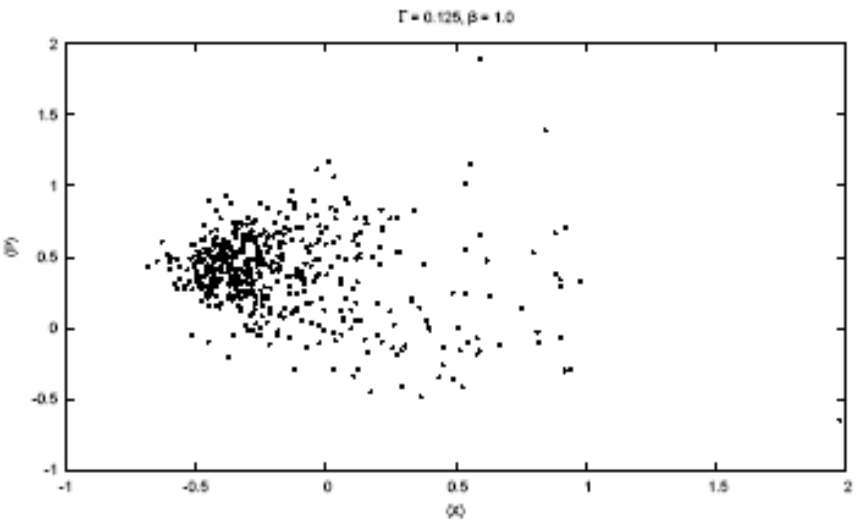


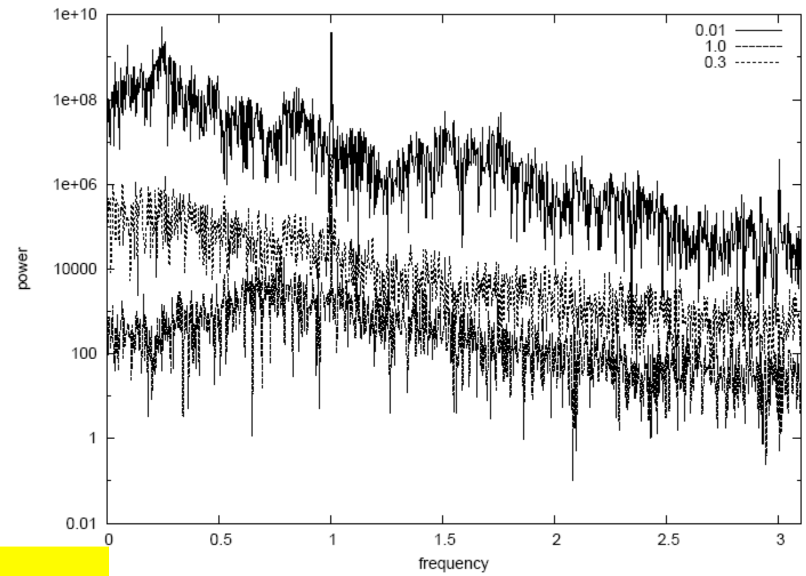


β=0.01



β=0.3





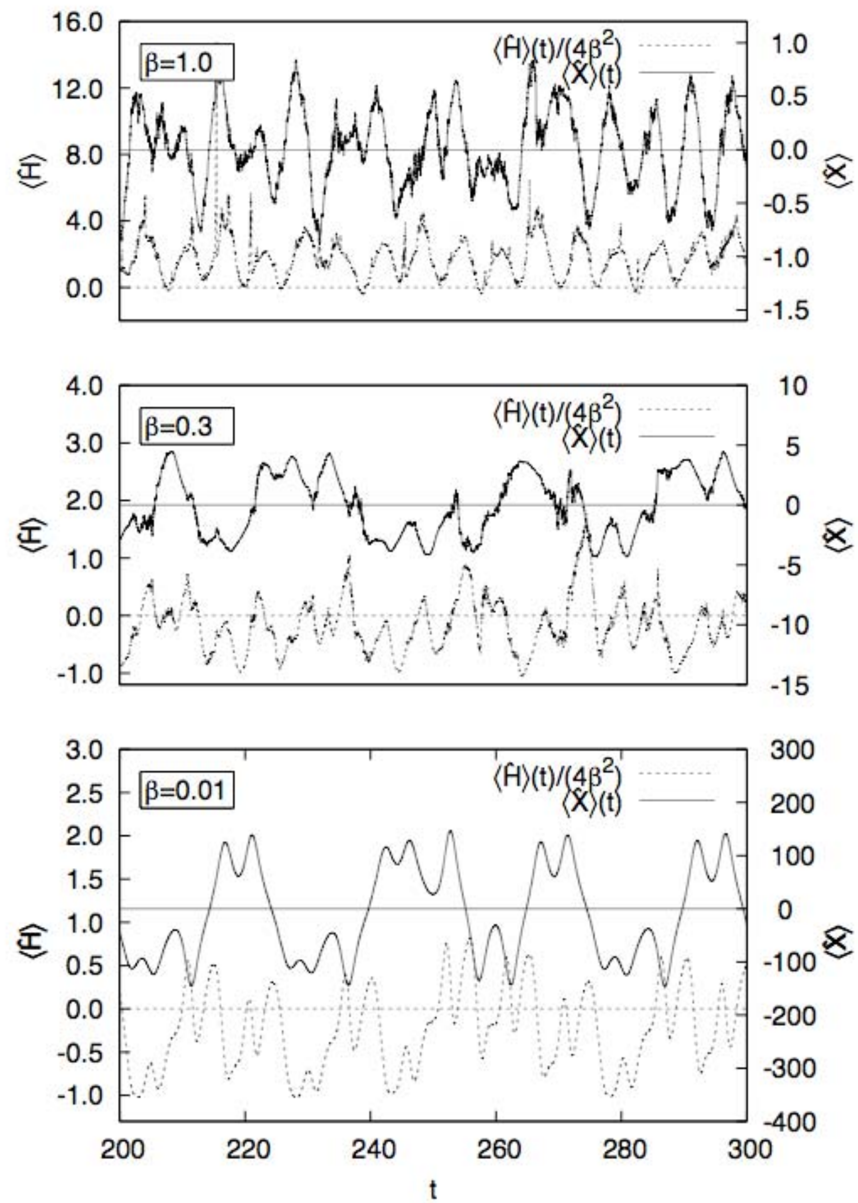
