Five is Different: Symmetry, Solvability and Entanglement in Quantum Few-Body Systems

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One-Dimensional Traps, Two-Body Interactions, Few-Body Symmetries:

I. One, Two, and Three Particles, arXiv: 1501.00215
II. N Particles, arXiv: 1505.00659

MY

TALK

Newton’s Cradle,

or Executive Ball Clicker

Quantum Newton's Cradle

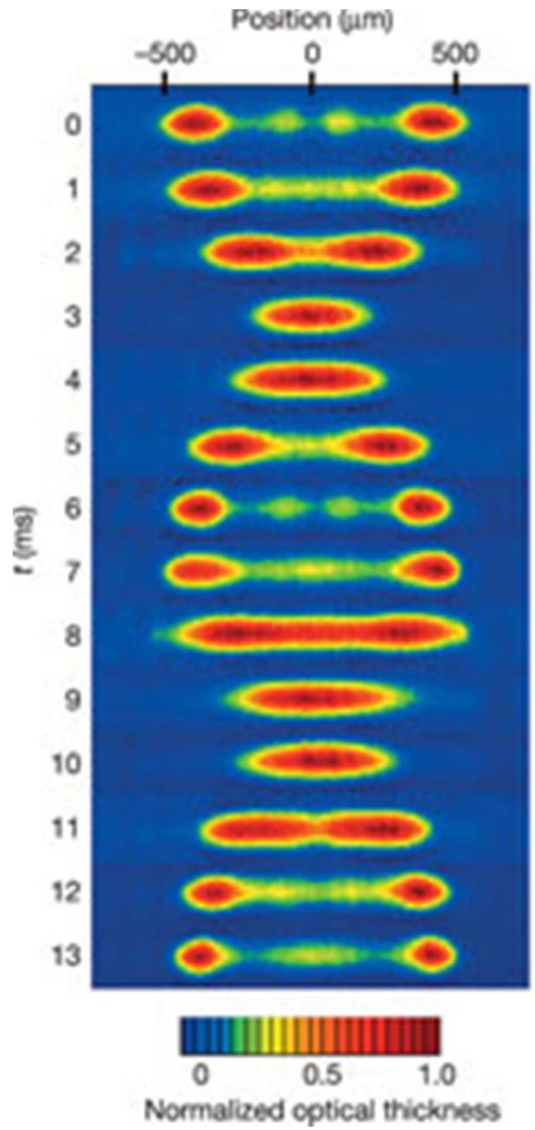
Schrodinger's Cradle?
Quantum abacus?

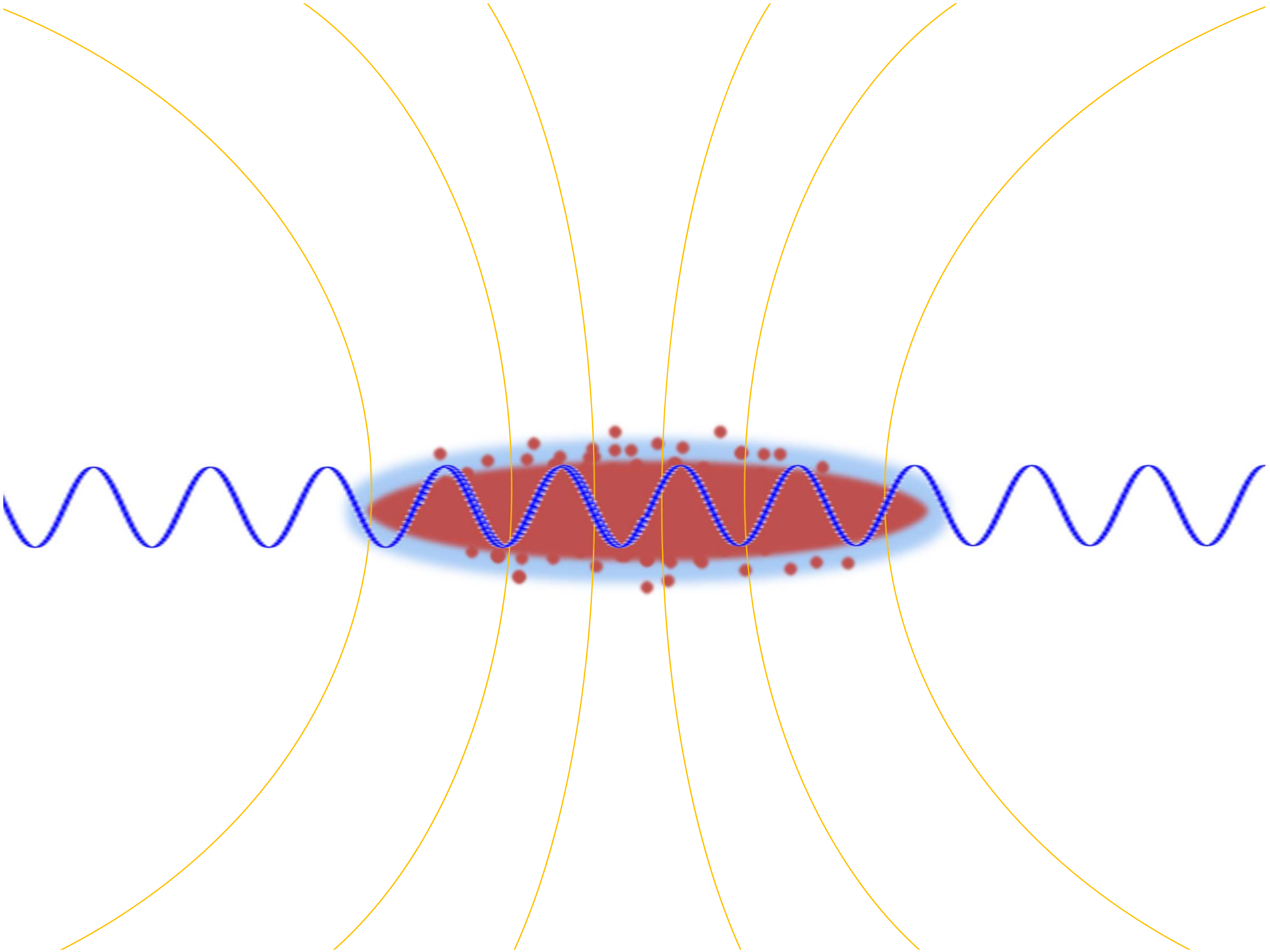
$$\hat{H} = \sum_{i=1}^{N} \left(-\frac{1}{2m} \frac{\partial^2}{\partial x_i^{\,2}} + V(x_i) \right) + g \sum_{\langle i,j \rangle} \delta\left(x_i - x_j \right)$$

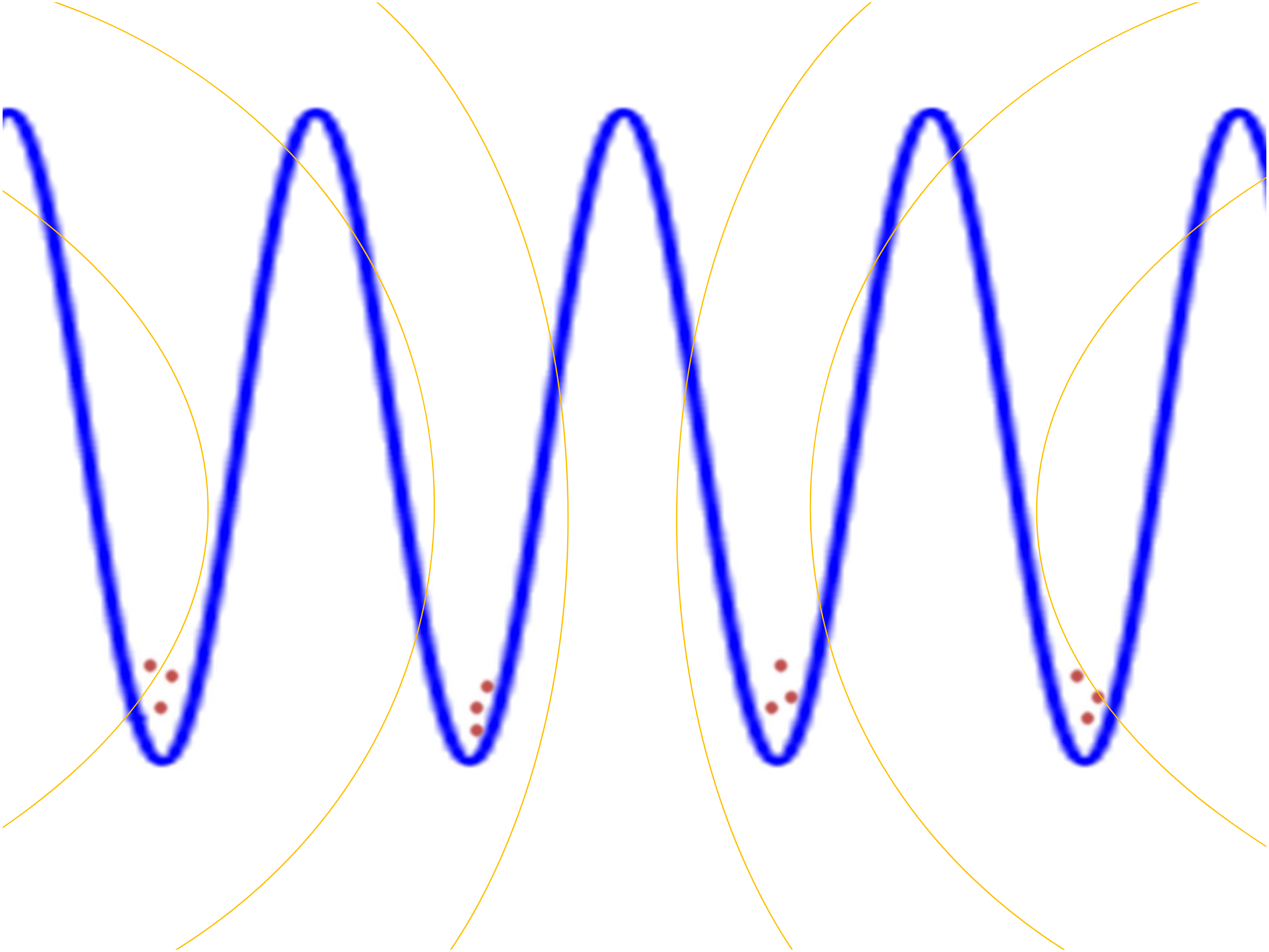
New features:

- Tunneling
- Identical particles
- Spins and/or internal components

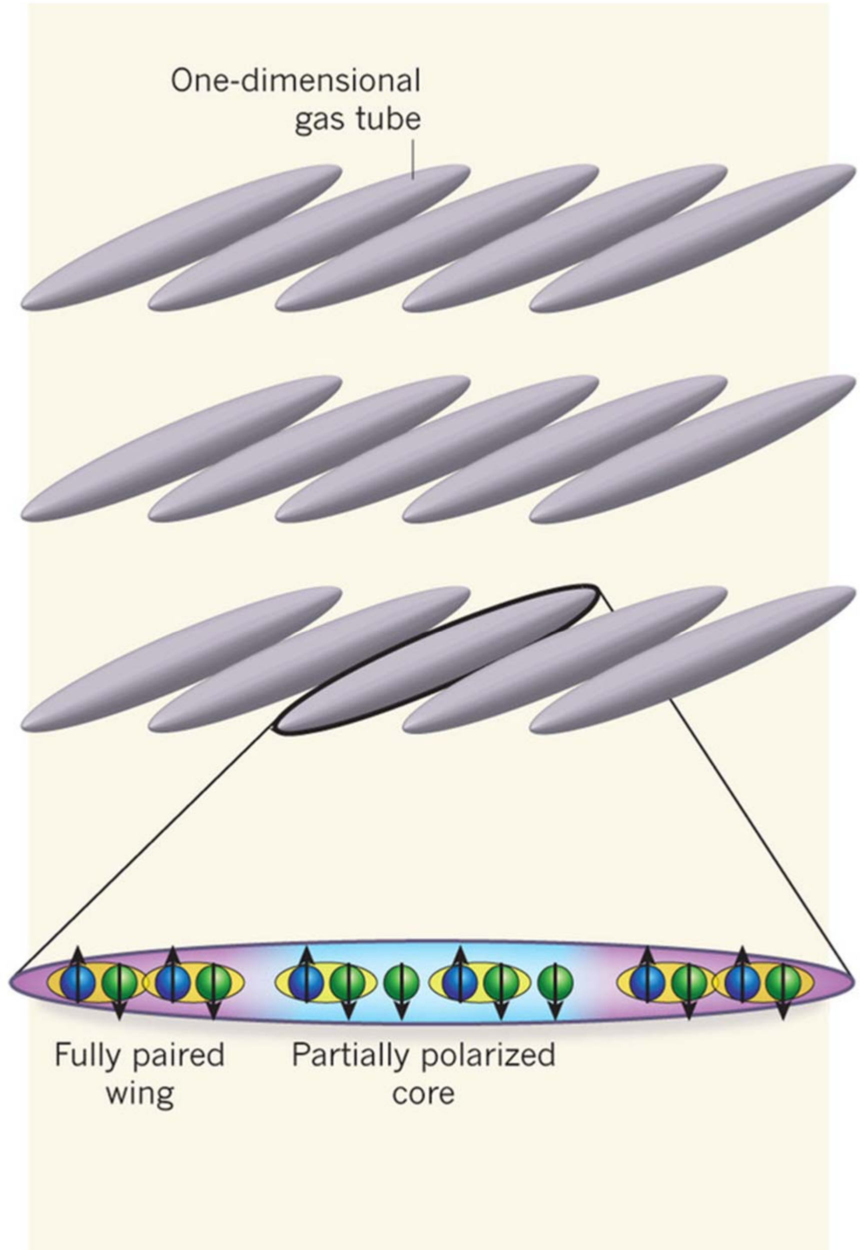
Toshiya Kinoshita, Trevor Wenger and David S. Weiss, Nature 440, 900-903 (13 April 2006)







