Quantum Mechanics without Hilbert Space

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TRANSFORMATION. COMMUNITY. *HOLY CROSS.*

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Overview

- The talk is about an algebraic-ish view of quantum mechanics, that trades off some “completeness” for “control”.
- Can think of it as complementary to Hilbert space constructions

OR

- As more basic (primary) than a Hilbert space representation
- This perspective helps me think about the Problem of Time (if time permits)
- I will showcase the power of this view-shift in the semiclassical regime

Motivation from quantum dynamics

States evolve

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \hat{H}|\psi\rangle$$

Operators evolve

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{A} = \left[\hat{A}, \hat{H}\right]$$

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Either way, expectation values obey

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi|\hat{A}|\psi\rangle = \frac{1}{i\hbar}\langle\psi|\left[\hat{A}, \hat{H}\right]|\psi\rangle$$

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This last equation can be viewed as an ODE on the function on the space of quantum states $\left\langle\hat{A}\right\rangle(|\psi\rangle) := \langle\psi|\hat{A}|\psi\rangle$

Since RHS depends on a different function $\left\langle\left[\hat{A},\hat{H}\right]\right\rangle(|\psi\rangle)$ we need additional equations to complete the system.

Quantum dynamics as a system of ODEs

- Suppose $\left[\hat{A},\hat{H}\right]=i\hbar\hat{A}_1$, $\left[\hat{A}_1,\hat{H}\right]=i\hbar\hat{A}_2$, etc.
- Then get an infinite system of coupled ODEs in functions $\left\langle\hat{A}\right\rangle, \left\langle\hat{A}_1\right\rangle, \left\langle\hat{A}_2\right\rangle \ldots$ which can be solved in principle

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left\langle\hat{A}\right\rangle = \left\langle\hat{A}_1\right\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\left\langle\hat{A}_1\right\rangle = \left\langle\hat{A}_2\right\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\left\langle\hat{A}_2\right\rangle = \ldots \\ \ldots \end{cases}$$

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<u>Some problems:</u>

- $\hat{A}$, $\hat{A}_1$, $\hat{A}_2$, …, may not be well defined (domain issues)
- When viewed as functions on the Hilbert space, $\left\langle \hat{A} \right\rangle, \left\langle \hat{A}_1 \right\rangle, \left\langle \hat{A}_2 \right\rangle$ … are not continuous

Need a slight change of perspective to make this work.

Main idea of the approach

- Work with some sufficiently large algebra of "observables"

$\mathcal{A}$ = unital, associative, complex *-algebra.

- ***Not the algebra of all possible operators on Hilbert space, rather a "quantization" of some classical algebra of functions, that resolve phase-space points.***

- <u>For example</u>, for a particle in 1-D, can take $\mathcal{A}$ to contain complex polynomials in $\hat{x}$ and $\hat{p}$, subject to $[\hat{x}, \hat{p}] = i\hbar\mathbf{1}$
- Let Γ_{Q} be the space of all complex-linear functionals on $\mathcal{A}$
- For example, a state $|\psi\rangle$ in any representation of $\mathcal{A}$ that is in the domain of all its elements induces a linear functional on $\mathcal{A}$ via

$$\omega\left(\hat{A}\right) := \langle\psi|\hat{A}|\psi\rangle$$

What kind of a space is Γ_{Q}?

- Not a Hilbert space, all of the following are included: $\langle\psi|\hat{A}|\psi\rangle$, $\mathrm{Tr}\left(\hat{\rho}\hat{A}\right)$, $\langle\psi|\hat{A}|\phi\rangle$, and other operations.
- It is a vector space $(\omega_1 + \omega_2)\left(\hat{A}\right) = \omega_1\left(\hat{A}\right) + \omega_2\left(\hat{A}\right)$
- Assume we restrict to normalized "states" where $\omega\left(\hat{\mathbf{1}}\right) = 1$ work with an affine subspace $(\omega_1 - \omega_2)\left(\hat{\mathbf{1}}\right) = 0$
- The states are not in general positive, typically need to impose $\omega\left(\hat{A}\hat{A}^*\right) \geq 0$ (for real spectra of observables)
- Options for linking to Hilbert space:

  - Treat Γ_{Q} as an auxiliary tool, while working with a specific Hilbert space, impose relevant restrictions on states

  - Any positive linear functional can be used to construct a Hilbert space representation (GNS construction)

Structure of Quantum Phase space Γ_{Q}

- Each $\hat{A} \in \mathcal{A}$ naturally corresponds to a function $\left\langle \hat{A} \right\rangle$ on Γ_{Q} (due to vector space duality between the two)

$$\left\langle \hat{A} \right\rangle (\omega) := \omega\left(\hat{A}\right)$$

- These functions are linear and hence continuous (relative to the vector space topology on Γ_{Q})

- They resolve points of Γ_{Q} (again due to duality)
- Analogy with classical mechanics goes further as Γ_{Q} is equipped with a Poisson bracket on these functions

$$\left\{\left\langle \hat{A} \right\rangle, \left\langle \hat{B} \right\rangle\right\}_{\mathrm{Q}} := \frac{1}{i\hbar}\left\langle \left[\hat{A}, \hat{B}\right]\right\rangle$$

- Extends to (all) other functions on Γ_{Q} (through requiring linearity in both arguments and the “product rule”)

All of this is directly relevant to quantum time evolution.

Analogue of Schrödinger equation on Γ_Q

- A distinguished Hamiltonian element $\hat{H}$ in $\mathcal{A}$ creates a one-parameter flow on Γ_Q, such that

  states evolve according to $$\frac{\mathrm{d}}{\mathrm{d}t}\omega\left(\hat{A}\right) := \frac{1}{i\hbar}\omega\left(\left[\hat{A},\hat{H}\right]\right)$$

- Along this flow functions on Γ_Q evolve as

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \left\{f,\left\langle\hat{H}\right\rangle\right\}_Q$$

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  (so long as $\hat{H},\ \hat{A},\ \hat{A}_1,\ \hat{A}_2,\ \ldots \in \mathcal{A}$)

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- Can it ever be solved? Yes, e.g. if decouples into finite subsystems, OR if can be perturbatively truncated.

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***Note:* all of this carries over if we replace $\hat{H}$ by e.g. a symmetry action generator.**

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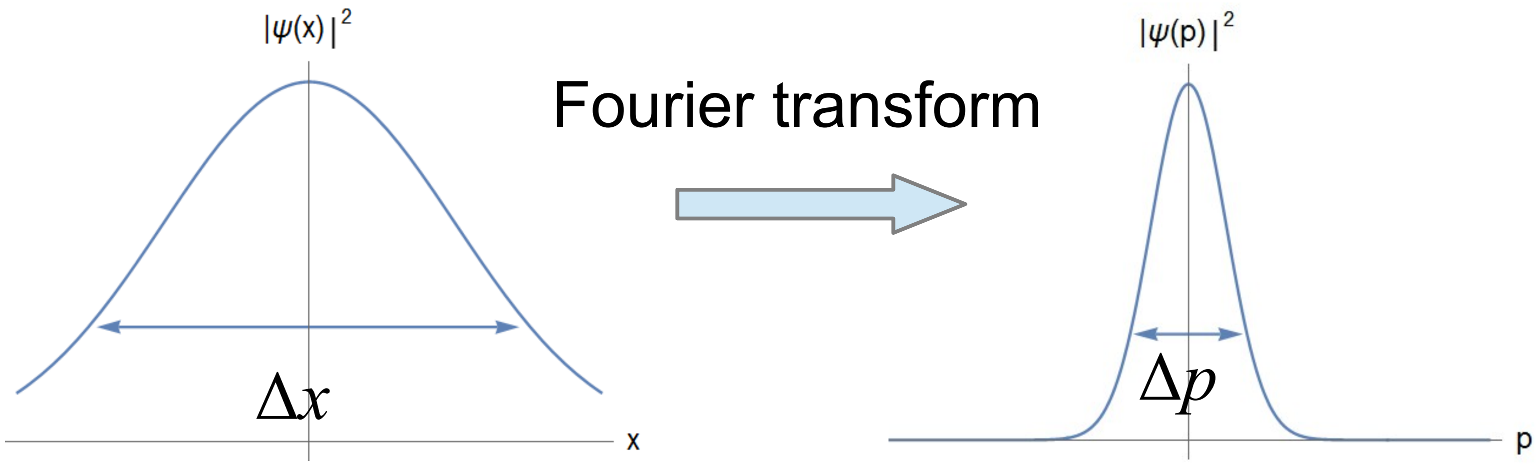
Application: Expansion in a semiclassical state

