WKB theory in phase space for simulating the weakly nonlinear dynamics of gravity waves

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Eddy – Mean-Flow Interactions in Fluids

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Using WKB theory in phase space for simulating the weakly nonlinear dynamics of internal gravity waves

1. Gravity waves in the atmosphere
   - interaction with the mean flow
   - parameterization problem
2. WKB theory and ray-tracing
   - a coupled WKB – mean-flow system
   - the caustics problem
3. The "phase-space" WKB – mean-flow model
   - Eulerian version (finite-volume model)
   - Lagrangian version ("ray-tracer")
4. Examples
   - hydrostatic wave packet
   - modulationally-unstable (non-hydrostatic) wave packet
   - waves reflected by a shear layer

Gravity waves in the atmosphere

pre-1900 Internal gravity waves known at least since Lord Rayleigh

- *Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density* (1883)
- Waves are dispersive, with upper limit on frequency
- To support waves, density must decrease with z (static stability)

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  Plumb & McEwan 1978: theory of QBO
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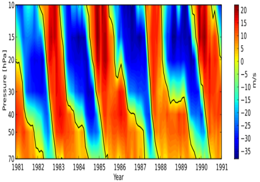
1980s Gravity wave parameterization in weather and climate models (Lindzen 1981, Holton 1982, Palmer 1986 etc.)

- generation (topography, convection, geostrophic adjustment)
- propagation (WKB theory)
- gravity wave drag



QBO

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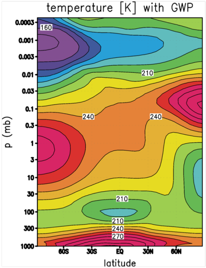

cold summer pole

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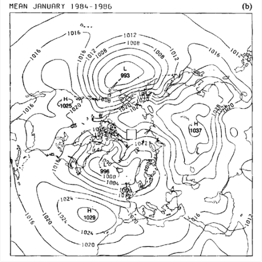
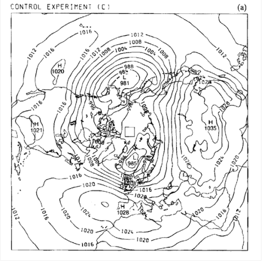


observations



no GW parameterization

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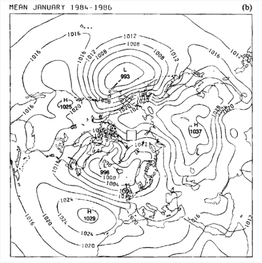

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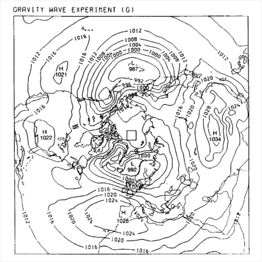

with GW parameterization

Parameterization

- Forcing due to gravity waves of the mean flow – gravity-wave "drag" (GWD) – is parameterized in weather and climate models (Lindzen 1981, Alexander & Dunkerton 1999, Warner & McIntyre 2001, Song & Chun 2008)
- A typical GWD parameterization scheme:
  - assumes a given source spectrum of waves
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  - uses linear theory and WKB theory to determine the positions of critical layers and the heights at which waves should overturn

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- A typical GWD parameterization scheme:
  - assumes a given source spectrum of waves
  - assumes an instantaneous background state of the atmosphere
  - uses linear theory and WKB theory to determine the positions of critical layers and the heights at which waves should overturn
  - neglects transience in the background and horizontal variation of background (single-column assumption)
  - neglects self-interaction (self-acceleration) of the wave field

Motivation

- Today's regional models (and even GCM: e.g. Watanabe et al. 2008) can resolve part of the gravity wave spectrum covered by parameterizations (do existing tuned sources and dissipation still apply?)
- Time scales of gravity wave group propagation not short compared to variations in background – such as solar tides (Senf & Achatz 2011)

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- Idea: to test how the waves and mean flow obtained by solving the full time-dependent WKB equations coupled to the equations for the mean flow compare with wave-resolving simulations (i.e. LES)

- ... towards an improved GWD parameterization scheme (with less tuning)
- Important consideration, for example, for use in simulations of different climate scenarios