Immanuel Bloch, Nature 467, 535–536 (30 September 2010)



Measurements and Knobs

- Tune trap shape, interaction strength fast or slow
- Many body
  - Trap loss rate from bound state formation
  - Release trap, wait a while…Fourier transform!
- Few body
  - RF spectroscopy of energy levels
  - Probe spatial states by tunneling rate
- Couple spatial degrees of freedom to spin degrees of freedom

These things are known, Khaleesi

$$\frac{\hat{H}}{\eta} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial q_i^{\,2}} + V^1(q_i)\right) + g\sum_{\langle i,j\rangle}\delta\left(\left|q_i - q_j\right|\right)$$

- Has different names depending on trap shape and two body interaction
- Exact solutions in terms of one-body states for
  - Girardeau (and generalizations) for Fermi-Bose mapping at unitary limit of contact interactions

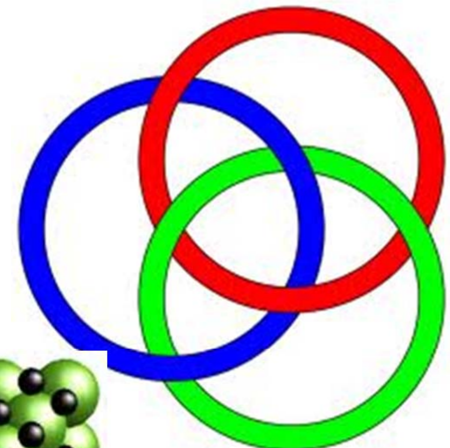
$$g = 0 \qquad\qquad 1/g = 0$$

- Some cases integrable and/or analytically solvable
  - Lieb-Liniger and Bethe Ansatz for contact interaction and homogeneous potential and periodic or hard wall boundary conditions
  - Busch et al for two bodies with contact interaction and harmonic traps

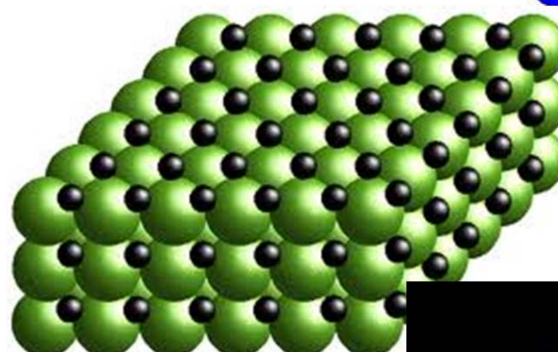
Why?

- Because we can!

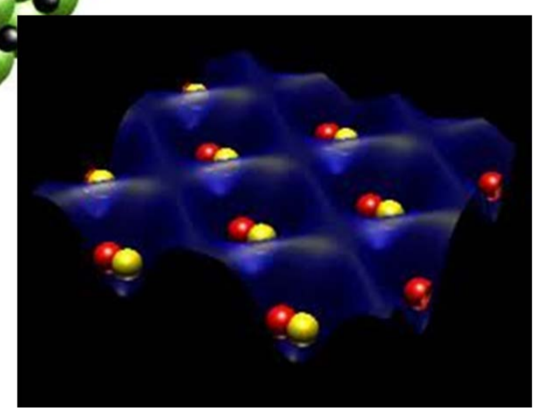
- Universality



- Quantum simulations



- Quantum information processors



Why me?

- Chance, contingency, stubbornness, vanity

- Entanglement in interacting systems with complicated degrees of freedom and constraints on physical observables

- ……symmetry…….solvability…algebraic solvability …..controllability… ..entanglement…...coherence… integrability…… …….the future!

Algebraic Solvability

- Quadratic equation

$$ax^2 + bx + c = 0$$

- Solution

$$(x - x_+)(x - x_-) = 0 \qquad x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Solvability
  - Requires radicals and complex numbers
  - Provides simplicity, information

Cubic equation

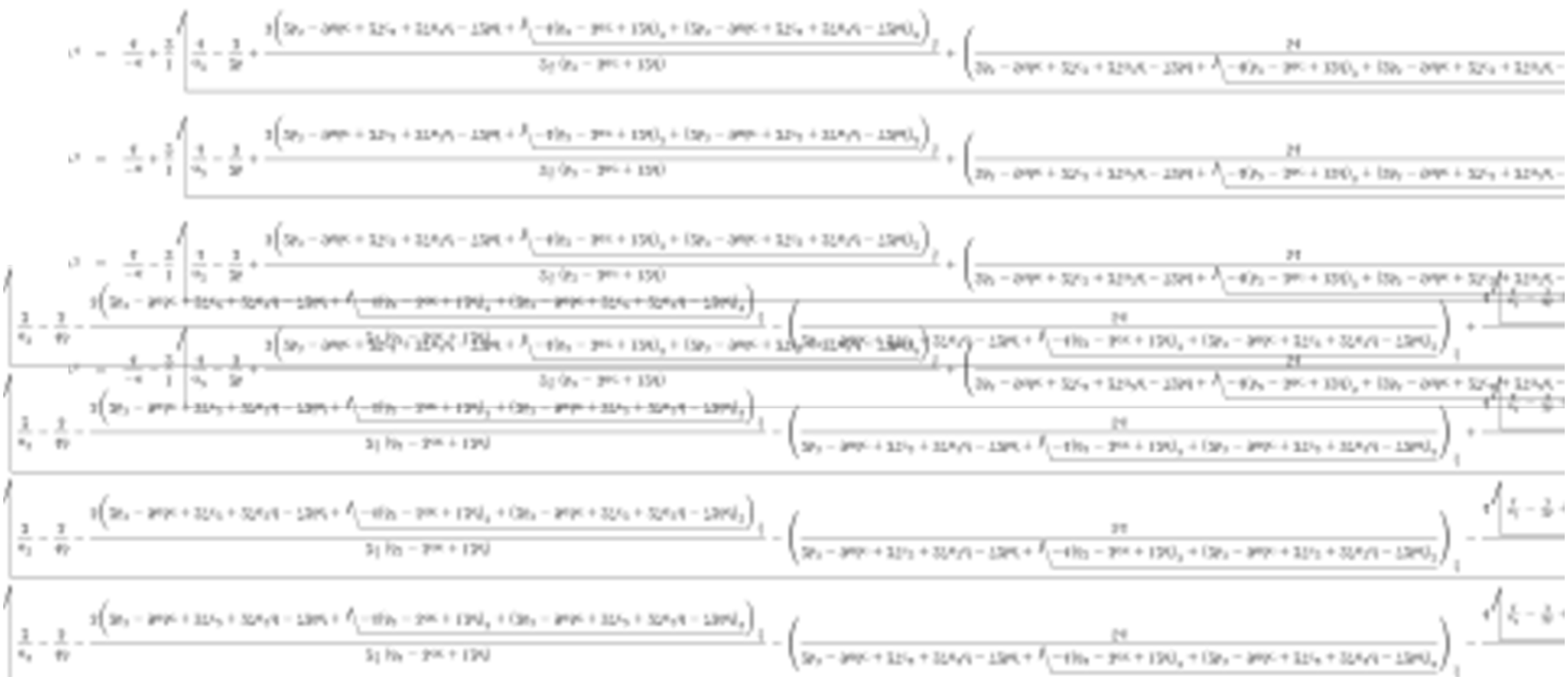
$$ax^3 + bx^2 + cx + d = 0$$

$$\begin{aligned}
x_1 = &-\frac{b}{3a} \\
&-\frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
&-\frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
x_2 = &-\frac{b}{3a} \\
&+\frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
&+\frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
x_3 = &-\frac{b}{3a} \\
&+\frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \\
&+\frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]}
\end{aligned}$$

Wikimedia commons

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$



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Quintic Equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

DNE

Not enough symmetry to solve quintic equation.
Thanks, Galois.

"Algebraic Universality"

- Identify symmetries for
  - Arbitrary trap shape and two-body interactions
  - Specific trap shapes (harmonic) and specific two-body interactions (contact)
  - Any number of internal components (spin)
- Find algebraic expressions for few-body observables in terms of (all or some)
  - One particle energies and wave functions
  - Two body matrix elements

Non-interacting

$$C_0^N = S_N \ltimes C_1^{\times N}$$

$$K_0^N = S_N \ltimes K_1^{\times N}$$

Two-body, Galilean invariant interactions

$$C^N = S_N \times C_1$$

$$K^N = S_N \times K_1$$

Contact interaction, unitary limit

$$C_\infty^N = S_N^{\times 2} \times C_1$$

$$K_\infty^N = S_N^{\times 2} \times K_1$$

Algebraically solvable for any *N* in terms *of* one-body properties only

Secret Motivation

Integrability, separability and entanglement

- Abstractly: Particles and Tailored Observables
- Directly: Few body systems as a resource for quantum information processing

Symmetry in the Quantum Abacus

$$\frac{\hat{H}}{\eta} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {q_i}^2} + V^1(q_i)\right) + g\sum_{\langle i,j\rangle}\delta\left(\left|q_i - q_j\right|\right)$$

- Configuration space symmetries
- Kinematic symmetries
- Dynamic symmetries
- Spin statistics

Symmetry in the Quantum Abacus

SYMMETRY!

$$\frac{\hat{H}}{\eta} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial q_i^{\,2}} + V^1(q_i)\right) + g\sum_{\langle i,j\rangle}\delta\left(\left|q_i - q_j\right|\right)$$