(for finitely generated algebra $\mathcal{A}$)

- <u>One definition:</u> "***a state "sharply peaked" about some point in the classical phase space.***"
- E.g. for a 1-D particle, a Gaussian can be narrow in both position and momentum



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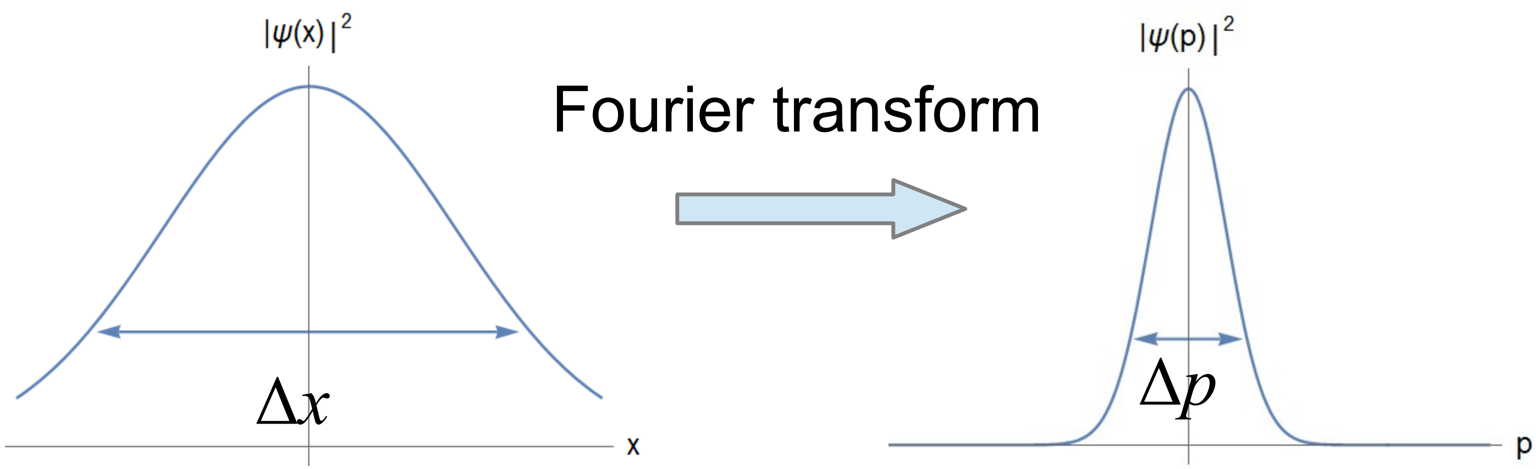

$$\Delta x \, \Delta p \geq \hbar/2$$

- Select a "small" scale R_x for x and R_p for p, with $R_x R_p = \hbar/2$ and work with re-scaled quantities $\bar{x} := \frac{x}{R_x}\sqrt{\hbar/2}$ and $\bar{p} := \frac{p}{R_p}\sqrt{\hbar/2}$, which have units of $\sqrt{\hbar}$
- So, there are Gaussians with $\Delta\bar{x}$ and $\Delta\bar{p} \sim \sqrt{\hbar}$

Semiclassical perturbation theory

Δx and Δp aren't the only parameters characterizing the spreading of the state. Define ***generalized moments***:

$$\Delta\left(x^a p^b\right) := \left\langle (\hat{x} - \langle \hat{x} \rangle)^a \, (\hat{p} - \langle \hat{p} \rangle)^b \right\rangle_{\mathrm{Weyl}}$$