Coupled one-dimensional wave – mean flow model

Waves

- 2D Boussinesq eqns with stratification linearized about mean wind $U(z,t)$:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \frac{\partial U}{\partial z}w' + \frac{\partial P'}{\partial x} = 0$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} - NB' + \frac{\partial P'}{\partial z} = 0$$

$$\frac{\partial B'}{\partial t} + U\frac{\partial B'}{\partial x} + Nw' = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

with:

pressure	$P = p/\rho$
stratification	$N^2(z) \equiv (g/\theta_0)\overline{\theta}_z$
buoyancy	$B' \equiv (g/N\theta_0)\theta'$

Mean-flow ("GCM")

- Mean flow forced by divergence of momentum flux associated with the waves:

$$\frac{\partial U}{\partial t} = -\frac{\partial}{\partial z}\overline{u'w'}$$

where:

$$\overline{f} \equiv \frac{1}{L}\int_0^L f(x,z,t)\mathrm{d}x$$

- N.B. Horizontal-mean vertical flux of buoyancy vanishes for a monochromatic gravity wave (N^2 independent of time)

WKB theory in one dimension

- Assume the stratification varies slowly in height and the mean wind varies slowly in height and/or time compared to the wave fields, i.e. that

$$\frac{1}{U_0}\frac{\partial U}{\partial t} \ll \frac{1}{u_0'}\frac{\partial u'}{\partial t}, \qquad \left[\frac{1}{U_0}\frac{\partial U}{\partial z}, \frac{1}{N_0}\frac{\mathrm{d}N}{\mathrm{d}z}\right] \ll \frac{1}{u_0'}\frac{\partial u'}{\partial z}$$

- Introduce "slow" time and height coordinates $\tau \equiv \epsilon t$, and $\zeta \equiv \epsilon z$ (ϵ is the scale-separation parameter)
- Assume the WKB ansatz:

$$\begin{pmatrix} u' \\ w' \\ B' \\ P' \end{pmatrix} = \mathrm{Re}\left(\sum_{j=0}^{\infty} \epsilon^j \begin{pmatrix} \hat{u}_j(\zeta,\tau) \\ \hat{w}_j(\zeta,\tau) \\ \hat{B}_j(\zeta,\tau) \\ \hat{P}_j(\zeta,\tau) \end{pmatrix} \exp\left\{i\left[kx + \frac{1}{\epsilon}\Theta(\zeta,\tau)\right]\right\}\right)$$

- Define the vertical wavenumber and frequency

$$m(\zeta,\tau) \equiv \frac{1}{\epsilon}\frac{\partial\Theta}{\partial z} = \frac{\partial\Theta}{\partial\zeta}, \quad \omega(\zeta,\tau) \equiv -\frac{1}{\epsilon}\frac{\partial\Theta}{\partial t} = -\frac{\partial\Theta}{\partial\tau}$$

- Horizontal wavenumber k is constant because the coefficients in the linear system have no explicit x-dependence.

WKB theory in one dimension

- Substitute WKB ansatz into the linear equations:

$\Rightarrow$ At $\mathcal{O}\left(\epsilon^0\right)$:

$$\hat{\omega}^2 = \frac{N^2 k^2}{k^2 + m^2}, \qquad [\hat{u}_0, \hat{w}_0, \hat{B}_0, \hat{P}_0] = a\left[-i\frac{\hat{\omega}}{k}, i\frac{\hat{\omega}}{m}, \frac{N}{m}, -i\frac{\hat{\omega}^2}{k^2}\right]$$

where $\hat{\omega} \equiv \omega - kU$ is the intrinsic frequency

The dispersion and polarization relations of plane gravity-waves with uniform N and U equal to their respective local values are satisfied at all points

$\Rightarrow$ At $\mathcal{O}\left(\epsilon^1\right)$:

$$\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial}{\partial \zeta}(c_g \mathcal{A}) = 0$$

where c_g is the group speed and $\mathcal{A} \equiv E/\hat{\omega}$ is the wave action density