- Configuration space symmetries

$V(q_i) \sim \sqrt{|q_i|}$

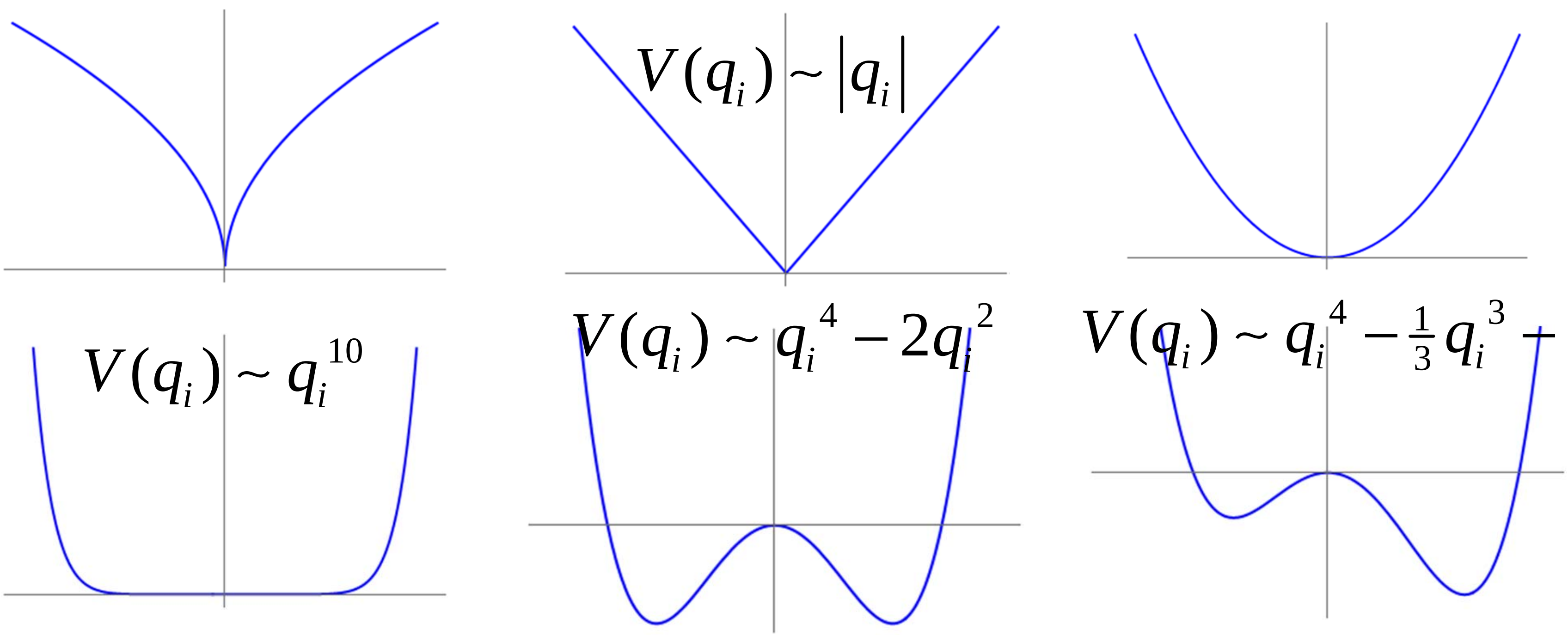
$V(q_i) \sim |q_i|$

$V(q_i) \sim q_i^{\,2}$

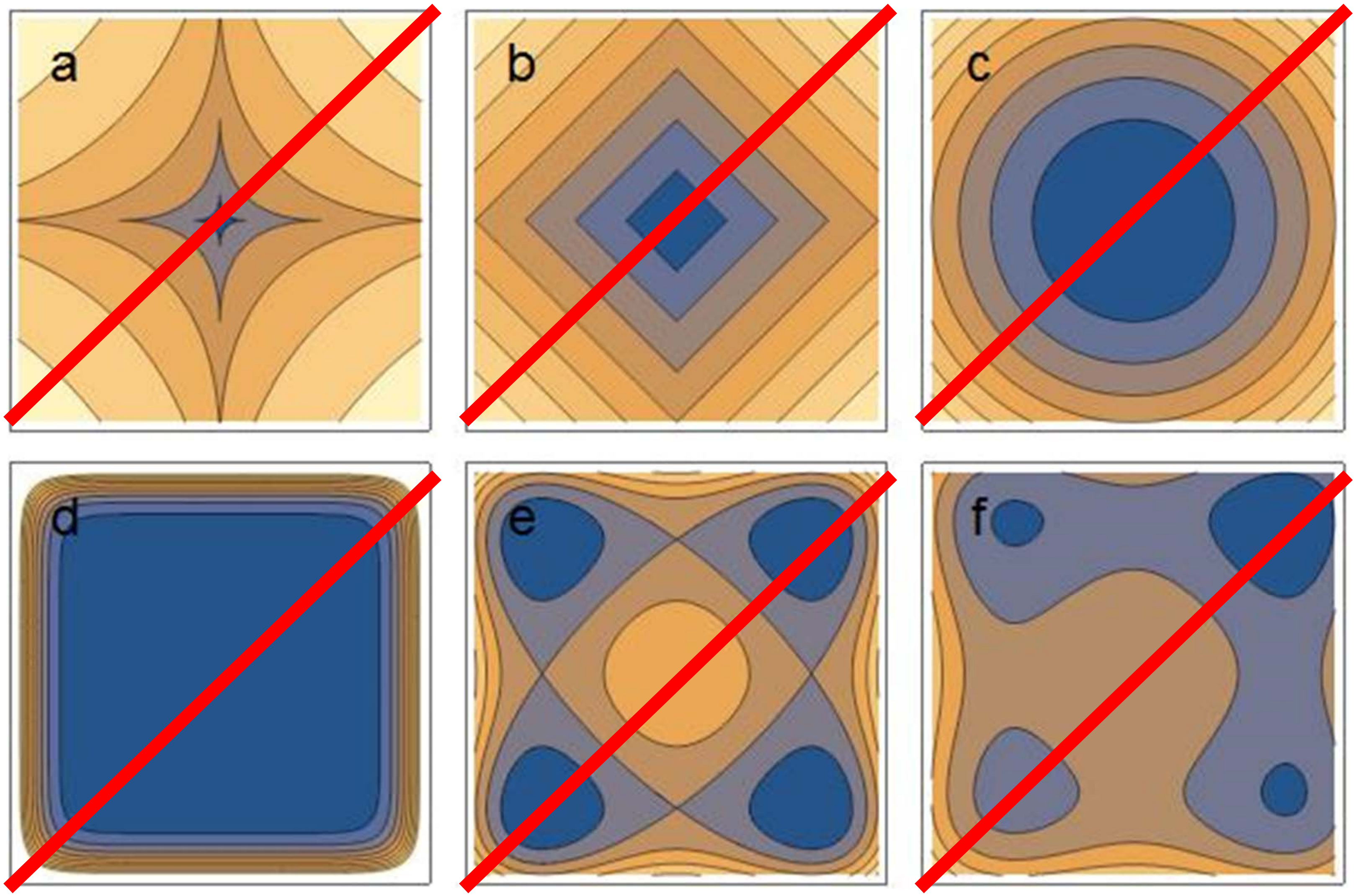
$V(q_i) \sim q_i^{\,10}$

$V(q_i) \sim q_i^{\,4} - 2q_i^{\,2}$

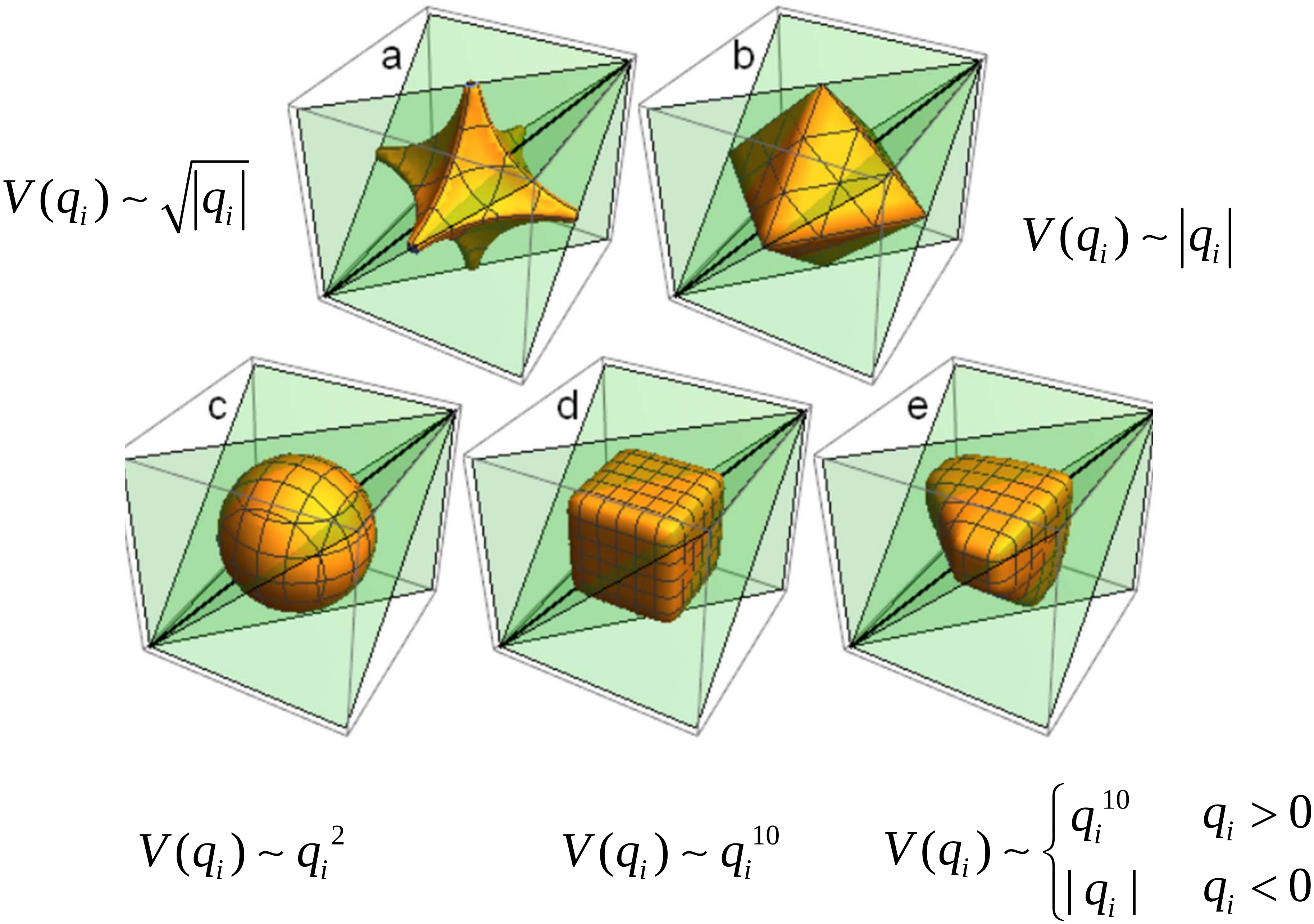
$V(q_i) \sim q_i^{\,4} - \tfrac{1}{3}q_i^{\,3} - 2q_i^{\,2}$



$V(q_i) \sim \sqrt{|q_i|}$ $V(q_i) \sim |q_i|$ $V(q_i) \sim q_i^{\,2}$



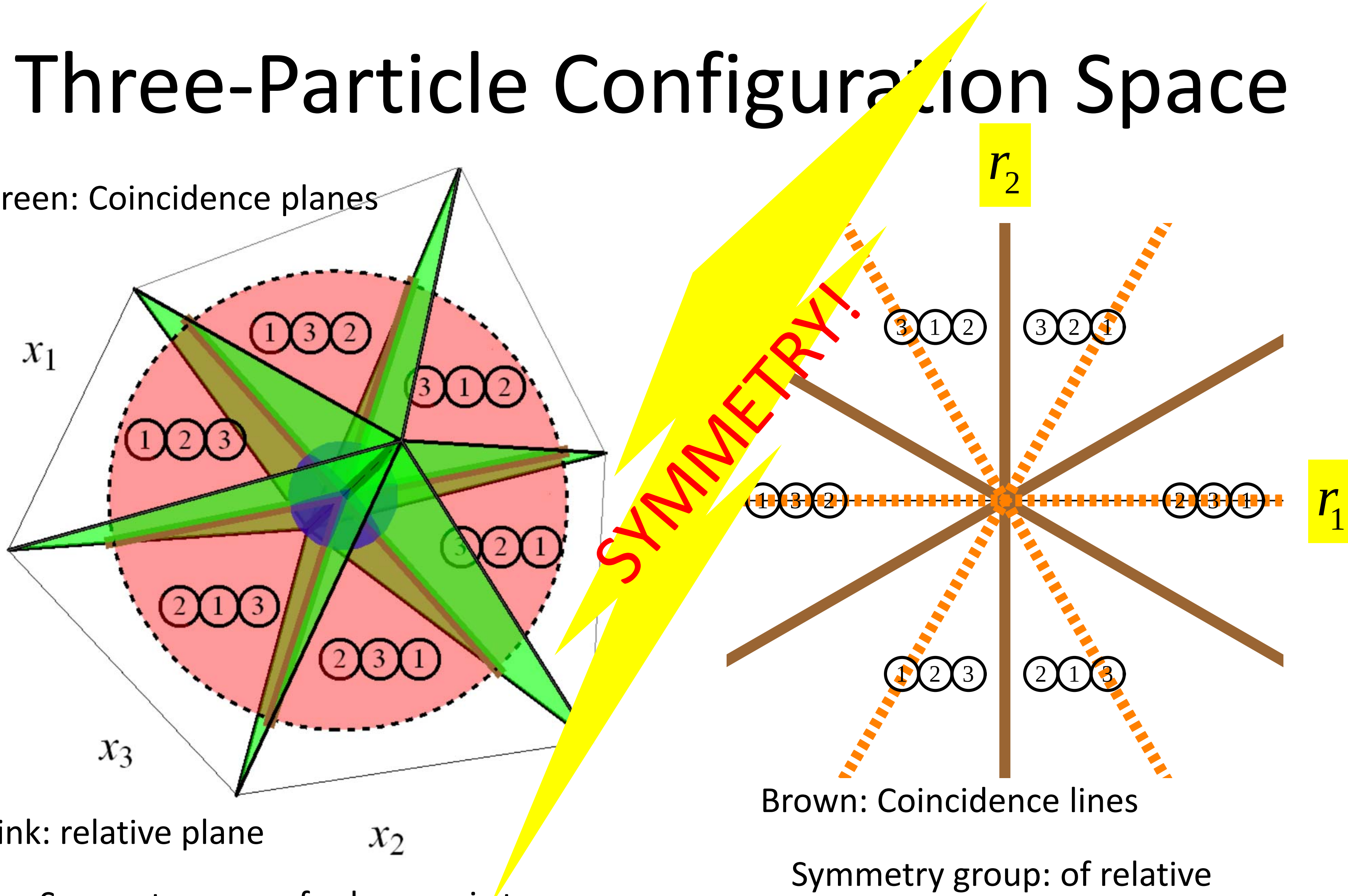
$V(q_i) \sim q_i^{\,10}$ $V(q_i) \sim q_i^{\,4} - 2q_i^{\,2}$ $V(q_i) \sim q_i^{\,4} - \frac{1}{3}q_i^{\,3} - 2q_i^{\,2}$