Terms inside are totally symmetrized

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Some useful properties:

- Specifying $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$ and moments is exactly equivalent to specifying values assigned to e.g. $\left\langle \hat{x}^a \hat{p}^b \right\rangle$ → ***specifies state***
- Neat relation to values assigned to symmetrized monomials

$$\left\langle \hat{x}^a \hat{p}^b \right\rangle_{\text{Weyl}} = \langle\hat{x}\rangle^a \langle\hat{p}\rangle^a \; + \; \sum_{n=0}^{a} \sum_{m=0}^{b} \frac{1}{a!b!} \frac{\partial^{n+m}(x^a p^b)}{\partial x^n \partial p^m}\Bigg|_{\substack{x = \langle\hat{x}\rangle \\ p = \langle\hat{p}\rangle}} \Delta\left(x^n p^m\right)$$

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In a semiclassical state assume $\Delta\left(x^a p^b\right) \propto \hbar^{\frac{a+b}{2}}$

Are there actual states like that?

→ Yes! e.g. a Gaussian

$$\Delta\left(x^a p^b\right) = \begin{cases} \frac{a!b!}{2^{a+b}\left(\frac{a}{2}\right)!\left(\frac{b}{2}\right)!}\hbar^{\frac{(a+b)}{2}}, & \text{for even } a \text{ and } b \\ 0, & \text{otherwise} \end{cases}$$

Truncation of dynamics

So, quantum time evolution equations look something like this

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}\rangle = \left\{\langle\hat{x}\rangle, \left\langle\hat{H}\right\rangle\right\}_Q \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}\rangle = \left\{\langle\hat{p}\rangle, \left\langle\hat{H}\right\rangle\right\}_Q \\ \frac{\mathrm{d}}{\mathrm{d}t}\Delta\left(x^2\right) = \left\{\Delta\left(x^2\right), \left\langle\hat{H}\right\rangle\right\}_Q \\ \ldots \end{cases}$$

Now we truncate the system by dropping all terms of order above some *N*, where $\mathrm{Order}\left(fg\hbar^n\right) := \mathrm{Order}(f) + \mathrm{Order}(g) + 2n$

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Key result: Order of $\{f, g\}_Q$ is at least as large as the order of f ***and*** the order of g. As a result:

- Truncated dynamical system is always finite
- Bracket $\{f, g\}_Q$ can be evaluated before ***or*** after truncation

Can use truncated Hamiltonian!

e.g. truncate above order 3

$$\left\langle\hat{H}\right\rangle = H_{\mathrm{class}} + H^{(2)} + H^{(3)} + H^{(4)} + \dots$$

(Main results go through for any finitely-generated algebra. [arXiv:1410.0704])

Example: $\hat{H} = \sqrt{\hat{p}_\alpha^2 + \hat{\alpha}^2}$ (M. Bojowald, A.T. [arXiv:0911.4950])

- Originally motivated by a cosmological model
- Here $\hat{\alpha}$ and $\hat{p}_\alpha$ are a canonical pair subject to $[\hat{\alpha}, \hat{p}_\alpha] = i\hbar$
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Solve using two methods:

I. **Standard QM**: use eigenstates of harmonic oscillator with square-root eigenvalues $\hat{H}|\phi_n\rangle = \sqrt{\hat{E}}|\phi_n\rangle = \sqrt{\lambda_n}|\phi_n\rangle$

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II. **Semiclassical truncation**:

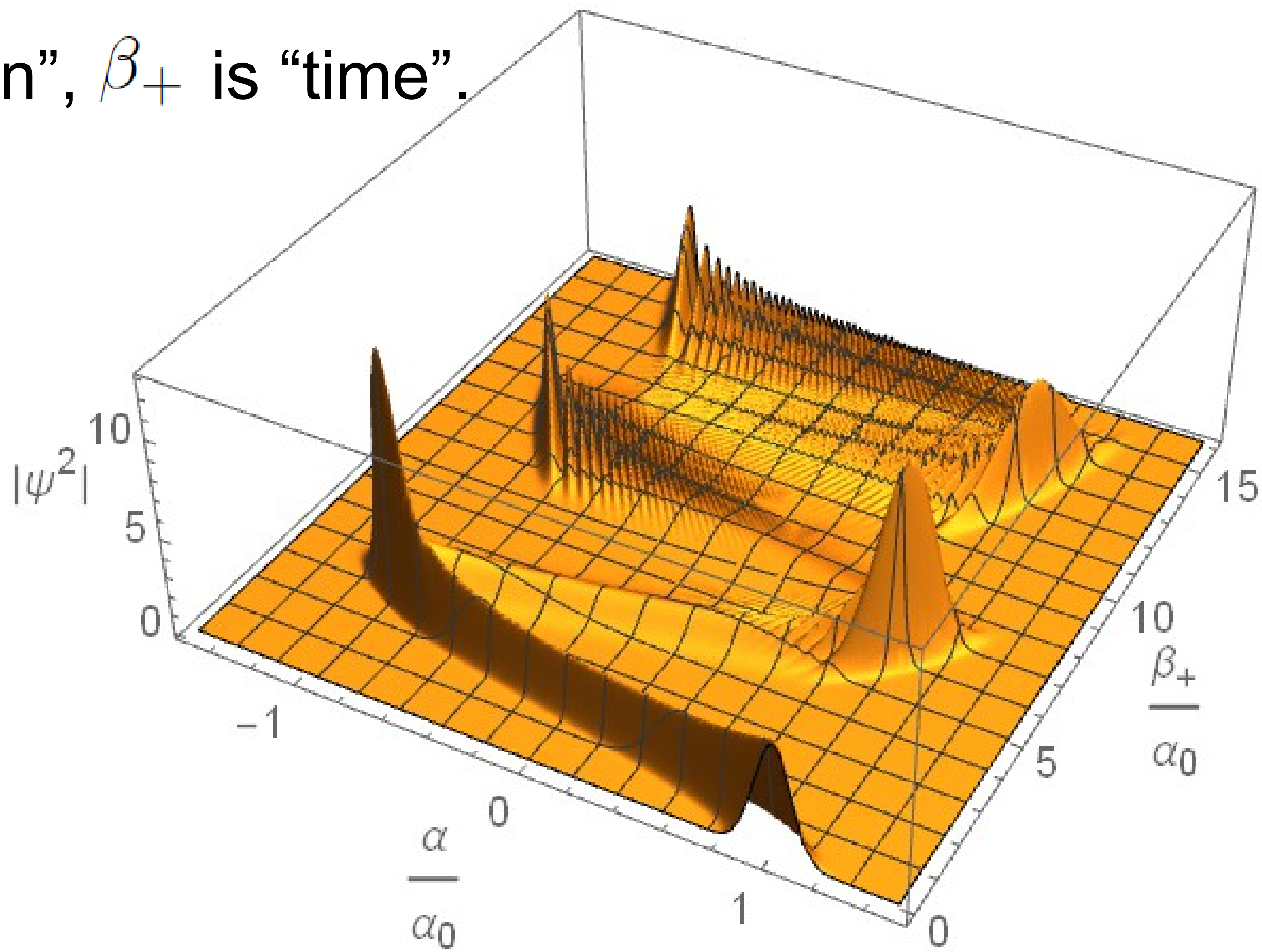
- Truncate at order 2, retain $\langle\hat{\alpha}\rangle, \langle\hat{p}_\alpha\rangle$, spreads, and covariance
- Expand $\langle\hat{H}\rangle = \left\langle\sqrt{\hat{E}}\right\rangle = \left\langle\sqrt{\langle\hat{E}\rangle + \widehat{\Delta E}}\right\rangle$ in $\langle\widehat{\Delta E}^n\rangle$, truncate
- Use truncated $\langle\hat{H}\rangle$ to compute dynamical equations via $\{,\}_Q$

Example: $\hat{H} = \sqrt{\hat{p}_\alpha^2 + \hat{\alpha}^2}$ (M. Bojowald, A.T. [arXiv:0911.4950])

- Evolve an initially Gaussian wavefunction

$$|\psi(\beta_+ = 0)\rangle = \sum_{n=0}^{\infty} \exp\left(-\frac{|z|^2}{2}\right) \frac{z^n}{\sqrt{n!}} |\phi_n\rangle$$

- Here α is "position", β_+ is "time".



