The Energy Interpretation of Interacting Static Black Holes

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The Energy Interpretation of Interacting Static Black Holes

Outline:

- BHs in extra dimensions
- BHs with electric charge

Black Holes in Extra Dimensions

Based on:

Scott Fraser, Doug Eardley

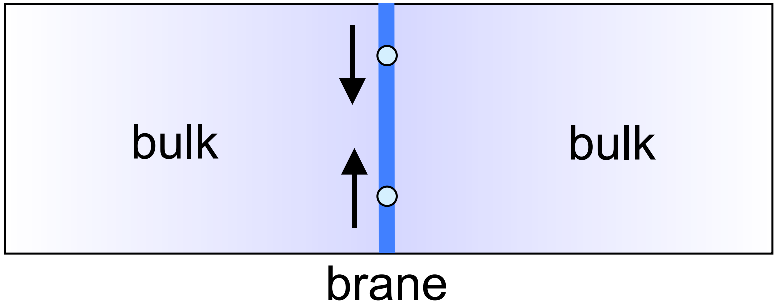
arXiv:1408.4425 [gr-qc]

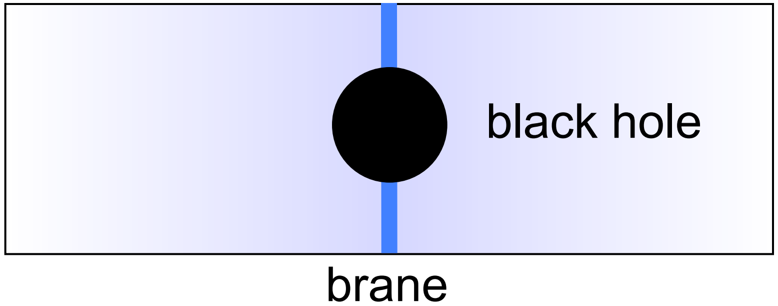
arXiv:1409.0884 [gr-qc]

(to appear in Phys. Rev. D)

Introduction: Extra Dimensions

- Our observed universe has 4 dimensions (3 spatial, 1 time).
- Use general relativity with **extra dimensions** (total $D > 4$).
- **Model**: observed universe = 4-dimensional surface (**brane**).
- Only gravity (including BHs) extends into the bulk ($D > 4$).
- Gravity acts 4-dim. (large distance), D-dim. (small distance)





Introduction: Extra Dimensions

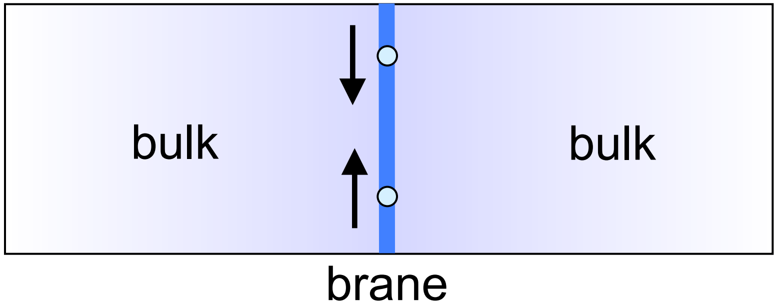
- Geometry of extra dimensions could explain the hierarchy:

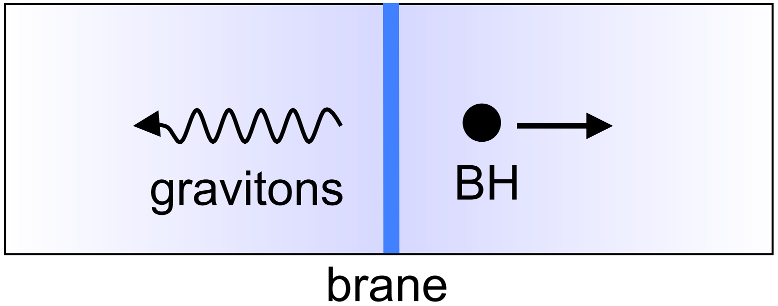
$$M_{4,\,\text{Planck}} / (\text{TeV}/c^2) \sim 10^{16} \qquad \text{if} \qquad M_{5,\,\text{Planck}} \sim \text{TeV}/c^2$$

(Henceforth: use units with $c = 1$, use mass = energy)

- If we live on a brane (explaining hierarchy): high-energy collisions on the brane would produce small black holes!

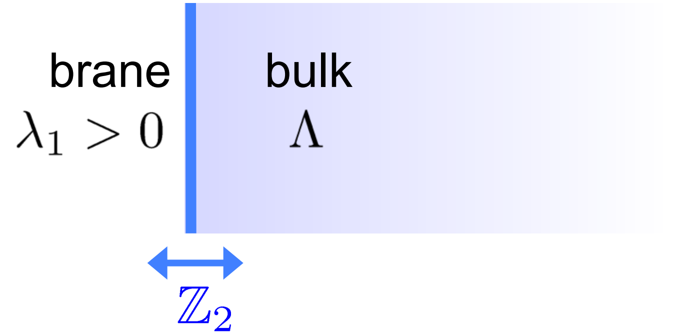
- Could the black hole fall into the bulk?