Amplitude of waves evolves so as to conserve total wave action

Ray equations

- From the dispersion relation

$$\Omega_{\pm}(m, z, t) \equiv kU \pm \frac{kN}{\sqrt{k^2 + m^2}}$$

and the definitions of m and ω follow the ray equations

$$\boxed{\frac{\mathrm{d}_g \zeta}{\mathrm{d}\tau} = \left(\frac{\partial \Omega_{\pm}}{\partial m}\right)_{\zeta,\tau} \equiv c_g, \quad \frac{\mathrm{d}_g m}{\mathrm{d}\tau} = -\left(\frac{\partial \Omega_{\pm}}{\partial \zeta}\right)_{m,\tau}, \quad \frac{\mathrm{d}_g \omega}{\mathrm{d}\tau} = \left(\frac{\partial \Omega_{\pm}}{\partial \tau}\right)_{\zeta,\tau}}$$

where $\frac{\mathrm{d}_g}{\mathrm{d}t} \equiv \left(\frac{\partial}{\partial \tau}\right)_{\zeta} + c_g \left(\frac{\partial}{\partial \zeta}\right)_{\tau}$ is the time derivative along a ray.

- The wave action equation in ray form is $\boxed{\frac{\mathrm{d}_g \mathcal{A}}{\mathrm{d}\tau} = -\mathcal{A}\frac{\partial c_g}{\partial \zeta}}$
- The ray equations can be solved as an initial value problem for the evolution of the wave field on a discrete set of "ray-points"
- Challenge is to compute the divergence of the group velocity c_g using information on the irregular distribution of ray-points

Caustics

- Wave-action equation $\mathrm{d}_g\mathcal{A}/\mathrm{d}t = -\mathcal{A}\,\partial c_g/\partial\zeta$ not well-posed in the presence of caustics: where wavenumber m (and hence c_g) becomes a multi-valued function of space

Example 1: Reflection

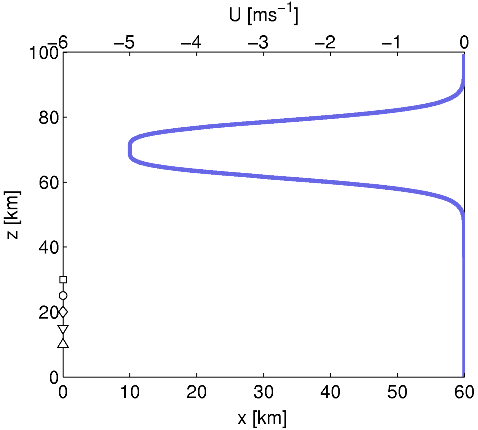
Background:

$$U(z) = -(5\ \mathrm{ms}^{-1})\ \mathrm{sech}\left[\frac{(z-z_1)^2}{(3\ \mathrm{km})^2}\right]$$

Waves:

$$k = 2\pi/(3\ \mathrm{km})$$
$$m_0 = -2\pi/(3\ \mathrm{km})$$
$$\omega = \Omega_+$$

(reflection level where $U(z) = \dfrac{\hat{\omega}_0 - N}{k}$)



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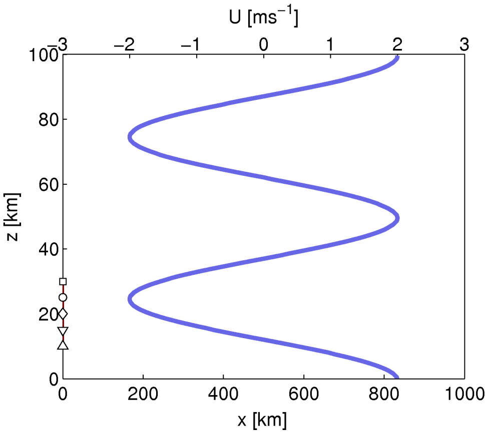
Example 2: Overtaking

Background:

$$U(z) = (2\ \mathrm{ms}^{-1}) \cos\left(\frac{2\pi z}{50\ \mathrm{km}}\right)$$

Waves:

$$k = 2\pi/(30\ \mathrm{km})$$
$$m_0 = -2\pi/(3\ \mathrm{km})$$
$$\omega = \Omega_+$$



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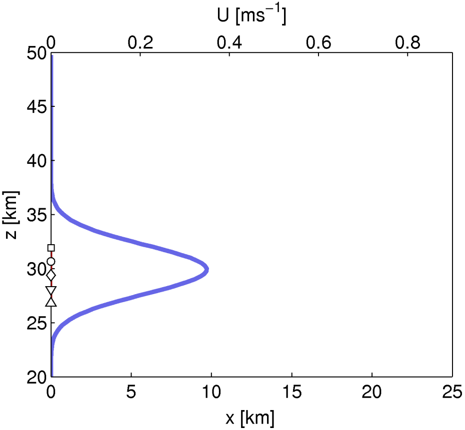
Example 3: Modulational instability