β=1.0

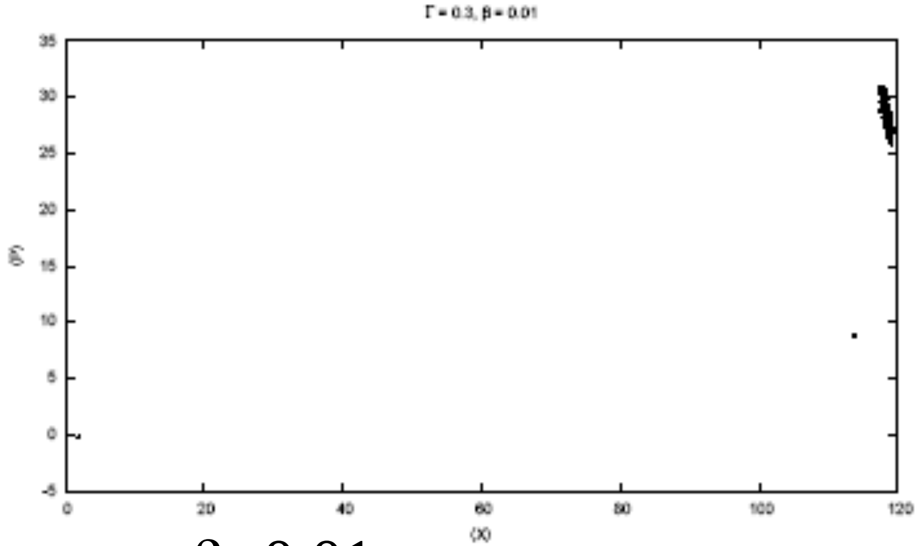
Poincare sections
Γ=0.125

Power spectrum

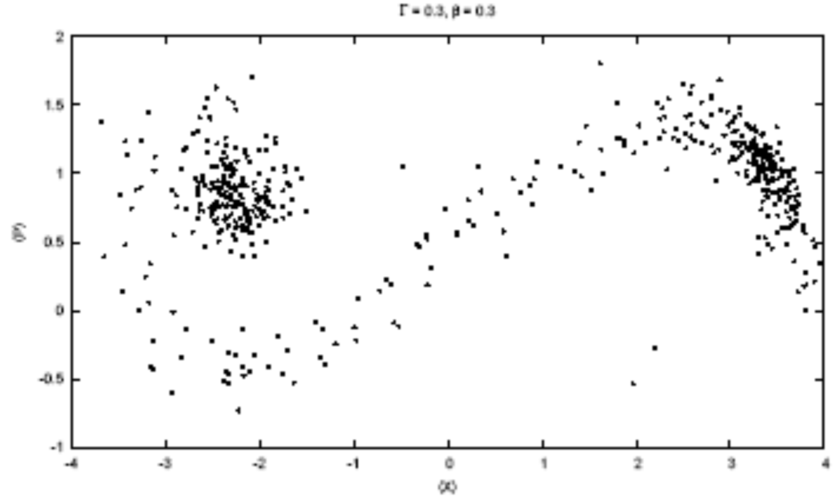