Randall-Sundrum Model (RS2)

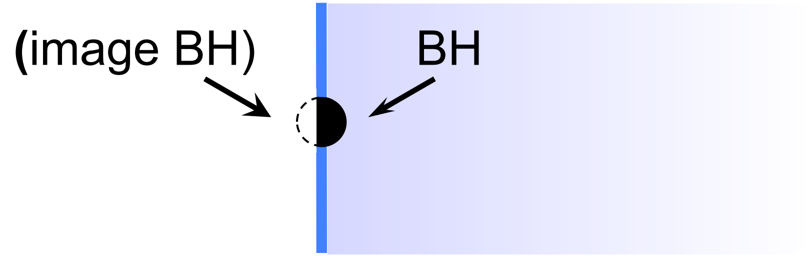
- **Brane: positive tension λ_1** (gravitationally repulsive)



- **Bulk: cosmological constant Λ**

- **Orbifold ($\mathbb{Z}_2$ mirror) symmetry:**

  - **Covering space**: **Identify** symmetric points across brane. All physics must be mirror-symmetric.

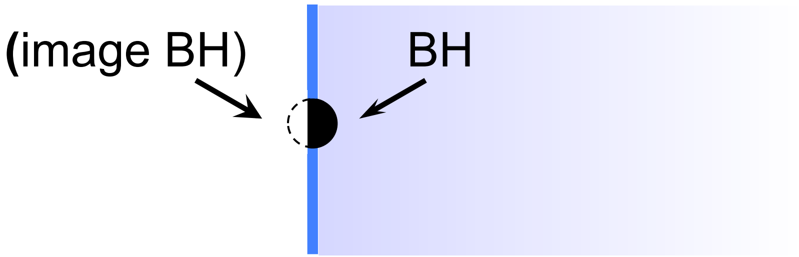


  - **Physical space**: on **one** side of the brane.

Small Static Black Holes in RS2

- Well-known (numerically): a static small BH on the **brane**.

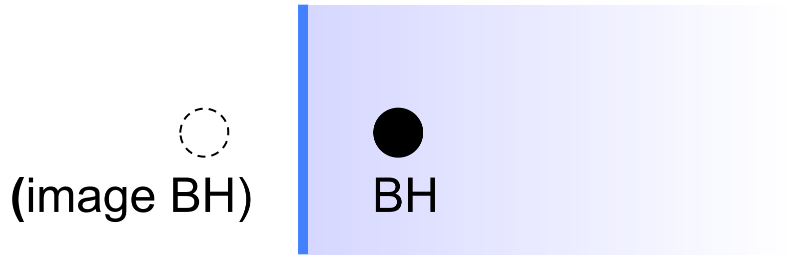
  [Kudoh et al, 2003]
  [Figueras and Wiseman, 2011]
  [Abdolrahimi et al, 2013]



- Is this BH stable?

- The BH experiences:
  - **repulsion** from brane (due to positive brane tension)
  - **attraction** to brane (due to orbifold image)

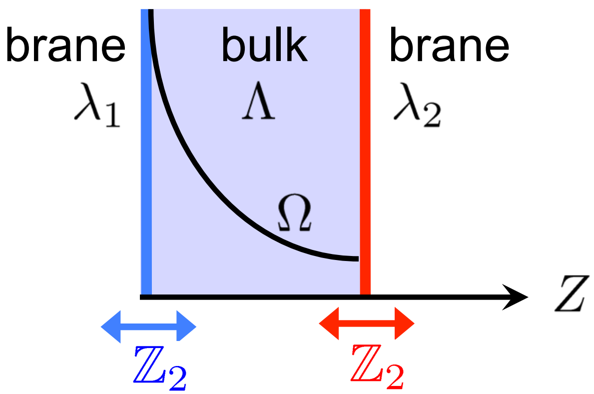
- Where these effects balance: expect a static small **bulk** BH



- Is this BH stable?

Randall-Sundrum Asymptotics

- **RS1** (two branes; can explain the hierarchy if $\Omega_1/\Omega_2 \sim 10^{16}$)