a
b
c
d
e
$V(q_i) \sim \sqrt{|q_i|}$
$V(q_i) \sim |q_i|$
$V(q_i) \sim q_i^2$
$V(q_i) \sim q_i^{10}$
$V(q_i) \sim \begin{cases} q_i^{10} & q_i > 0 \\ |q_i| & q_i < 0 \end{cases}$

Three-Particle Configuration Space

Green: Coincidence planes



Pink: relative plane

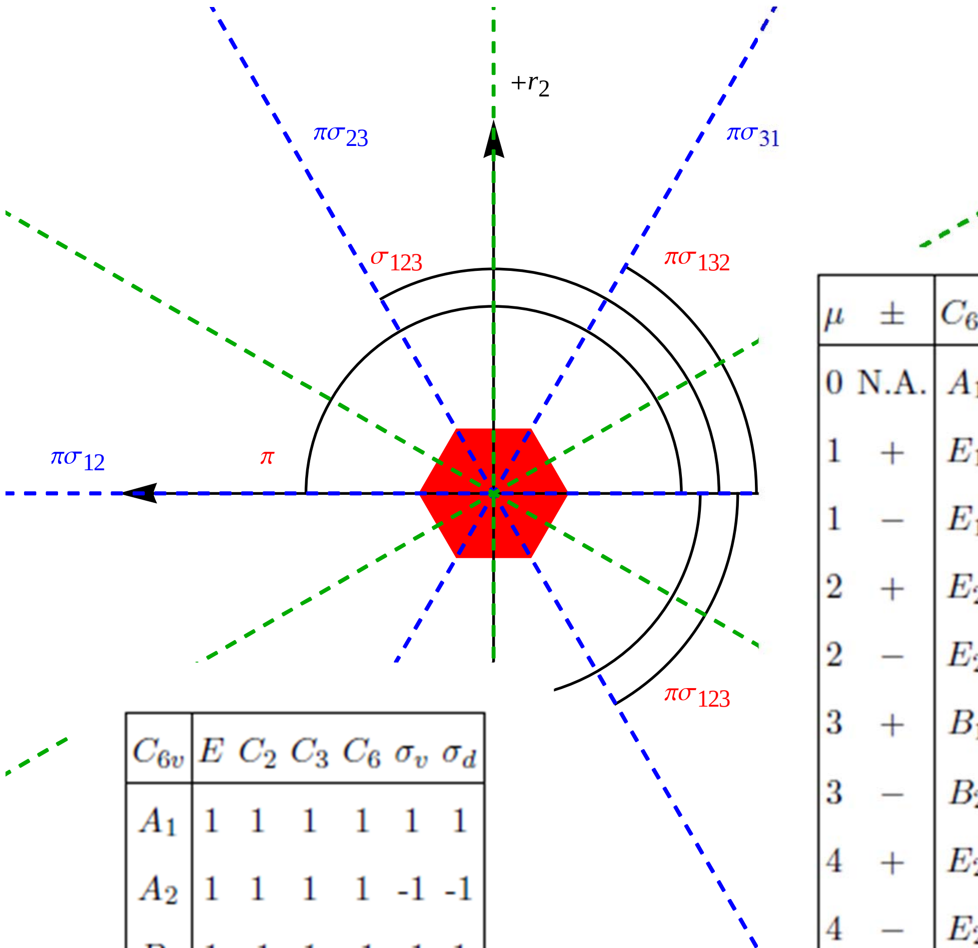
Brown: Coincidence lines

Symmetry group for harmonic trap:

$$S_3 \times O(1) \times O(1) \sim D_{6h}$$

Symmetry group: of relative configuration space

$$S_3 \times O(1) \sim C_{6v}$$



C_{6v}	E	C_2	C_3	C_6	σ_v	σ_d
A_1	1	1	1	1	1	1
A_2	1	1	1	1	-1	-1
B_1	1	-1	1	-1	-1	1
B_2	1	-1	1	-1	1	-1
E_1	2	-2	-1	1	0	0
E_2	2	2	-1	-1	0	0

μ	$\pm$	C_{6v}	C_{2v}	Possibilities
0	N.A.	A_1	A_1	BBB, BBX, XYZ
1	+	E_1	B_1	FFX, XYZ
1	−	E_1	B_2	BBX, XYZ
2	+	E_2	A_1	BBX, XYZ
2	−	E_2	A_2	FFX, XYZ
3	+	B_1	B_1	FFF, FFX, XYZ
3	−	B_2	B_2	BBB, BBX, XYZ
4	+	E_2	A_1	BBX, XYZ
4	−	E_2	A_2	FFX, XYZ
5	+	E_1	B_1	FFX, XYZ
5	−	E_1	B_2	BBX, XYZ
6	+	A_1	A_1	BBB, BBX, XYZ
6	−	A_2	A_2	FFF, FFX, XYZ

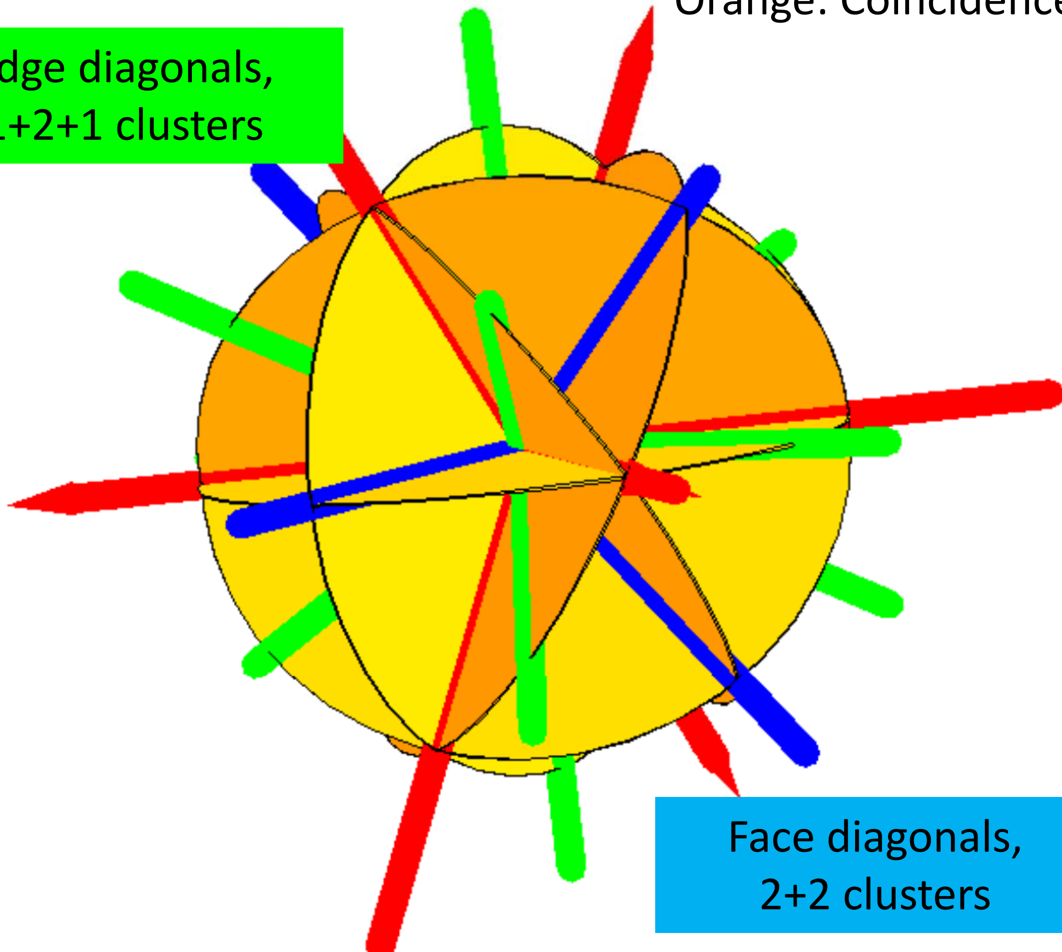
$g \in C_{6v}$	$g \in S_3 \times Z_2$	$\varphi \to \varphi'$
E	$\hat{e}$	φ
σ_v	$\hat{\sigma}_{12}$	$-\varphi + \pi$
	$_{23}$	$-\varphi + \frac{\pi}{3}$
	$_{31}$	$-\varphi - \frac{\pi}{3}$
	$_{231}$	$\varphi - \frac{2\pi}{3}$
	$_{312}$	$\varphi + \frac{2\pi}{3}$
	$\hat{\pi}$	$\varphi + \pi$
	$\hat{\sigma}_{12}$	$-\varphi$
	$\hat{\sigma}_{23}$	$-\varphi - \frac{2\pi}{3}$
	$\hat{\sigma}_{31}$	$-\varphi + \frac{2\pi}{3}$
	$_{231}$	$\varphi + \frac{\pi}{3}$
	$_{312}$	$\varphi - \frac{\pi}{3}$

ansformation designa

Four-Particle
Relative Configuration Space

Orange: Coincidence planes

Edge diagonals,
1+2+1 clusters



Face diagonals,
2+2 clusters

Body diagonals are
projections of particle axes,
3+1 clusters

Interacting configuration
ymmetry group:

$$S_4 \times O(1) \sim O_h$$

$$(12)\ldots \to 6\sigma_d$$

$$(123)\ldots \to 8C_3$$

$$(12)(34)\ldots \to 3C_2$$

$$(1234)\ldots \to 6S_4$$

$$\pi \to i$$

$$\pi(12)\ldots \to 6C_2'$$

$$\pi(123)\ldots \to 8S_6$$

$$\pi(12)(34)\ldots \to 3\sigma_h$$

$$\pi(1234)\ldots \to 6C_4$$

Non-interacting

$$\mathrm{C}_0^N = \mathrm{S}_N \ltimes \mathrm{C}_1{}^{\times N} \qquad \mathrm{K}_0^N = \mathrm{S}_N \ltimes \mathrm{K}_1{}^{\times N}$$