Background:
Time-dep. mean flow induced by Gaussian wave packet

$$U_0(z) = (.4\ \mathrm{ms}^{-1})\exp\left[-\frac{1}{2}\left(\frac{z-z_0}{3\ \mathrm{km}}\right)^2\right]$$

Waves:

$$k = 2\pi/(2\ \mathrm{km})$$
$$m_0 = -2\pi/(2.9\ \mathrm{km})$$
$$\omega = \Omega_+$$



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Example 4: Critical layer

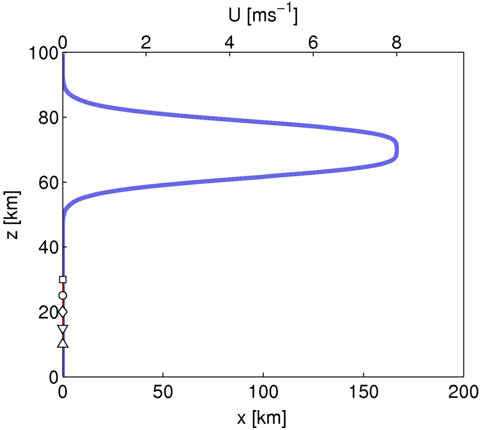
Background:

$$U(z) = (8\ \mathrm{ms}^{-1})\ \mathrm{sech}\left[\frac{(z - z_1)^2}{(3\ \mathrm{km})^2}\right]$$

Waves:

$$k = 2\pi/(3\ \mathrm{km})$$
$$m_0 = -2\pi/(3\ \mathrm{km})$$
$$\omega = \Omega_+$$

"Caustic at infinity"



Phase-space WKB model

- A solution to the caustics problem is to define a wave-action density on a phase space of position ζ and wavenumber m

$$\mathcal{N}(\zeta, m, \tau) = \int \mathrm{d}\alpha \left[\mathcal{A}_\alpha(\zeta, \tau)\delta(m_\alpha - m)\right]$$

  where each value of α corresponds to a particular WKB solution with a different $\mathcal{A}$ and m at each ζ.

- References:
  - Dewar 1970, Dubrulle & Nazarenko 1997
  - For internal waves: Bühler & McIntyre 1999, Hertzog et al. 2000
  - Weakly nonlinear coupled version: Muraschko et al. 2014
  - Related to methods used in forecasting surface waves in the ocean
- Caustics cannot occur because rays with different wavenumbers will be at different phase-space positions.

Wave action density equation

- Differentiate $\mathcal{N}$ with respect to τ, keeping m and ζ fixed:

$$\frac{\partial \mathcal{N}}{\partial \tau} = \int \mathrm{d}\alpha \left[\frac{\partial \mathcal{A}_\alpha}{\partial \tau} \delta(m_\alpha - m) + \mathcal{A}_\alpha \frac{\partial}{\partial m_\alpha} \delta(m_\alpha - m) \frac{\partial m_\alpha}{\partial t} \right]$$

- Using the identity

$$\int f(x) \frac{\partial}{\partial x} \delta(x - x_0) \mathrm{d}x = - \int f(x) \frac{\partial}{\partial x_0} \delta(x - x_0) \mathrm{d}x$$

and the ray equations, this becomes

$$\frac{\partial \mathcal{N}}{\partial \tau} = \int \mathrm{d}\alpha \left[- \frac{\partial}{\partial \zeta} (c_{g\alpha} \mathcal{A}_\alpha) \delta(m_\alpha - m) \right. \\ \left. - \mathcal{A}_\alpha \frac{\partial}{\partial m} \delta(m_\alpha - m) \left(\dot{m}_\alpha - c_{g\alpha} \frac{\partial m_\alpha}{\partial \zeta} \right) \right]$$

where

$$\dot{m}_\alpha = - \left. \frac{\partial \Omega}{\partial \zeta} \right|_{m = m_\alpha(\zeta, \tau)} \quad \text{and} \quad c_{g\alpha} = \left. \frac{\partial \Omega}{\partial m} \right|_{m = m_\alpha(\zeta, \tau)}$$