$$ds^2 = \Omega^2 \left(\eta_{ab} dx^a dx^b + dZ^2\right)$$

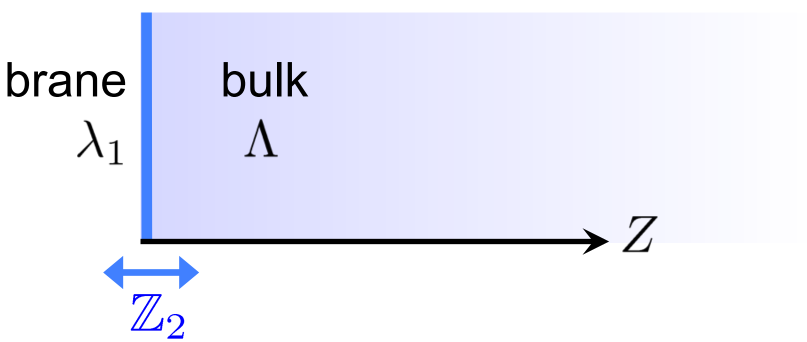
$$\Omega = \ell / Z$$

$$\lambda_1 = -\lambda_2 = \frac{2(D-2)}{8\pi G_D \ell}$$

$$P = -\frac{\Lambda}{8\pi G_D} = \frac{(D-1)(D-2)}{16\pi G_D \ell^2}$$

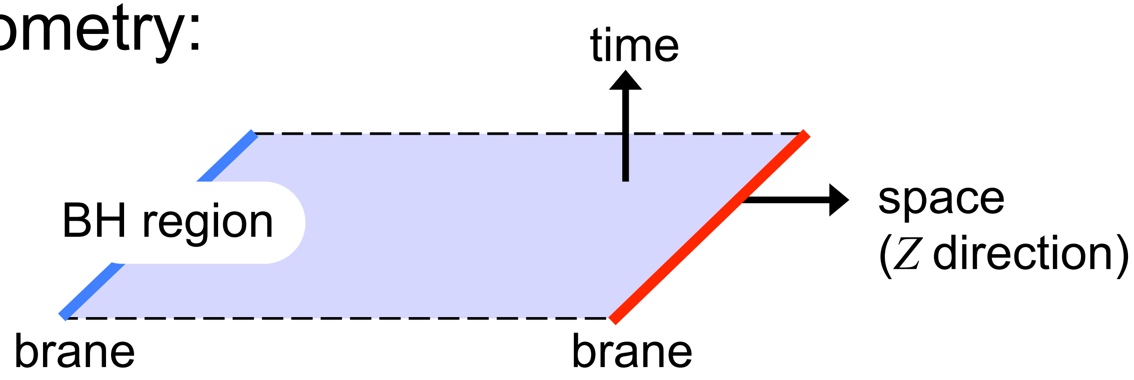
- Interbrane length: $L_b = \ell \ln(\Omega_1/\Omega_2)$

- **RS2** (single-brane limit; can't solve hierarchy problem)



A Static BH Extremizes Mass

- Spatial geometry:



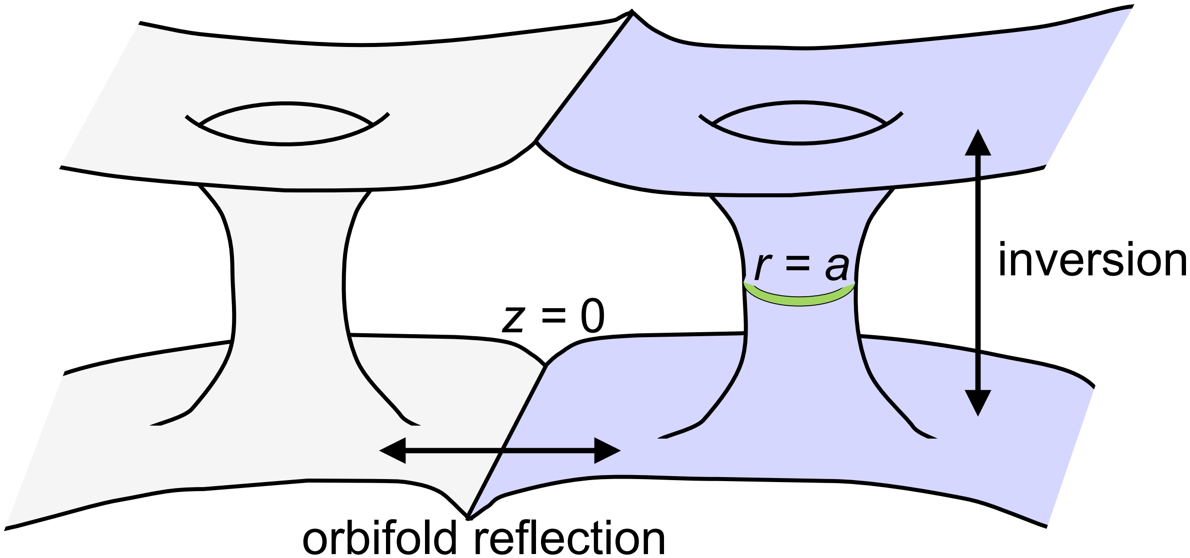
- Keep most parameters fixed: $(L_b\,,\,\ell\,,\,\Omega_1,\,\Omega_2)$
- A static BH obeys a First Law: $\delta M = \frac{\kappa}{8\pi G_5}\,\delta A$
- So: if a BH is static, it **extremizes** mass M, for fixed area A.
- Apparent horizon (AH): best approximation to event horizon.

- **We prove a Variational Principle** (for BH initially at rest): If mass M is **extremized** at fixed area A, the BH is **static**.

Geometry for BH (initially at rest)

Visualize spatial geometry as:

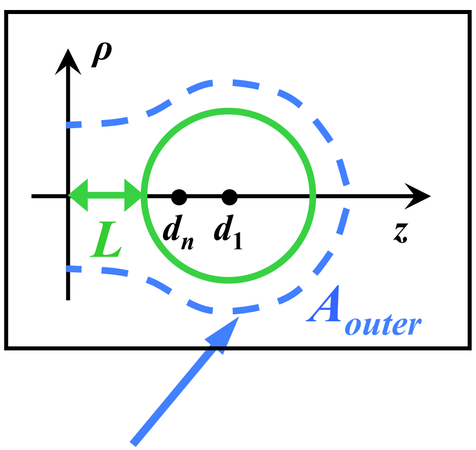
(if BH is **far** from brane)