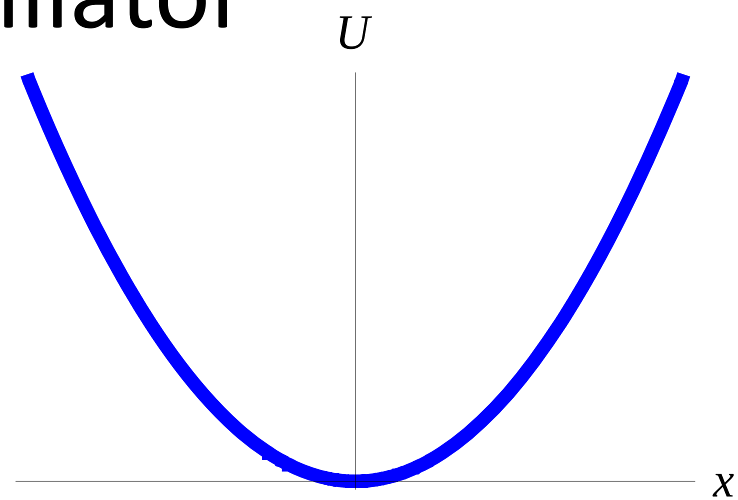
Two-body, Galilean invariant interactions

$$\mathrm{C}^N = \mathrm{S}_N \times \mathrm{C}_1 \qquad \mathrm{K}^N = \mathrm{S}_N \times \mathrm{K}_1$$

Contact interaction, unitary limit

$$\mathrm{C}_\infty^N = \mathrm{S}_N{}^{\times 2} \times \mathrm{C}_1 \qquad \mathrm{K}_\infty^N = \mathrm{S}_N{}^{\times 2} \times \mathrm{K}_1$$

Example of Kinematic Symmetry: Harmonic Oscillator



- Harmonic Hamiltonian:

$$H = \frac{1}{2m} p^2 + \frac{1}{2} k x^2 = E$$

- Classical solution:

$$\omega = \sqrt{\frac{k}{m}} \qquad x(t) = A\cos\left(\omega t + \phi_0\right)$$

$$A = \sqrt{\frac{2E}{k}} \qquad p(t) = -mA\omega\sin\left(\omega t + \phi_0\right)$$

One particle, one dimension: Classical

Initial conditions

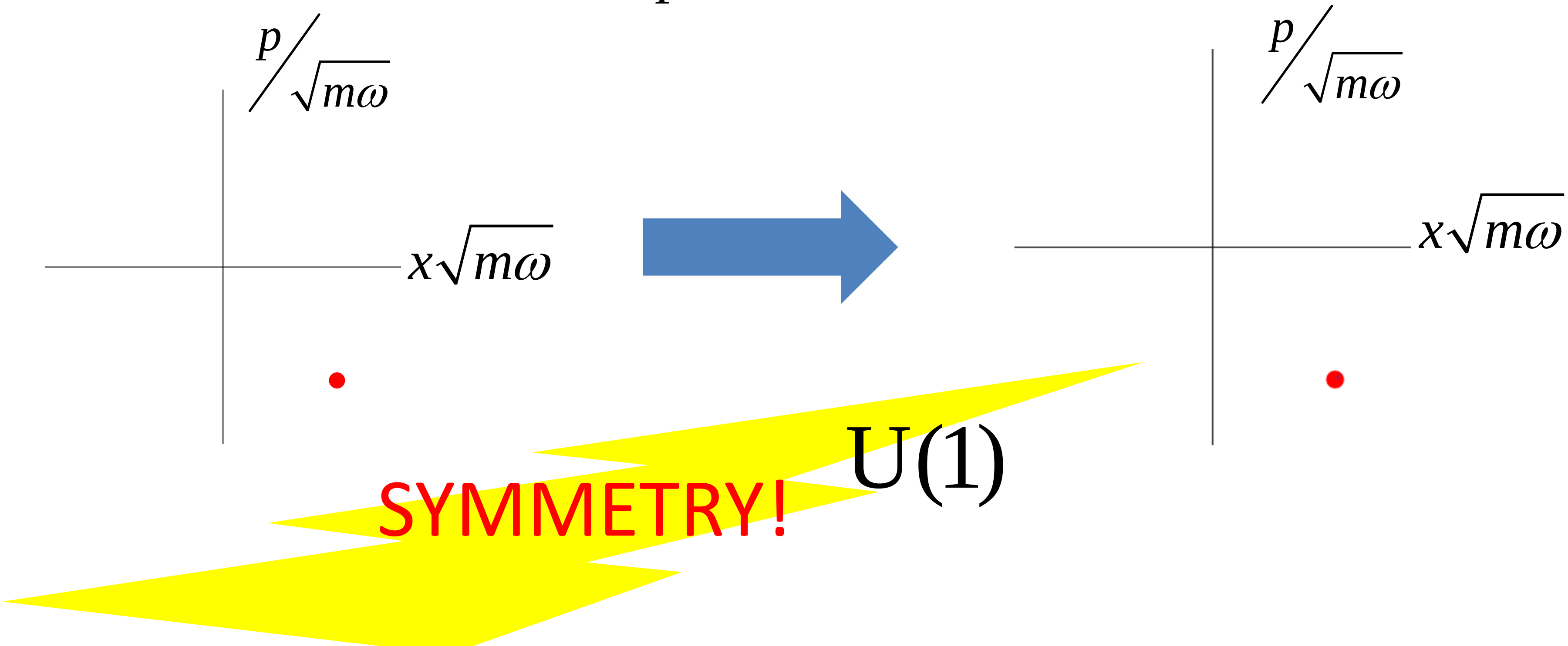
$$\left(x_0, p_0\right) \in \mathbb{R}^2$$

Dynamic equations

$$\dot{x} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial x}$$

Time-dependent state of system

$$\left(x(t), p(t)\right)$$



Stationary States of Harmonic Trap

- Three one-body harmonic Hamiltonians

$$H_0 = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{1}{2}k\left(x_1^2 + x_2^2 + x_3^2\right)$$

SYMMETRY!

U(3)

- Product states are stationary

$$\Phi_{n_1 n_2 n_3}(x_1, x_2, x_3) = \phi_{n_1}(x_1)\phi_{n_2}(x_2)\phi_{n_3}(x_3)$$

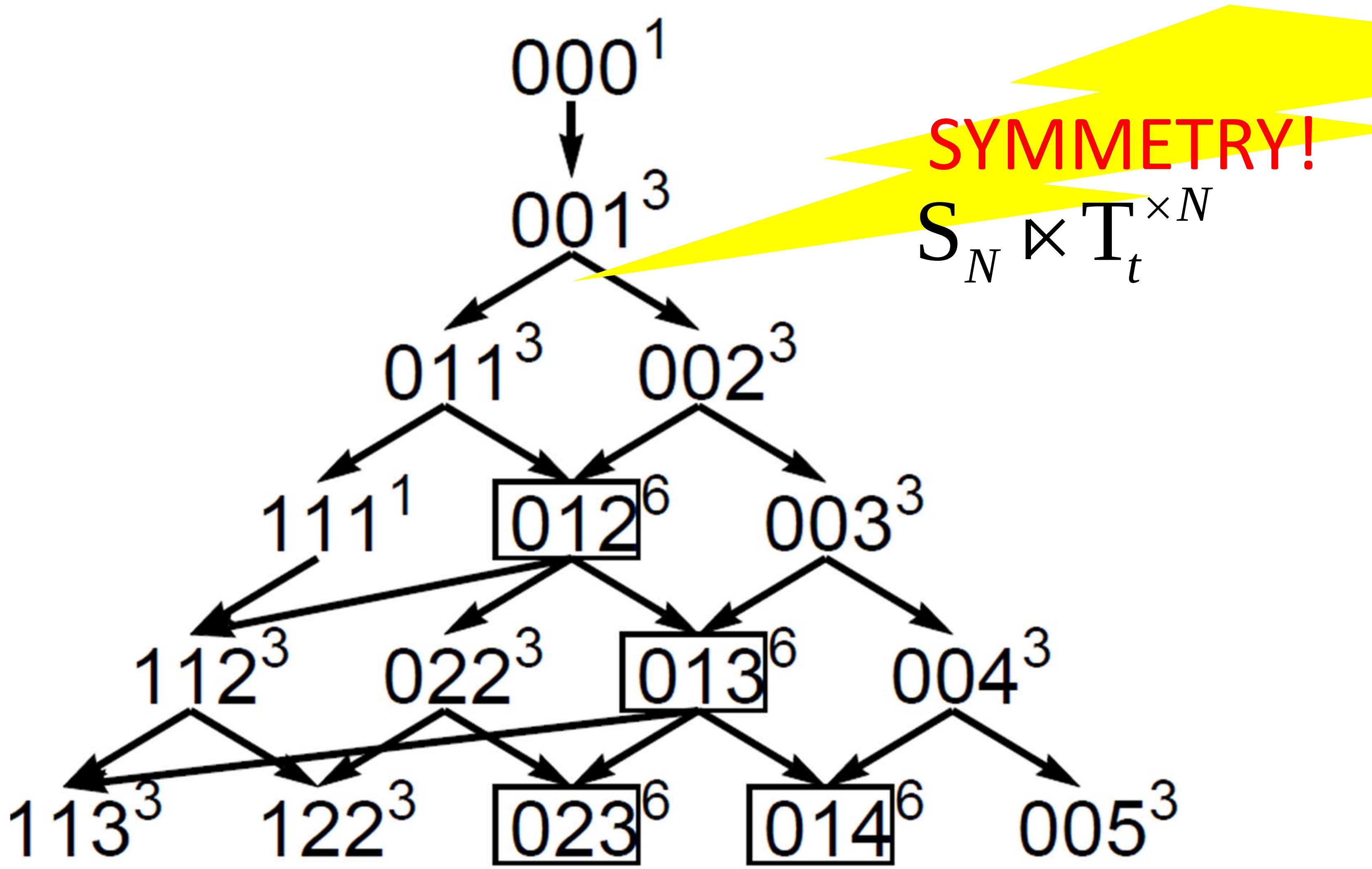
$$\phi_n(x) = \frac{(\pi\sigma^2)^{-1/4}}{\sqrt{2^n n!}} H_n\left(x/\sigma\right)e^{-x^2/2\sigma^2}$$

- Energy degeneracy

$$E = \hbar\omega(n_1 + n_2 + n_3 + \tfrac{3}{2})$$

energy	# of states
3/2	1
5/2	3
7/2	6
9/2	10

Non-interacting three particle spectrum



Identical Particles: Spin Statisics

SYMMETRY!

Bosons: symmetric under exchange of particles $f(x_1, x_2) = f(x_2, x_1)$

Fermions: antisymmetric under exchange of particles $f(x_1, x_2) = -f(x_2, x_1)$

$$\Phi_{000}(x_1, x_2, x_3) \Rightarrow \begin{cases} \phi_0(x_1)\phi_0(x_2)\phi_0(x_3) & \text{bosons} \\ 0 & \text{fermions} \end{cases}$$