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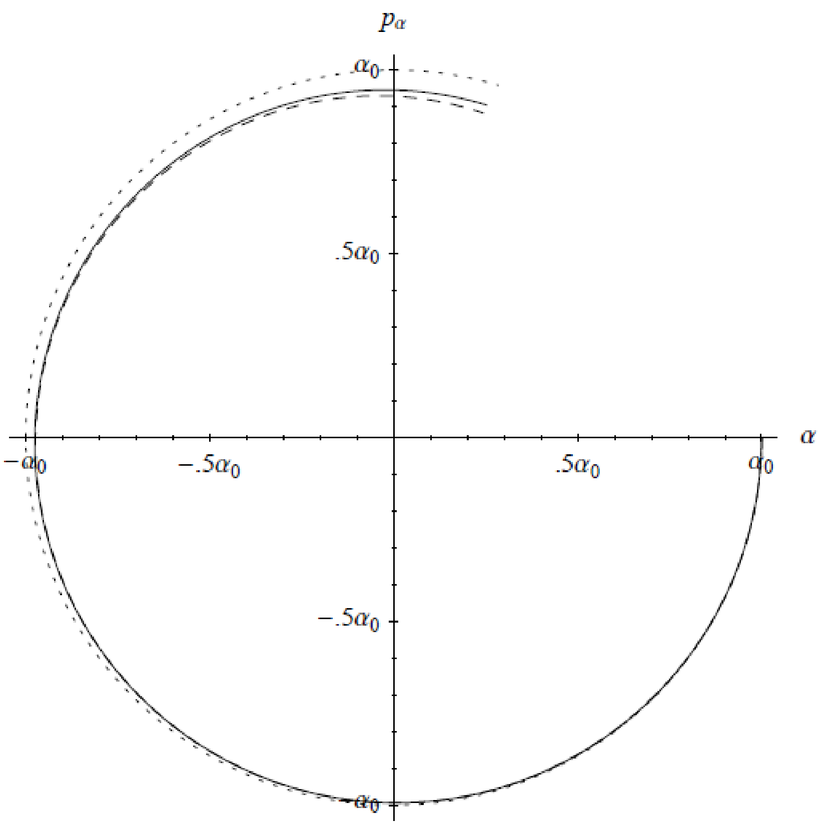
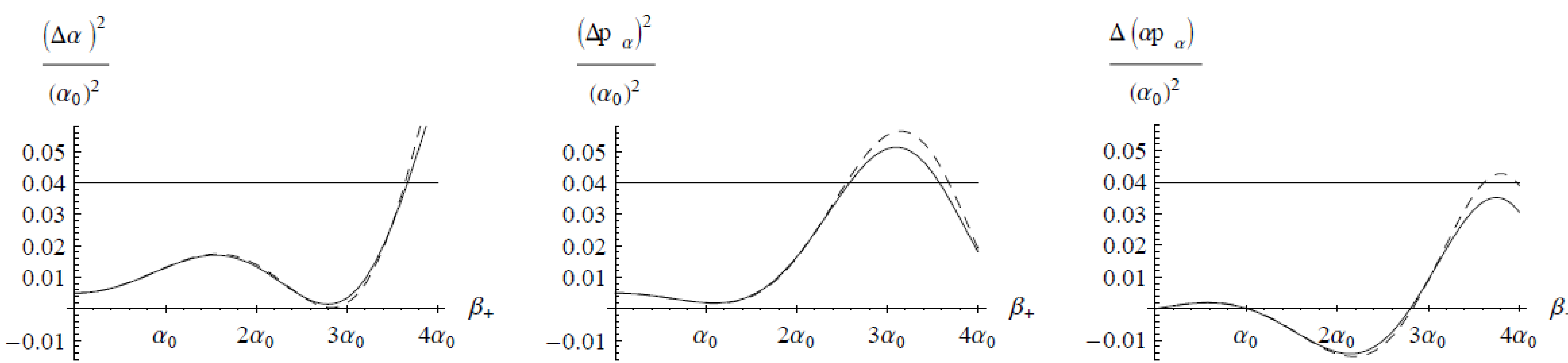