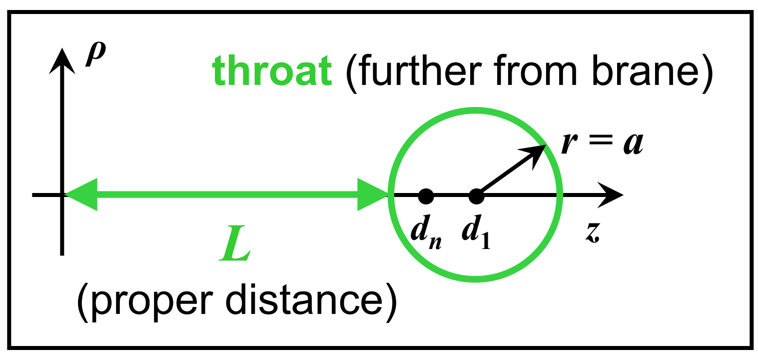


- Small black hole: $a \ll \ell$
- Trial geometry: $ds^2 = \psi^{4/(D-3)} d\mathbf{x}^2 \quad , \quad \mathbf{x} = (\vec{\rho}, z) \quad , \quad Z = \ell + z$
- Equation for geometry: $\nabla_{\mathrm{f}}^2 \psi = \frac{(D-1)(D-3)}{4\ell^2} \, \psi^{(D+1)/(D-3)}$
- At throat ($r = a$): $0 = r\partial_r \psi + \left(\frac{D-3}{2}\right) \psi$
- At brane ($z = 0$) : $0 = \partial_z \psi + \left(\frac{D-3}{2\ell}\right) \psi^{(D-1)/(D-3)}$

Black Holes (near the brane)

- Approximate $\ell \to \infty$: $\nabla_{\mathrm{f}}^2 \psi = 0 \quad , \quad \partial_z \psi \Big|_{z=0} = 0$
- Method of images [Misner 1963]: $\psi = 1 + \sum_{n=1}^{\infty} \left(\frac{q_n}{|\mathbf{x}+\mathbf{d}_n|^{D-3}} + \frac{q_n}{|\mathbf{x}-\mathbf{d}_n|^{D-3}} \right)$



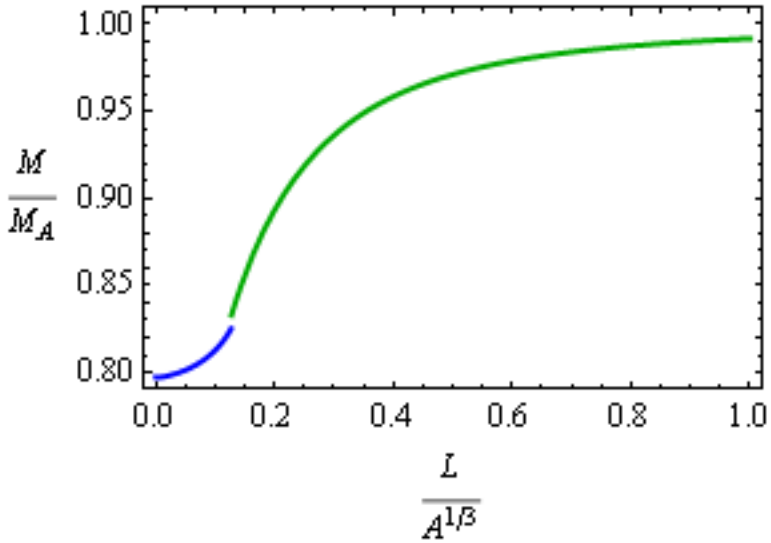


$$a = c \operatorname{csch} \mu_0$$
$$d_n = c \coth n\mu_0$$
$$q_n = (c \operatorname{csch} n\mu_0)^{D-3}$$

- **Find numerically**: A_{outer} , $\bar{\mu}_0$ if throat very near brane ($\mu_0 < \bar{\mu}_0$)
- Find analytically: $M = \frac{(D-2)\omega_{D-2}}{4\pi G_D} \sum_n q_n$, A_{throat} , L_D , $L_5 = c$
- Constant area $A = A_{\mathrm{AH}} = (A_{throat}$ or $A_{outer})$: $c(\mu_0) = \left[\frac{A}{f(\mu_0)} \right]^{1/(D-2)}$

Static BH, Stability, Binding Energy

Mass M (constant A)



$L \to 0$:

$M \to M_0$

large L:

$$M \simeq M_A - \frac{2}{3\pi}\frac{G_5 (M_A)^2}{(2L)^2}$$

rest energy:

$$M_A = \frac{3\pi}{8G_5}\left(\frac{A}{2\pi^2}\right)^{2/3}$$

- Each point: an initially static BH (**on brane**, or **off brane**).
- **The BH on the brane at $L \to 0$:**
- Variational Principle: this mass extremum is a **static** BH.
- It's **stable** against translations (mass is a local minimum).
- **High** binding energy: for brane with $\mathbb{Z}_n$ orbifold symmetry,

$$E_B = M_A - M_0 = \left[n^{1/(D-2)} - 1\right] M_0$$

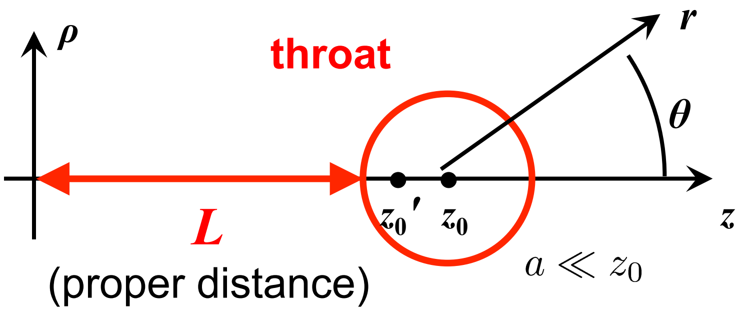
Black Holes (far from brane)

- Farther from brane, the approximation $\ell \to \infty$ breaks down.
- Solve the nonlinear problem with a perturbation series:

$$\psi = \psi_0 + (\psi_0)^2 \phi_1 + (\psi_0)^2 \sum_{i \geq 2} \phi_i \quad , \quad \psi_0 = \frac{\ell}{\ell + z}$$