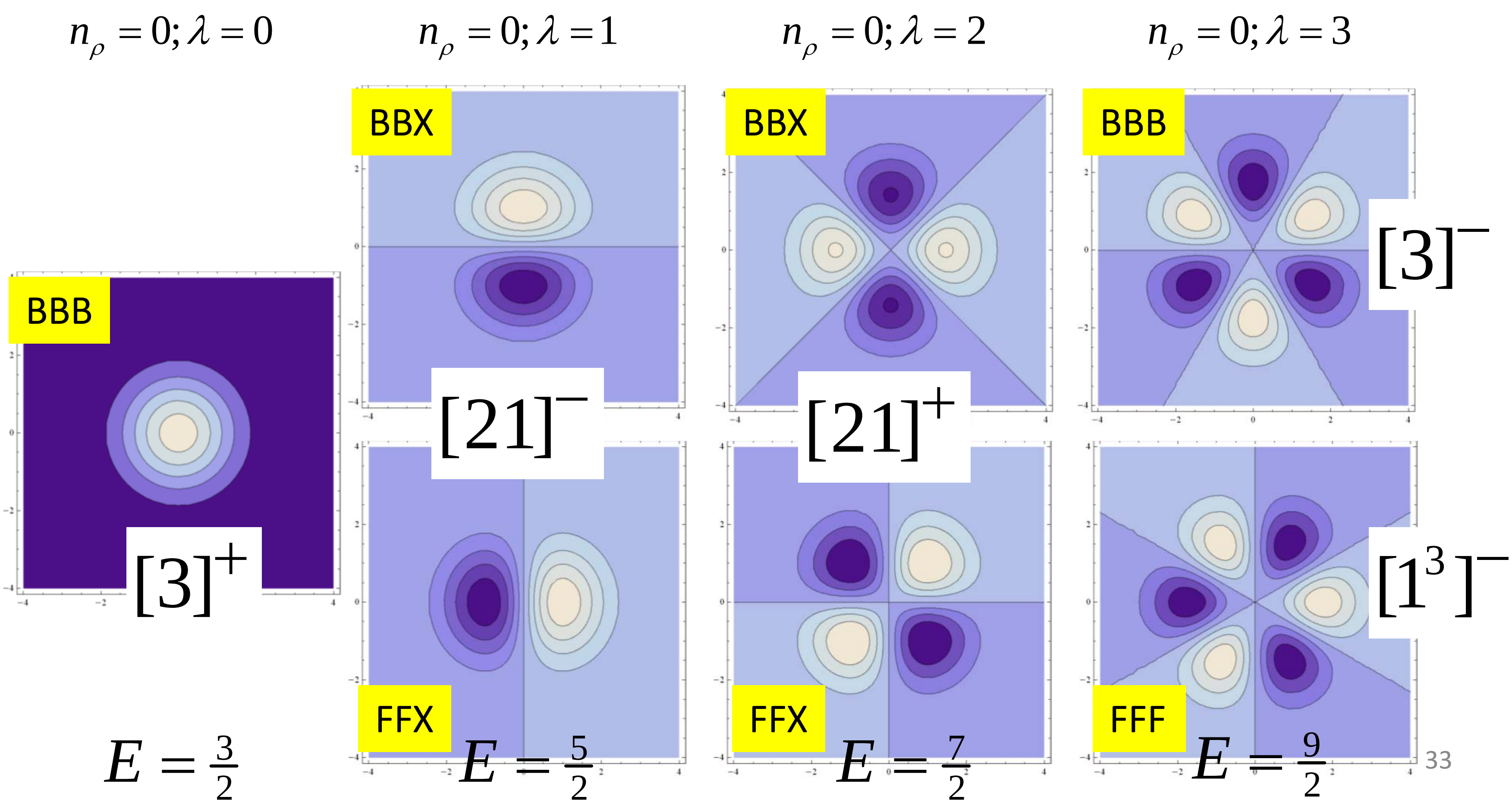
$$\Phi_{001}(x_1, x_2, x_3) \Rightarrow \begin{cases} \frac{1}{\sqrt{3}}(\phi_1(x_1)\phi_0(x_2)\phi_0(x_3) + \phi_0(x_1)\phi_1(x_2)\phi_0(x_3) + \phi_0(x_1)\phi_0(x_2)\phi_1(x_3)) \\ 0 \end{cases}$$

$$\Phi_{012}(x_1, x_2, x_3) \Rightarrow \begin{cases} \frac{1}{\sqrt{6}}(\phi_0\phi_1\phi_2 + \phi_0\phi_2\phi_1 + \phi_1\phi_0\phi_2 + \phi_1\phi_2\phi_0 + \phi_2\phi_0\phi_1 + \phi_2\phi_1\phi_0) \\ \frac{1}{\sqrt{6}}(\phi_0\phi_1\phi_2 - \phi_0\phi_2\phi_1 - \phi_1\phi_0\phi_2 + \phi_1\phi_2\phi_0 + \phi_2\phi_0\phi_1 - \phi_2\phi_1\phi_0) \end{cases}$$

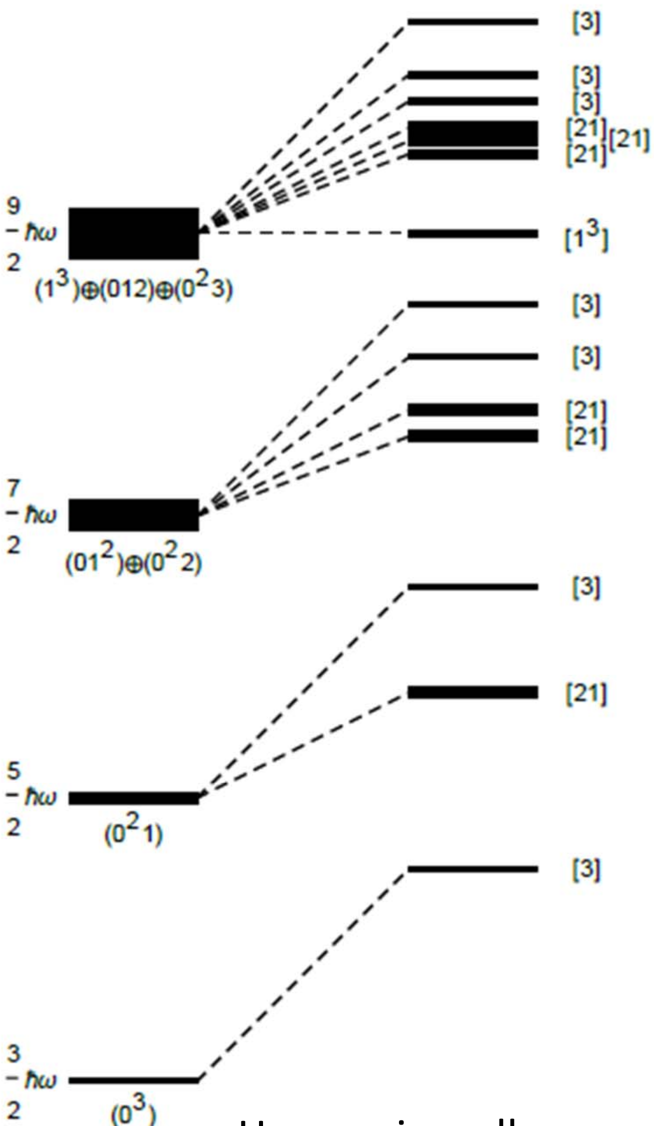
And fourothers....

Three Particles: Harmonic Trap, Not Interacting

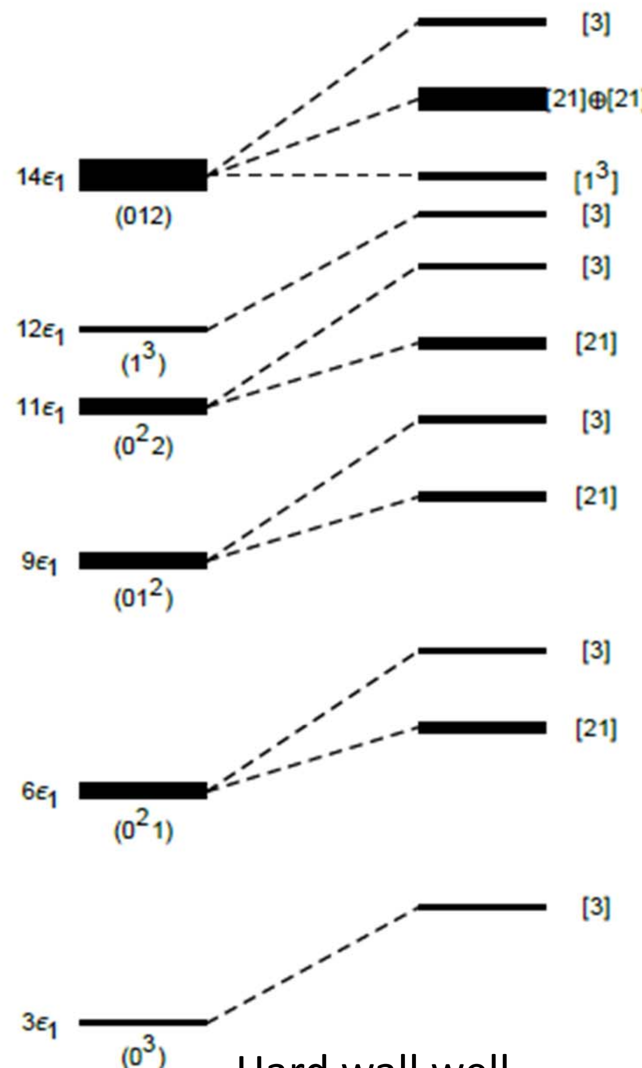
Wave functions of lowest energy for each symmetric group/relative parity