Wave action density equation

- Adding and subtracting $c_{g\alpha}\mathcal{A}_\alpha$ times the ζ derivative of the delta function in the first term in the integrand yields

$$\frac{\partial \mathcal{N}}{\partial \tau} = \int \mathrm{d}\alpha \left\{ -\frac{\partial}{\partial \zeta}\left[c_{g\alpha}\mathcal{A}_\alpha \delta(m_\alpha - m)\right] - c_{g\alpha}\mathcal{A}_\alpha \frac{\partial}{\partial m}\delta(m_\alpha - m)\frac{\partial m_\alpha}{\partial \zeta} \right.$$
$$\left. - \mathcal{A}_\alpha \frac{\partial}{\partial m}\delta(m_\alpha - m)\left(\dot{m}_\alpha - c_{g\alpha}\frac{\partial m_\alpha}{\partial \zeta}\right)\right\}$$

$$= \int \mathrm{d}\alpha \left\{ -\frac{\partial}{\partial \zeta}\left[c_{g\alpha}\mathcal{A}_\alpha \delta(m_\alpha - m)\right] - \dot{m}_\alpha \mathcal{A}_\alpha \frac{\partial}{\partial m}\delta(m_\alpha - m)\right\}$$

- Since $\dot{m}_\alpha$ and $\mathcal{A}_\alpha$ are functions of ζ and τ (and not m), they may be absorbed into the m partial derivative in the second term in the integrand, and since the integration over α commutes with both the ζ and m partial derivatives, we have

$$\frac{\partial \mathcal{N}}{\partial \tau} = -\frac{\partial}{\partial \zeta}\int \mathrm{d}\alpha \left[c_{g\alpha}\mathcal{A}_\alpha \delta(m_\alpha - m)\right] - \frac{\partial}{\partial m}\int \mathrm{d}\alpha \left[\dot{m}_\alpha \mathcal{A}_\alpha \delta(m_\alpha - m)\right]$$

Wave action density equation

- The final step is to use the identity

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\mathrm{d}x = \int_{-\infty}^{\infty} f(x_0)\delta(x-x_0)\mathrm{d}x$$

so that $c_{g\alpha}$ and $\dot{m}_\alpha$ may be replaced by $c_g(\zeta, m, \tau)$ and $\dot{m}(\zeta, m, \tau)$, both independent of α.

- We thus have, finally,

$$\frac{\partial \mathcal{N}}{\partial \tau} = -\frac{\partial}{\partial \zeta}\left\{c_g \int \mathrm{d}\alpha \left[\mathcal{A}_\alpha \delta(m_\alpha - m)\right]\right\} - \frac{\partial}{\partial m}\left\{\dot{m} \int \mathrm{d}\alpha \left[\mathcal{A}_\alpha \delta(m_\alpha - m)\right]\right\}$$

or

$$\boxed{\frac{\partial \mathcal{N}}{\partial \tau} + \frac{\partial}{\partial \zeta}(c_g \mathcal{N}) + \frac{\partial}{\partial m}(\dot{m}\mathcal{N}) = 0}$$

A conservation law for wave-action density in phase space!

Momentum flux and energy

- Horizontal mean momentum flux associated with a monochromatic wave packet may be written in terms of wave action (using the polarization and dispersion relations)

$$\overline{u'_\alpha w'_\alpha} = -\frac{N m_\alpha k}{(k^2 + m_\alpha^2)^{\frac{3}{2}}} |k \mathcal{A}_\alpha|$$

- Phase-space model assumes different spectral components do not interact with one another (except through interaction with the mean flow)
- The momentum flux is then an integral over m:

$$\overline{u'w'} = -Nk \int_{-\infty}^{\infty} \frac{m}{(k^2 + m^2)^{\frac{3}{2}}} |k\mathcal{N}| \mathrm{d}m$$

- Wave-energy density $E \equiv \frac{1}{2}(|u'|^2 + |w'|^2 + |B'|^2)$ is

$$E = \int_{-\infty}^{\infty} \hat{\omega} \mathcal{N}(\zeta, m, \tau) \mathrm{d}m = N \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 + m^2}} |k\mathcal{N}(\zeta, m, \tau)| \mathrm{d}m$$