- All main results from: ϕ_1 Perturbations: $\phi_i \ (i \geq 2)$

- For $\rho \gg \ell, z, z_0$: $\phi_1(\rho, z) \simeq \dfrac{2G_5 m}{\ell \rho}$ (determines mass m)



$$L = \int_0^{z_0 - a} dz\, \psi \simeq -\ell \ln \psi_0(z_0)$$

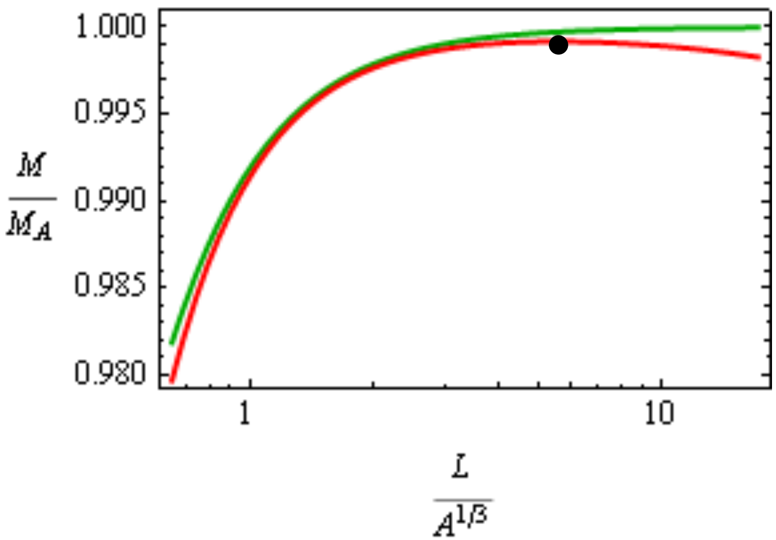
- Mass in terms of area A: $M \simeq \psi_0(z_0)\, M_A - \frac{2}{3\pi} \frac{G_5 (M_A)^2}{(2z_0)^2 \psi_0(z_0)}$

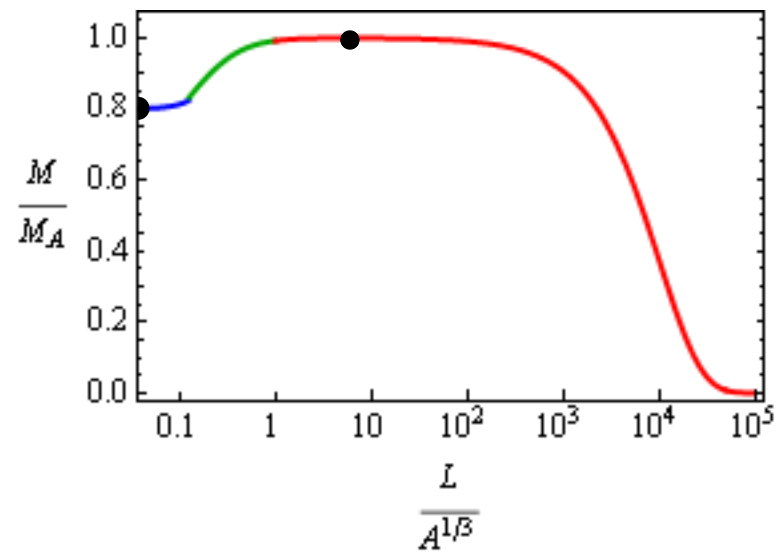
Static BH, Stability, Binding Energy

Mass M (constant A)

$A^{1/3} = 10^{-4}\ell$

Mass M (constant A)





- Mass extremum at: $L_{\rm ext} \simeq (z_0)_{\rm ext} \simeq \left(\frac{G_5 M_A \ell}{3\pi}\right)^{1/3}$
- Variational Principle: this extremum is a **static** black hole.
- It is **unstable** to translations (mass is a local maximum).
- **Small** contribution to binding energy from brane repulsion:

$$E_B = M_{\rm ext} - M_0 \simeq \left[2^{1/3} - 1 - \frac{3}{2}\left(\frac{2G_5 M_A}{3\pi\ell^2}\right)^{1/3}\right] M_0$$