Weak Perturbation



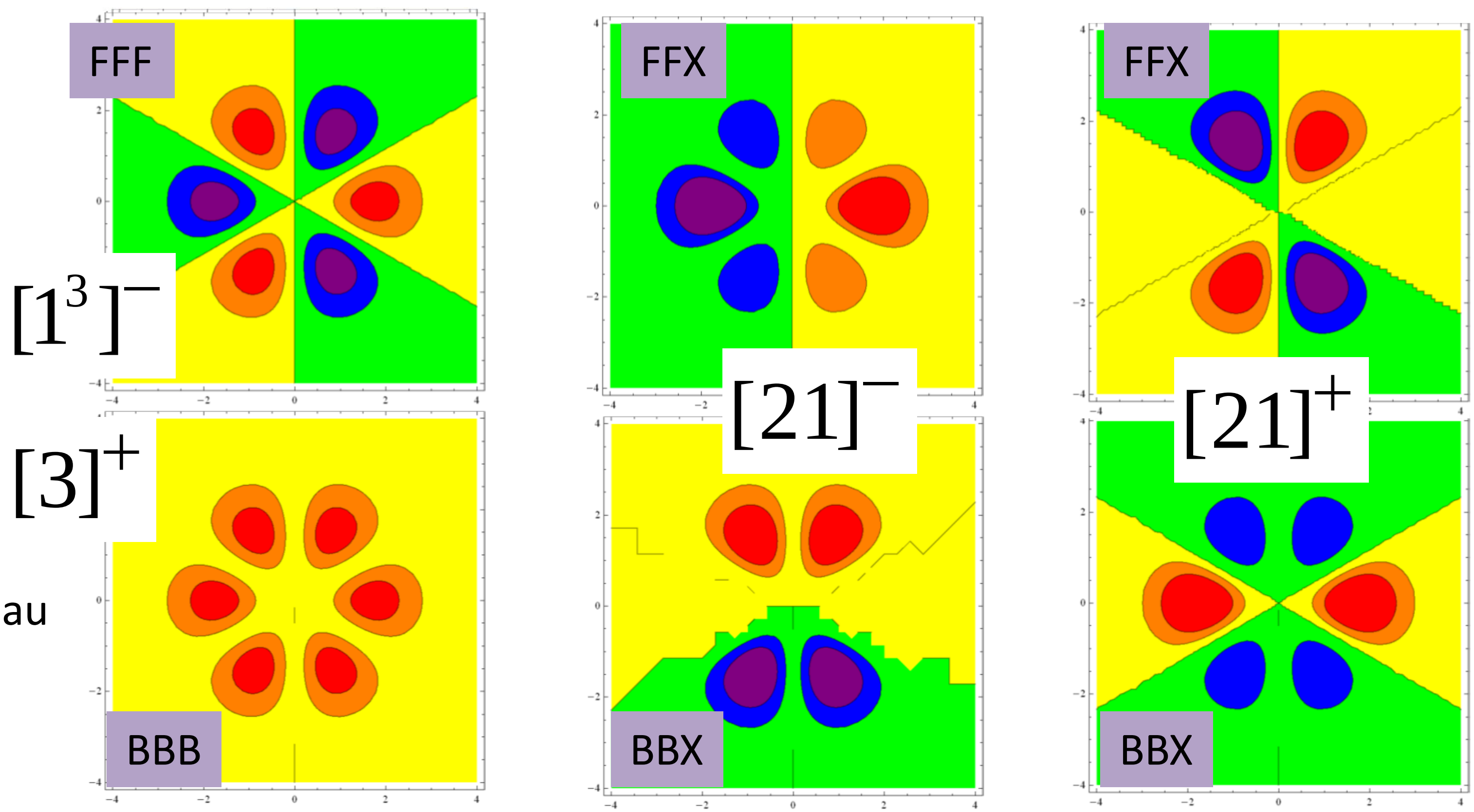
Harmonic well



Hard wall well

Three Particles: Unitary Limit

Starting from free fermionic solutions, one can build the rest

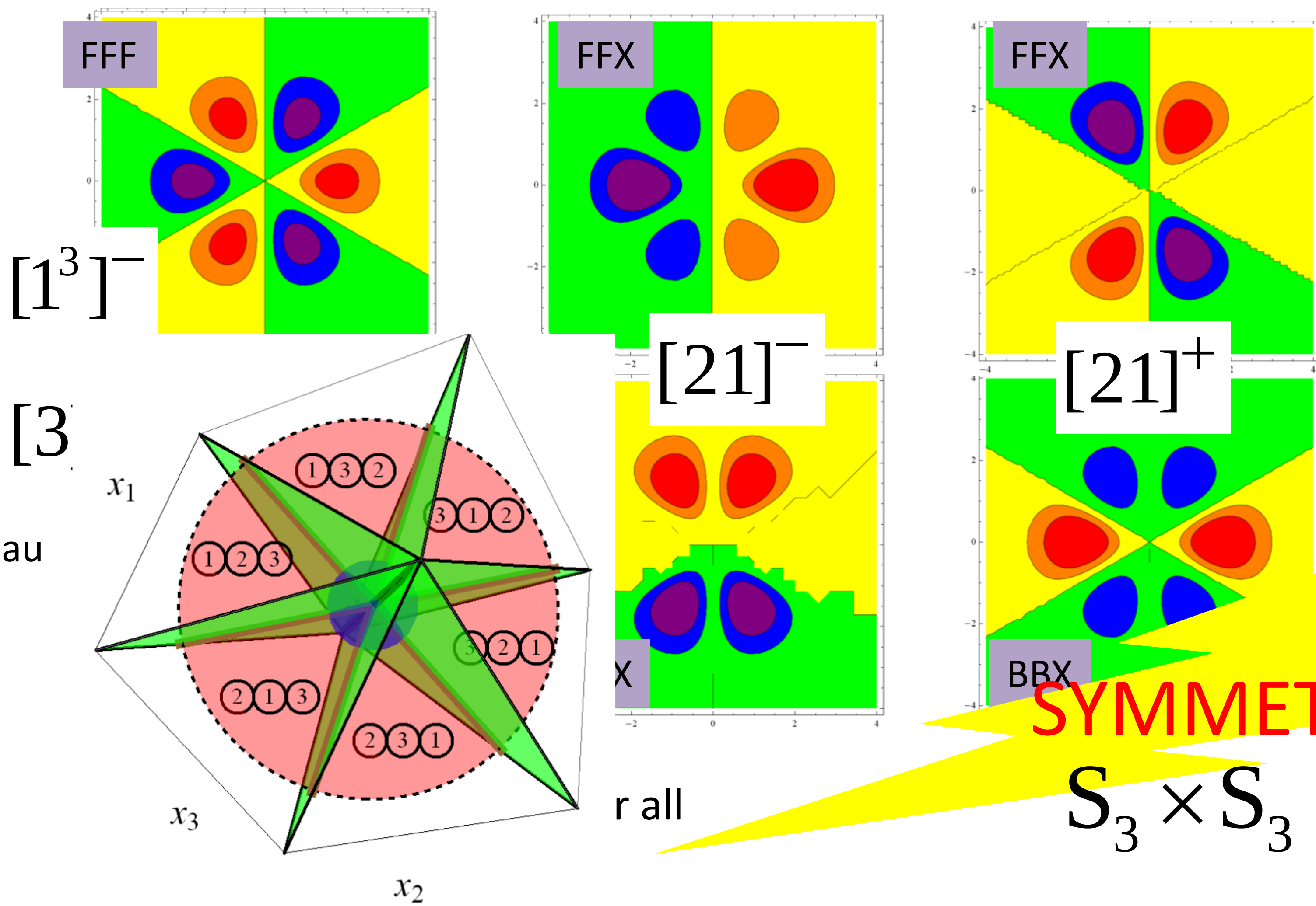


Girardeau

$E = \frac{9}{2}$ for all

Three Particles: Unitary Limit

Starting from free fermionic solutions, one can build the rest



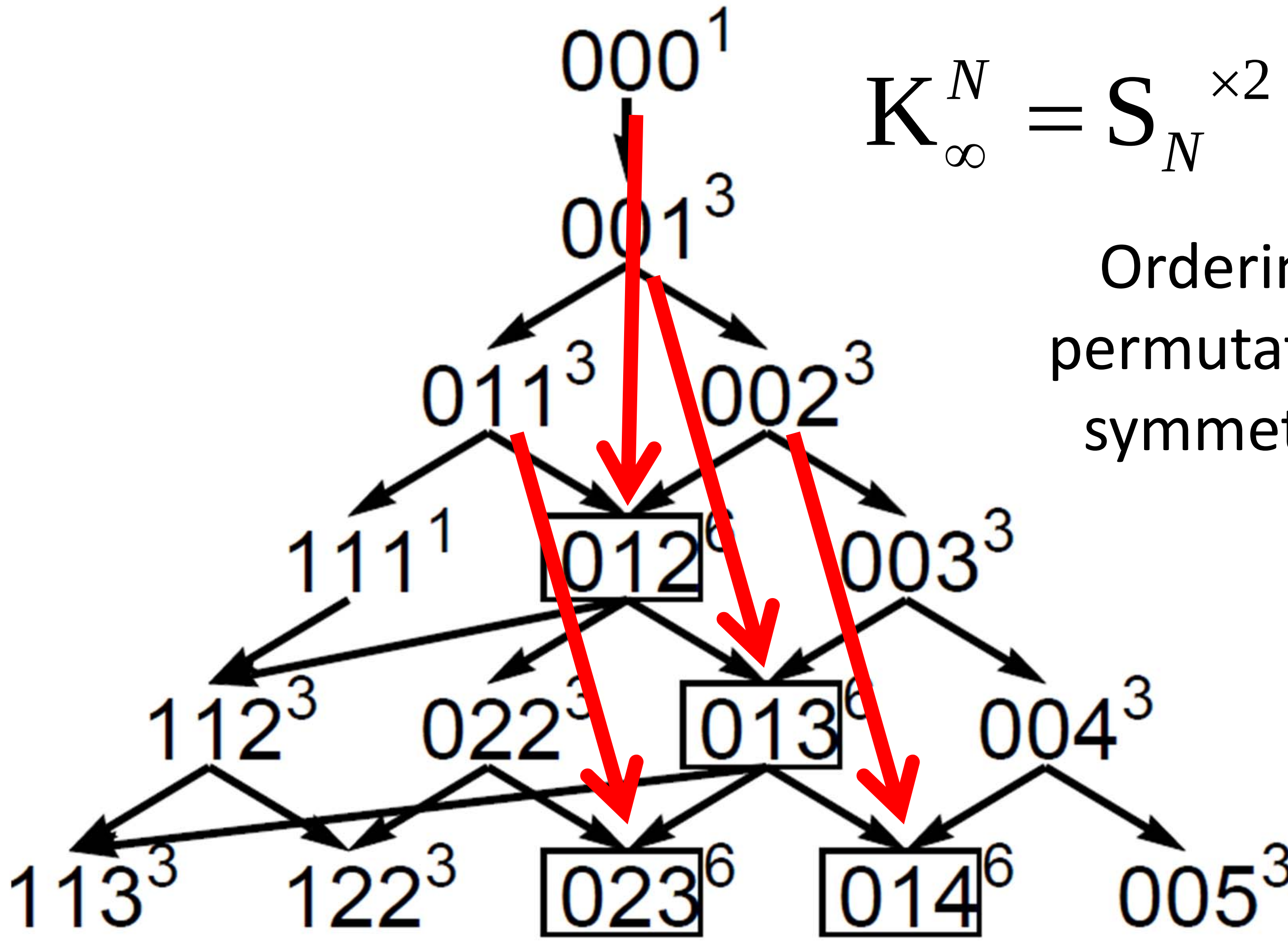
Girardeau

r all

SYMMETRY!