Energy and position dynamics
Γ=0.125

Crossing over <X> =0 with negative energy indicates tunneling

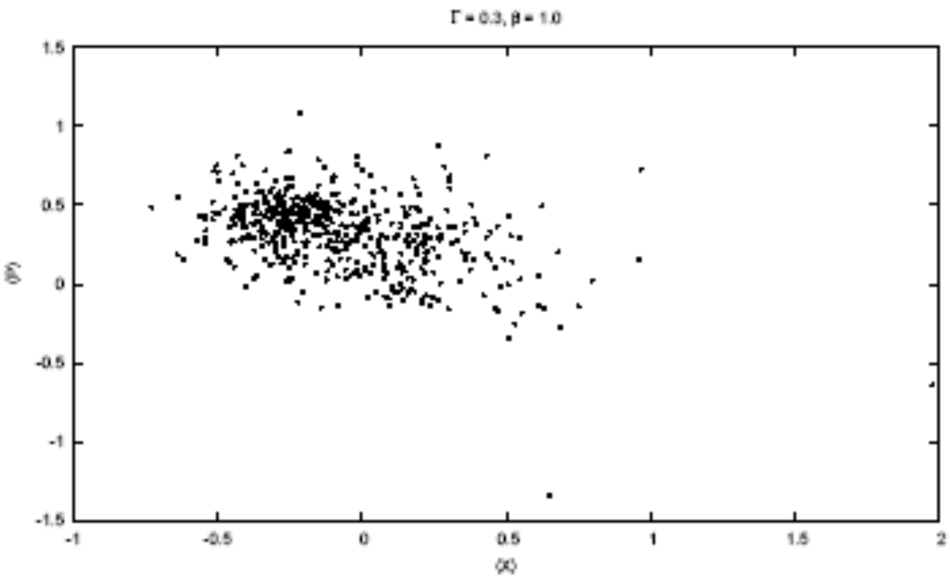




β=0.01