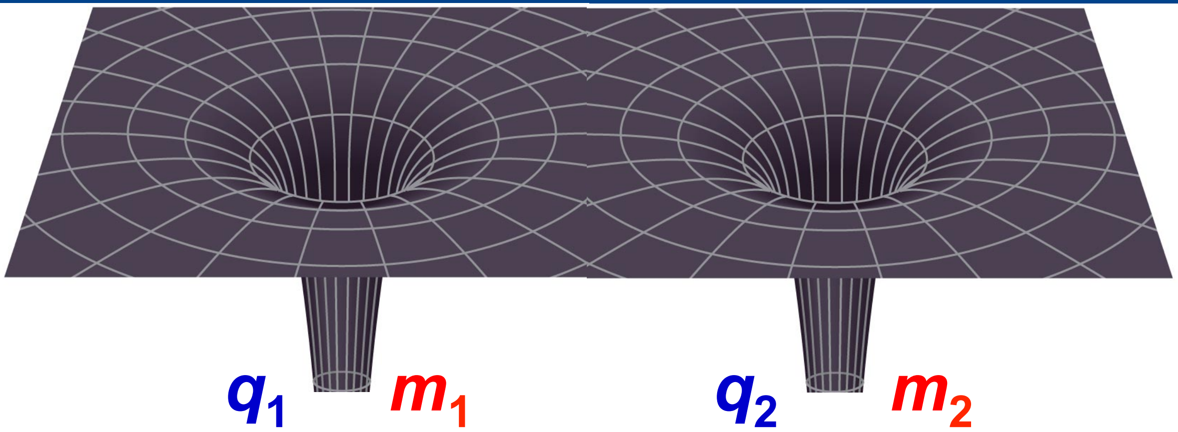
Black Holes with Electric Charge

To appear on arXiv

Scott Fraser, Shaker Funkhouser *

* First prize, 2015 CSU Student Research Competition

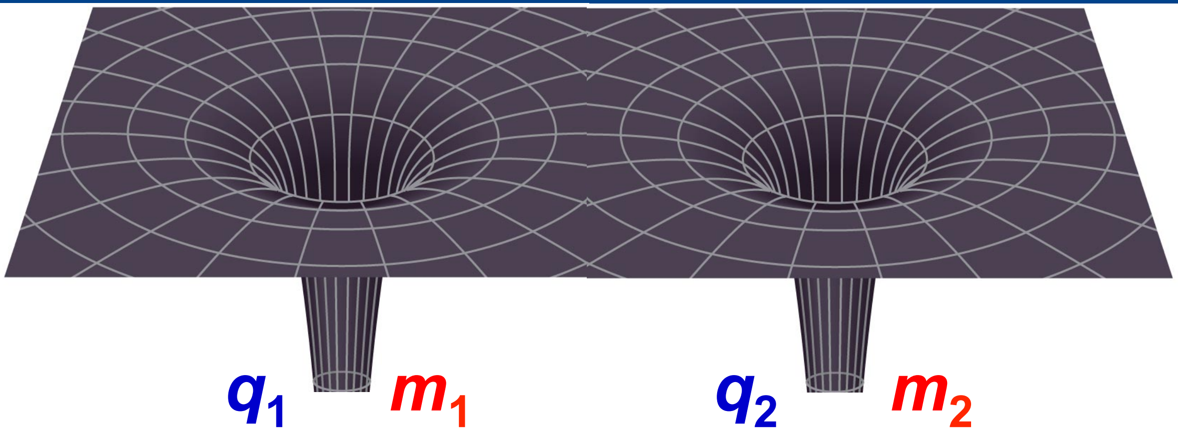
Introduction: A Set of Static BHs



- Intuition: the black holes are static due to balanced forces.
- **Masses *m*: gravitationally attract.**
- **Electric charges *q* (same sign): repel.**

- For nearly 50 years: this interpretation has prevailed.
- But in general relativity, gravity is due to spacetime geometry, not a force.

Introduction: A Set of Static BHs



- Known static condition, in units with $G = 1$ and $1/(4\pi\varepsilon_0) = 1$:

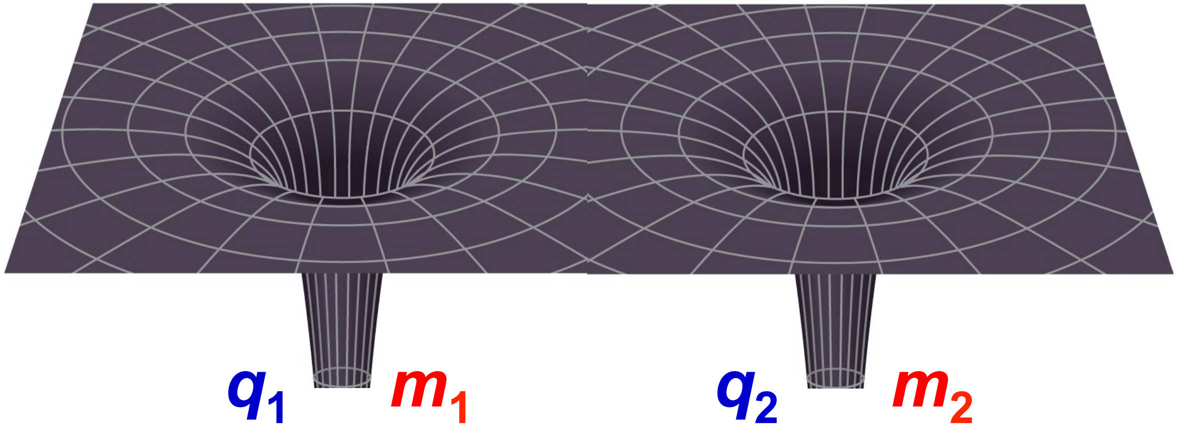
$$|q_1| = m_1 \quad \text{and} \quad |q_2| = m_2$$

Concisely: $|q_i| = m_i \quad (i = 1,2)$

[Majumdar and Papapetrou, 1947]
[Hartle and Hawking, 1972]

- We rediscover this condition using energy.