Phase-space WKB model 1: Eulerian model

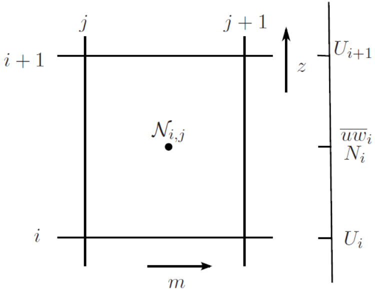
- Solves the conservation law

$$\frac{\partial \mathcal{N}}{\partial \tau} + \frac{\partial}{\partial \zeta}(c_g \mathcal{N}) + \frac{\partial}{\partial m}(\dot{m}\mathcal{N}) = 0$$

  using finite volume scheme MUSCL on 2D position-wavenumber grid
- Wave action density fluxes computed using ray equations for c_g and $\dot{m}$
- The momentum flux is

$$\overline{u'w'}_i = -\sum_j \frac{N_i m_j k}{(k^2 + m_j^2)^{\frac{3}{2}}} |k\mathcal{N}_{i,j}| \Delta_m$$

- $\partial U/\partial t$ is computed using a finite difference approximation to the spatial derivative.



Phase-space WKB model 2: Lagrangian model

- The phase-space flow $(c_g, \dot{m})$ is nondivergent:

$$\frac{\partial c_g}{\partial \zeta} + \frac{\partial \dot{m}}{\partial m} = \frac{\partial^2 \Omega}{\partial \zeta \partial m} - \frac{\partial^2 \Omega}{\partial m \partial \zeta} = 0$$

- Flow is therefore area preserving.

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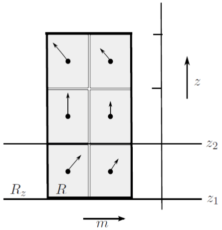
- Flow is therefore area preserving.
- "Ray tracer" solves

$$\frac{D_r \mathcal{N}}{D\tau} \equiv \frac{\partial \mathcal{N}}{\partial t} + c_g \frac{\partial \mathcal{N}}{\partial \zeta} + \dot{m} \frac{\partial \mathcal{N}}{\partial m} = 0$$

  on discrete "ray points" that move through phase space with velocity $(c_g, \dot{m})$

- The region R of nonzero $\mathcal{N}$ is approximated by rectangles.

Time t_0



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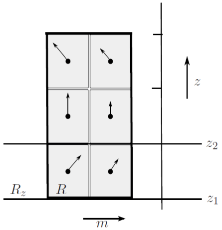
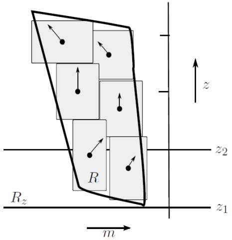
- The height and width of the rectangles change with time.

Time t_0



Time t_1



Phase-space WKB model 2: Lagrangian model

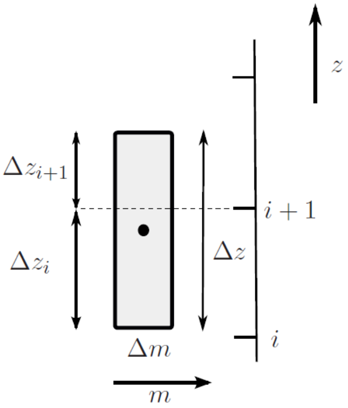
- The rectangles attached to ray particles are used to compute the momentum flux.
- Vertical mean momentum forcing in interval $z_i < \zeta < z_i + \Delta z$ is sum of contributions from all ray particles (index j):

$$\overline{u'w'}_i = -\frac{1}{\Delta z}\int_{R\cap R_i} \frac{N_i km}{(k^2+m^2)^{\frac{3}{2}}}|k\mathcal{N}|\mathrm{d}m\mathrm{d}\zeta$$

$$= -\sum_j \frac{1}{\Delta z}\int_{R_j\cap R_i} \frac{N_i km}{(k^2+m^2)^{\frac{3}{2}}}|k\mathcal{N}_j|\mathrm{d}m\mathrm{d}\zeta$$

$$= -\sum_j \left(\frac{\Delta z_i^j}{\Delta z}\right)\int_{m_{j1}}^{m_{j2}} \frac{N_i km}{(k^2+m^2)^{\frac{3}{2}}}|k\mathcal{N}_j|\mathrm{d}m$$

$$= \sum_j \left(\frac{\Delta z_i^j}{\Delta z}\right) N_i k|k\mathcal{N}_j| \left[\frac{1}{\left(k^2+m_{j2}^2\right)^{\frac{1}{2}}} - \frac{1}{\left(k^2+m_{j1}^2\right)^{\frac{1}{2}}}\right]$$