β=0.3



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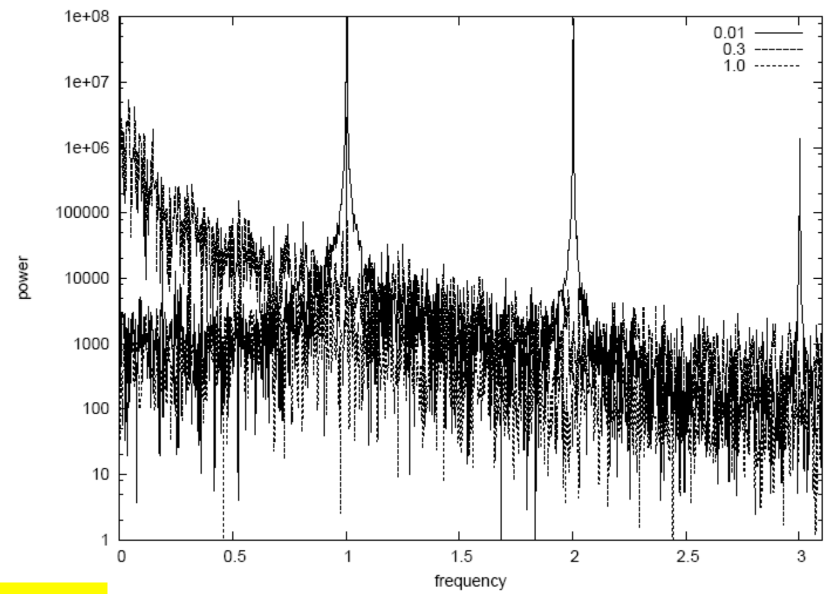
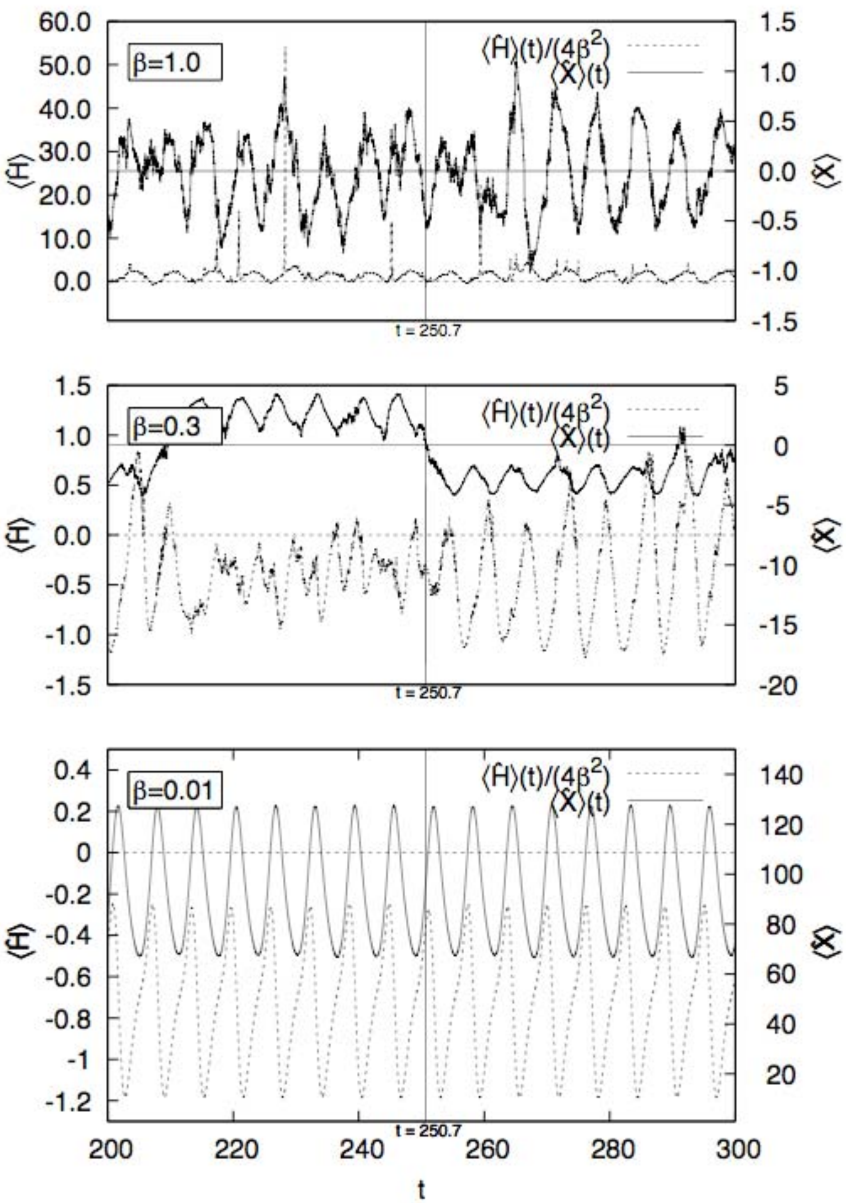