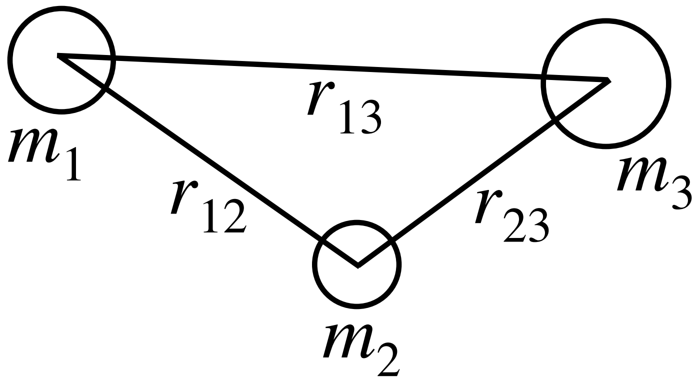
Static BHs Extremize Energy



- Our result:
  The first **energy** interpretation of the static condition:
  $$|q_i| = m_i \quad (i = 1, 2, \ldots, N)$$

- Our method:
  Prove that $|q_i| = m_i$ extremizes the energy of a known geometry with **arbitrary** same-sign charges: $|q_i| \leq m_i$

A Set of N Initially Static BHs

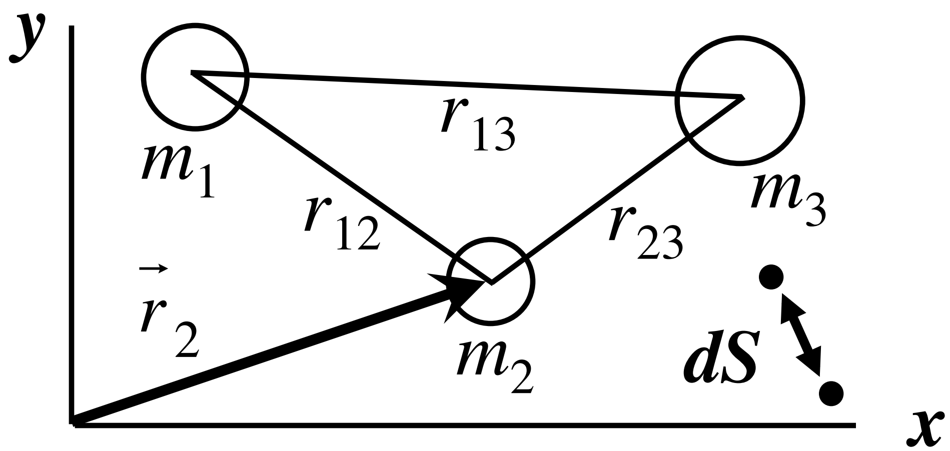


example shown: $N = 3$

Procedure:

- Review known geometry for *N* initially static BHs.
- Calculate each black hole's area.
- Convenient to use an expansion in large distances $\boldsymbol{r_{ij}}$.
- Evaluate total energy and extremize it.
- Show: the extremum yields the static condition $|\boldsymbol{q_i}| = \boldsymbol{m_i}$.

The known geometry



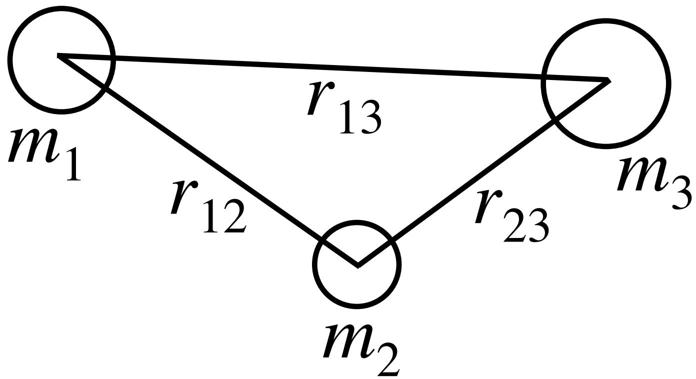
- Known geometry with $|\boldsymbol{q_i}| \leq \boldsymbol{m_i}$: $dS^2 = f^2\left(dx^2 + dy^2 + dz^2\right)$

  [Brill and Lindquist, 1963]

$$f = \left(1 + \sum_{i=1}^{N} \frac{\alpha_i}{\left|\vec{r} - \vec{r}_i\right|}\right)\left(1 + \sum_{i=1}^{N} \frac{\beta_i}{\left|\vec{r} - \vec{r}_i\right|}\right)$$

- Constants α_i and β_i are related to $\boldsymbol{m_i}$, $\boldsymbol{q_i}$, total energy $\boldsymbol{E}$.

The known geometry



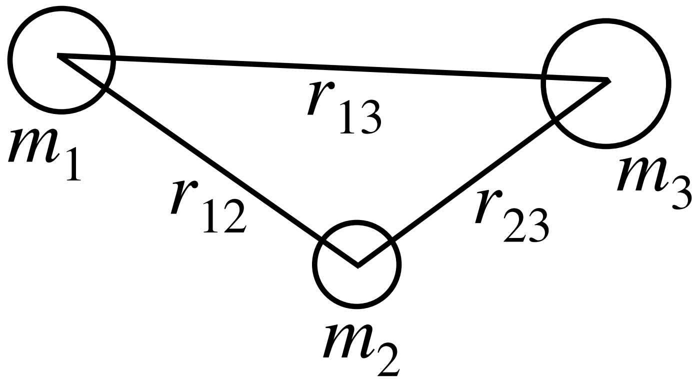
- Constants $\boldsymbol{\alpha_i}$ and $\boldsymbol{\beta_i}$ are related to $\boldsymbol{m_i}$, $\boldsymbol{q_i}$, total energy $\boldsymbol{E}$.