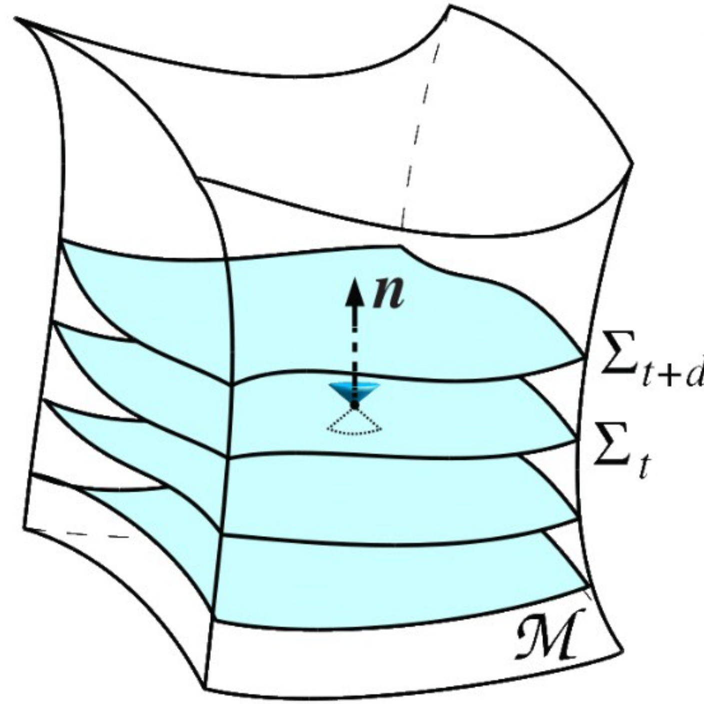
Evolve initially Gaussian state

Left: Classical (dotted), wavefunction (solid) and effective (dashed) phase space trajectories, evolved for $0 \leq \beta_+ \leq 5\alpha_0$

Below: wavefunction (solid) and effective (dashed) evolution of second order moments



The Problem of Time



- In Hamiltonian formulation of GR, energy is restricted to vanish (c.f. Sean's talk) ↔ time-reparameterization invariance.
- In quantum terms there is a constraint

$$\hat{C}_H|\psi\rangle = 0$$

- But what about $\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle$?
- Where did time-evolution go?

Idea: “Time is relations between co-evolving degrees of freedom.”

How can this work out in quantum theory?

The Parable of the Parameterized Particle

- Schrödinger equation of a quantum particle

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t)$$

- Can be re-written as a "Hamiltonian constraint"

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)\psi(x,t) = 0$$

- Solutions are the same, but...

The Parable of the Parameterized Particle

- Schrödinger equation of a quantum particle

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A state on $L^2(\mathbb{R}, \mathrm{d}x)$

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$-\hat{p}_t$

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- Solutions are the same, but... ***there are important subtleties!***

So, going from constraint to time-evolution involves demoting one observable DoF to a parameter.

Can this be done for a more general $\hat{C}_H|\psi\rangle = 0$?

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- **Notice** the solution is not normalizable on $L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}t)$

  $\rightarrow$ it is a distribution (a "bra").

- But... it can still be used to assign values to $\hat{x}, \hat{p}, \hat{t}, \hat{p}_t$

  if paired with a suitable "ket" $\rightarrow$ defines a "state" on the polynomial algebra generated by $\hat{x}, \hat{p}, \hat{t}, \hat{p}_t$

- Γ_{Q} ***already contains solutions to the Hamiltonian constraint!***

Thank you!