$S_3 \times S_3$

Bose-Fermi Mapping:
Unitary limit of contact interaction

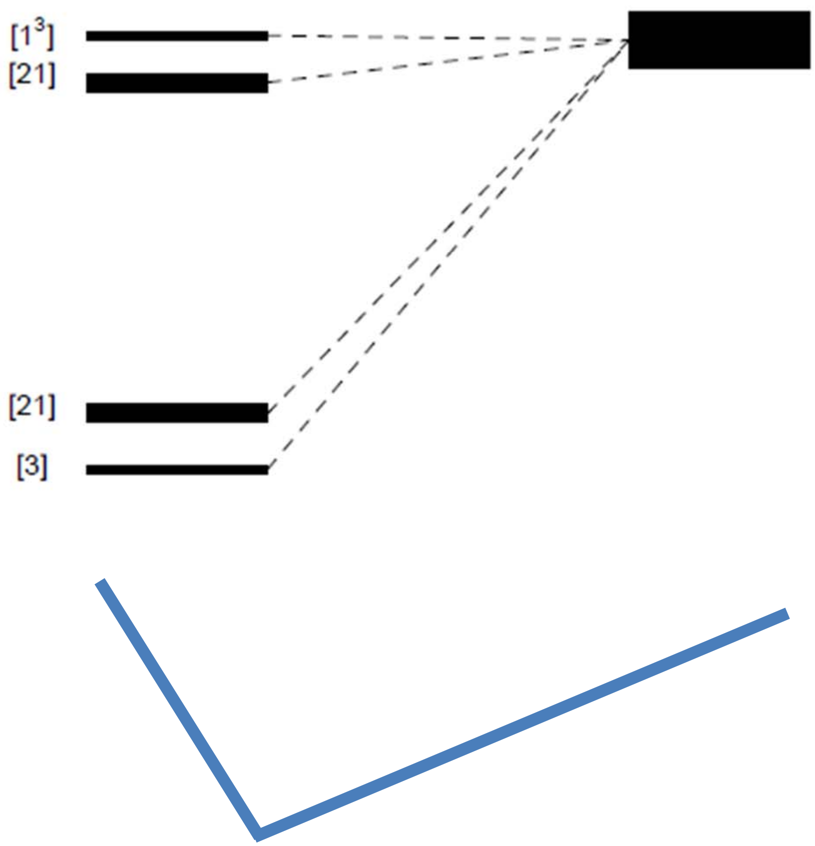


$$K_{\infty}^{N} = S_{N}^{\times 2} \times K_{1}$$

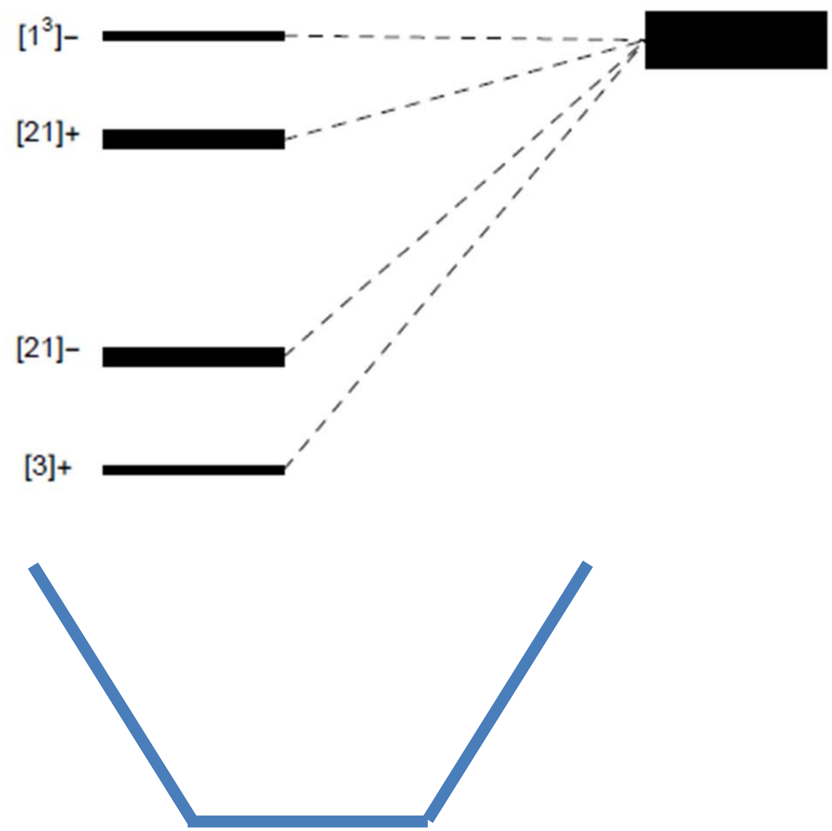
Ordering permutation symmetry

Near unitary, weak tunneling

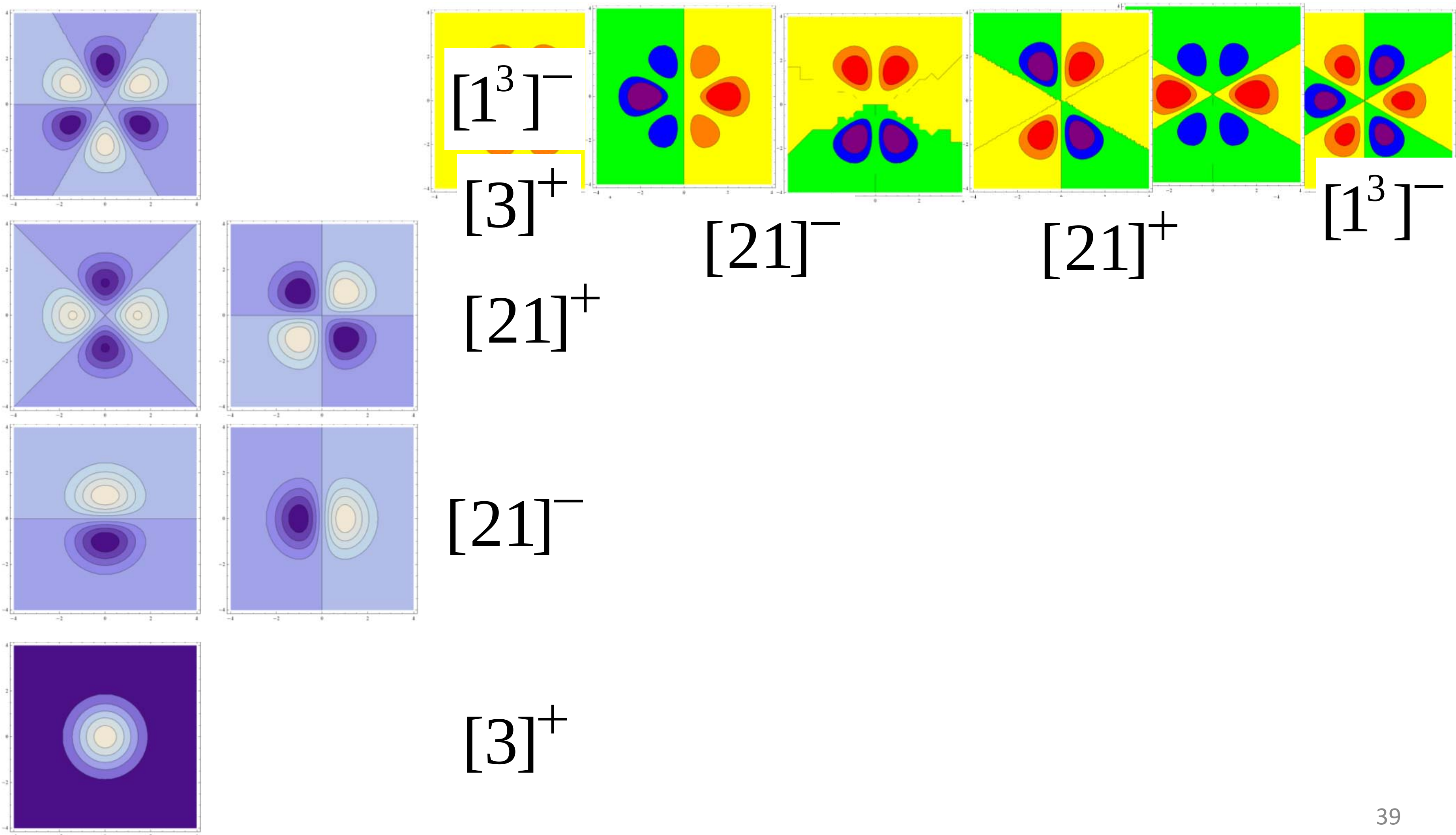
Two parameters



One parameter



Three Particle:
Symmetric trap, adiabatic mapping



Two particles

- Need least info about trap to generate algebraic solutions, splittings are linear

Three particles

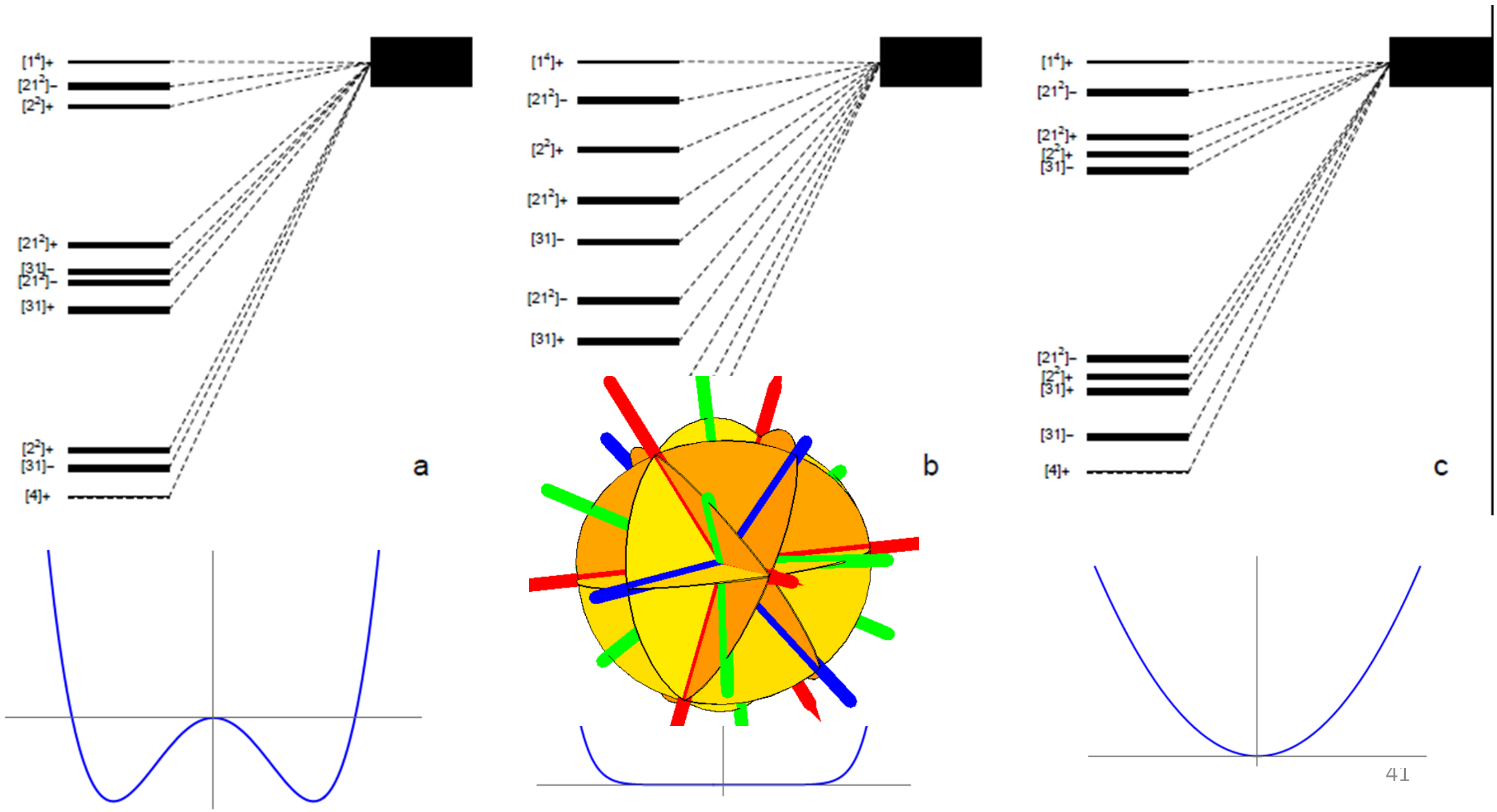
- Need a little more info, but splittings are trap-independent and quadratic at worst

Four particles

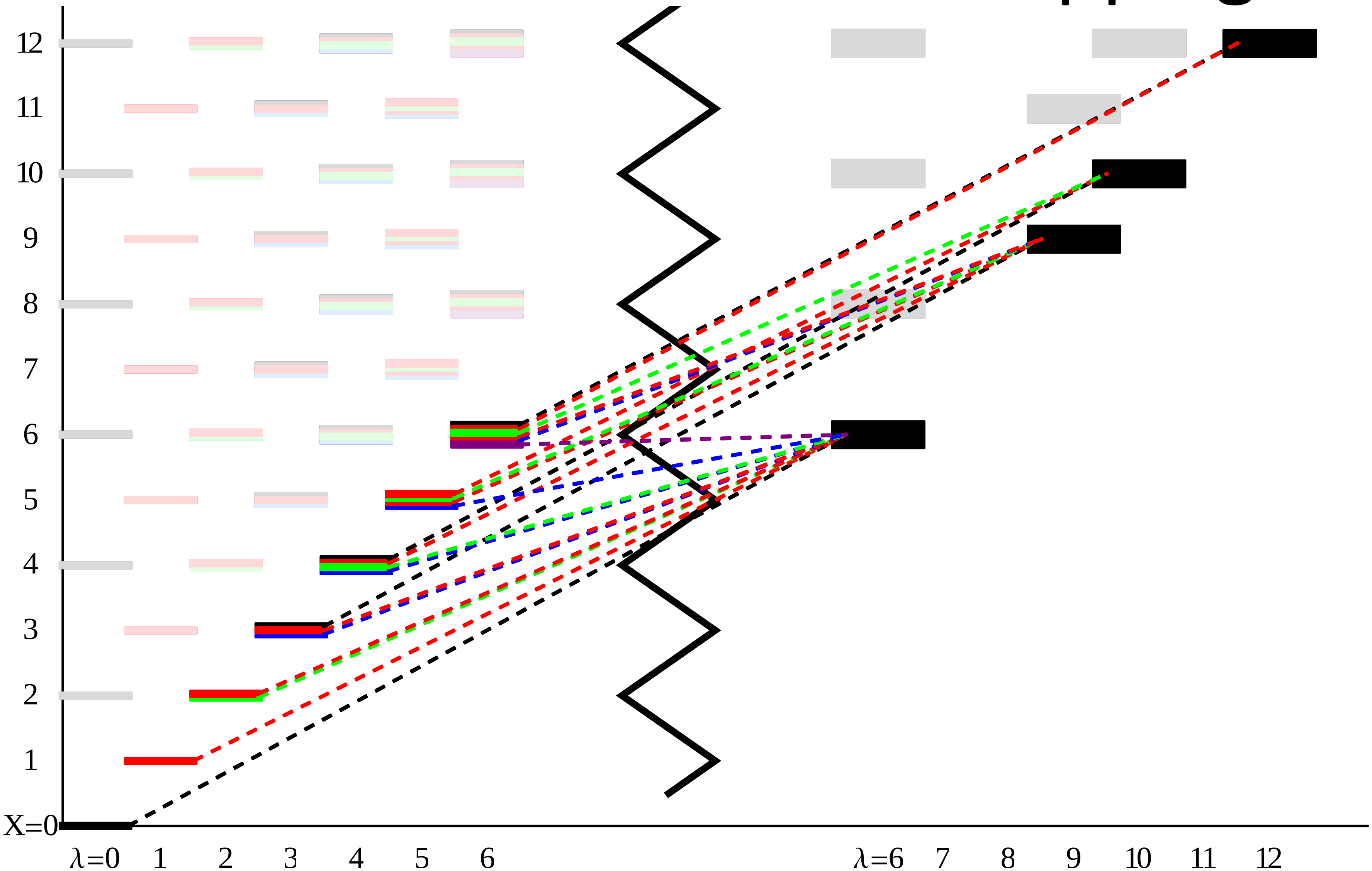
- Need even more information about non-interacting spectrum and wave functions
- Weak interactions and near-unitary interactions still algebraically solvable, but need cubic solutions for asymmetric traps

Near unitary, weak tunneling:

Two parameters for four particles in symmetric trap



Four Particle Adiabatic Mapping



Caveat emptor

Five particles?

- Meh.
- Need even more information about non-interacting spectrum and wave functions
- But even then, weak interactions not algebraically solvable
- Near-unitary interactions only algebraically solvable for symmetric traps (barely)

Conclusions

- Integrability in few body systems is rare and beautiful
  - Especially with more than five particles
- Group theoretical methods provide tools for spectroscopic analysis
  - There's a kaleidoscope in my atom trap!
- When you lose algebraic solvability, you lose practical control
  - Unusable entanglement!

Thanks: students; colleagues; Universality in Few-Body Physics Workshop at INT, UW;
DC NASA Spacegrant Consortium