$$m_i = \alpha_i + \beta_i + \sum_{j \neq i} \frac{\left(\alpha_i \beta_j + \alpha_j \beta_i\right)}{r_{ij}}$$

$$q_i = \beta_i - \alpha_i + \sum_{j \neq i} \frac{\left(\beta_i \alpha_j - \beta_j \alpha_i\right)}{r_{ij}}$$

$$E = \sum_{i=1}^{N} \left(\alpha_i + \beta_i\right)$$

The known geometry



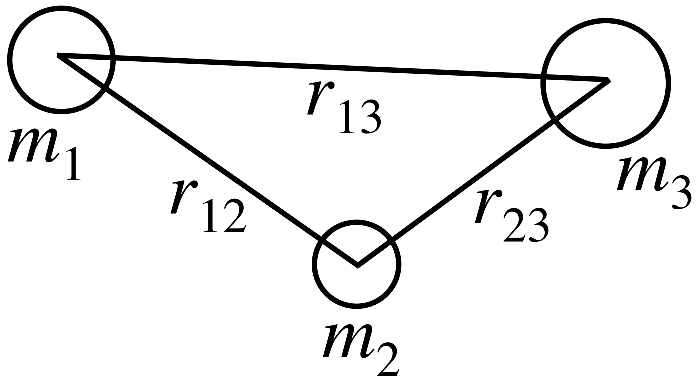
- Constants α_i and β_i are related to m_i , q_i , total energy E.

$$\alpha_i = \frac{(m_i - q_i)}{2}\left[1 - \frac{1}{2}\sum_{j \neq i}\frac{\left(m_j + q_j\right)}{r_{ij}}\right] + \left(\text{terms with } \frac{1}{(r_{ij})^n} \quad, n \geq 2\right)$$

$$\beta_i = \frac{(m_i + q_i)}{2}\left[1 - \frac{1}{2}\sum_{j \neq i}\frac{\left(m_j - q_j\right)}{r_{ij}}\right] + \left(\text{terms with } \frac{1}{(r_{ij})^n} \quad, n \geq 2\right)$$

$$E = \sum_{i=1}^{N}\left(\alpha_i + \beta_i\right)$$

Black hole area



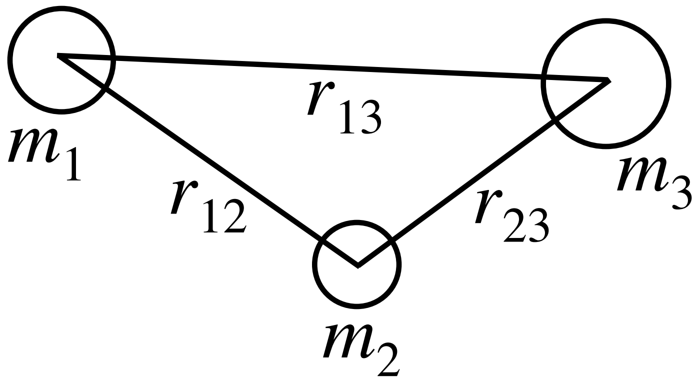
- Calculate the area $\boldsymbol{A_i}$ of black hole i:

$$A_i = 4\pi R_i^{\,2} f^2$$

$$R_i^{\,2} = \alpha_i \beta_i \left(1 + \sum_{j \neq i} \frac{\alpha_j}{r_{ij}}\right)^{-1} \left(1 + \sum_{j \neq i} \frac{\beta_j}{r_{ij}}\right)^{-1}$$

$$f = \left(1 + \frac{\alpha_i}{R_i} + \sum_{j \neq i} \frac{\alpha_j}{r_{ij}}\right)\left(1 + \frac{\beta_i}{R_i} + \sum_{j \neq i} \frac{\beta_j}{r_{ij}}\right) + \left(\text{terms with } \frac{1}{(r_{ij})^n},\ n \geq 2\right)$$

Black hole area



- Evaluate area $\boldsymbol{A_i}$ in terms of mass and charge:

$$\sqrt{\frac{A_i}{4\pi}} = m_i + \sqrt{m_i^2 - q_i^2} + \mathbf{0} + \left(\text{terms with } \frac{1}{(r_{ij})^n},\ n \geq 2 \right)$$

(terms involving $1/r_{ij}$ cancel)

- Solve for mass: $$m_i = \sqrt{\frac{\pi}{A_i}} \left(\frac{A_i}{4\pi} + q_i^2 \right) + \mathbf{0}$$

Evaluate Energy

- Recall: $E = \sum_{i=1}^{N} (\alpha_i + \beta_i)$ and $m_i = \sqrt{\frac{\pi}{A_i}} \left(\frac{A_i}{4\pi} + q_i^2 \right)$ + **0**
- Find:

$$E = \sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \sum_{j>i} \frac{(q_i q_j - m_i m_j)}{r_{ij}} + \left(\text{terms with} \frac{1}{(r_{ij})^n}, n \geq 2 \right)$$

- Extremize: $\boldsymbol{E(A_i , q_i , r_{ij})}$ while holding $\boldsymbol{q_i}$ and $\boldsymbol{r_{ij}}$ constant.
- One extremum is *N* conditions: $\frac{\partial E}{\partial A_i} = 0$
- Find: $\frac{\partial E}{\partial A_i} = 0$ yields the static condition $\boldsymbol{|q_i| = m_i}$.

Conclusions and Outlook

- **Black holes in RS models:**
  - Static BHs extremize energy (variational principle).
  - Small BHs on the brane are strongly bound: an important result for LHC experiments.

- **Charged black holes:**
  - By extremizing energy, we found a new interpretation of the long known static condition, $|q_i| = m_i$.
  - Proof at higher orders is in progress.