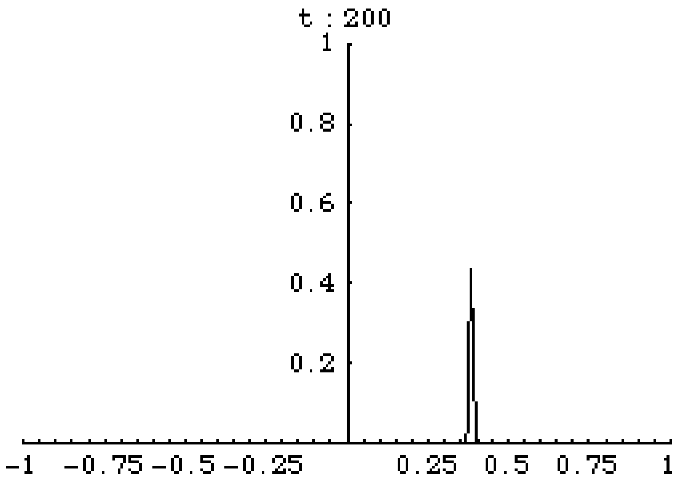
Poincare sections, Γ=0.3

Power spectrum

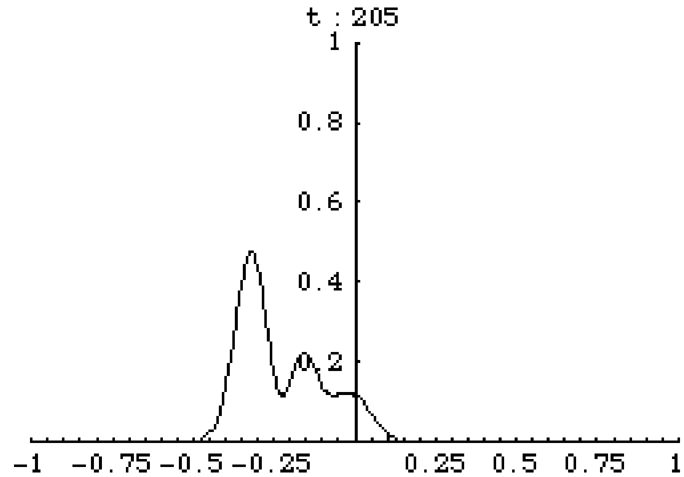
Energy and position dynamics
Γ=0.3

Crossing over <X> =0 with negative energy indicates tunneling

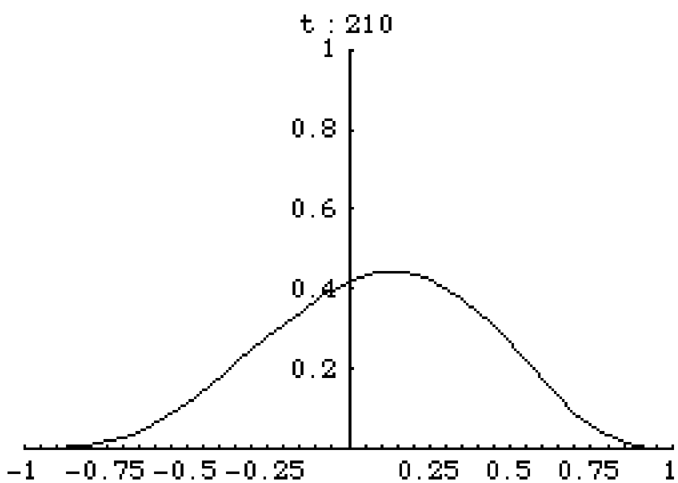




β=0.01



β=0.3



β=1.0

Barrier crossing at different β values

Summary

- Classically chaotic dynamics can be recovered for open quantum systems
- There is no chaos in deep quantum limit even here
- Quantum chaos different from classical limit exists
- Quantum chaos away from classical limit involves pure quantum effects such as tunneling
- Quantum chaos away from classical limit exists in the absence of classical chaos (quantum-induced chaos)
- The quantum-classical transition is non-monotonic

  - **A. Kapulkin and AKP Phys. Rev. (2007) submitted**
  - **C. Amey, A. Steege, A. Kapulkin, A. Pattanayak (in preparation)**

  - A. Gammal and AKP Phys Rev E 75, 036221 (2007)
  - P. Sripakdeevong, A. Gammal, AKP (in preparation)

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