Test case: Quasimonochromatic wave packet

- Deceptively simple test case: Gaussian wave packet

$$b'(x,z,t=0) = A_b(z)\cos(kx+m_0 z)$$
$$u'(x,z,t=0) = A_b(z)\frac{m_0}{k}\frac{\hat{\omega}_0}{N_0^2}\sin(kx+m_0 z)$$
$$w'(x,z,t=0) = -A_b(z)\frac{\hat{\omega}_0}{N_0^2}\sin(kx+m_0 z)$$

where $b' = NB$ is buoyancy, m_0 is a constant, and

$$A_b(z) = a_0\frac{N_0^2}{m_0}\exp\left[-\frac{(z-z_0)^2}{2\sigma^2}\right]$$

- The waves are statically stable for $|a_0| < 1$.
- Initialization of phase-space wave-action density

$$\mathcal{N}(m,z,t=0) = \begin{cases} \dfrac{A_b^2(z)}{2N_0^2\hat{\omega}_0}\dfrac{1}{\Delta m_0}\ , & \text{for } m_0 - \frac{1}{2}\Delta m_0 < m < m_0 + \frac{1}{2}\Delta m_0 \\ 0\ , & \text{otherwise} \end{cases}$$

Example 1: Hydrostatic wave packet

Background:

Uniform stratification

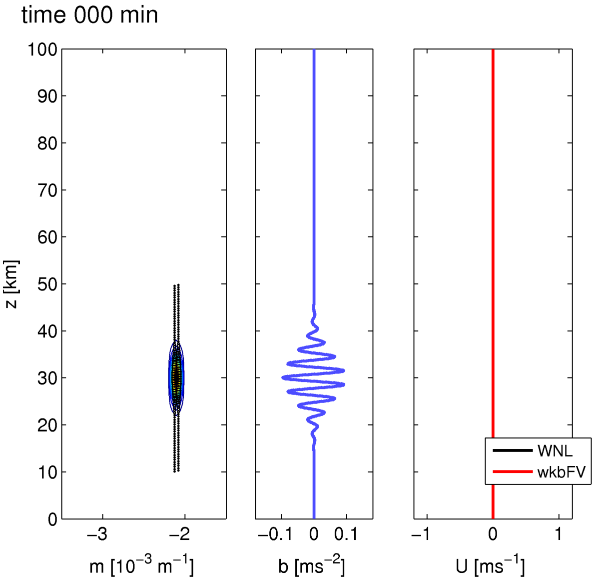
$$N = N_0 = 0.02\ \mathrm{s}^{-1}$$

No initial mean flow

$$U(t_0) = 0$$

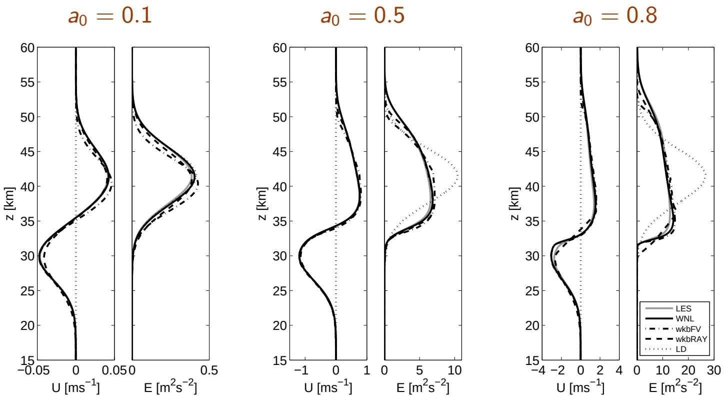
Waves:

$$k = 2\pi/(3\ \mathrm{km})$$
$$m_0 = -2\pi/(30\ \mathrm{km})$$
$$\omega = \Omega_+$$
$$a_0 = 0.1, \underline{0.5}, 0.8$$



Example 1: Hydrostatic wave packet

- Wave energy and induced mean flow at 200 min. for different amplitudes:



- WKB models (FV and RAY) compare well with weakly nonlinear wave-resolving model (WNL) and fully nonlinear model INCA (LES)
- Dotted line is linear solution without feedback on mean flow

Example 2: Refraction by a variable stratification

Background:

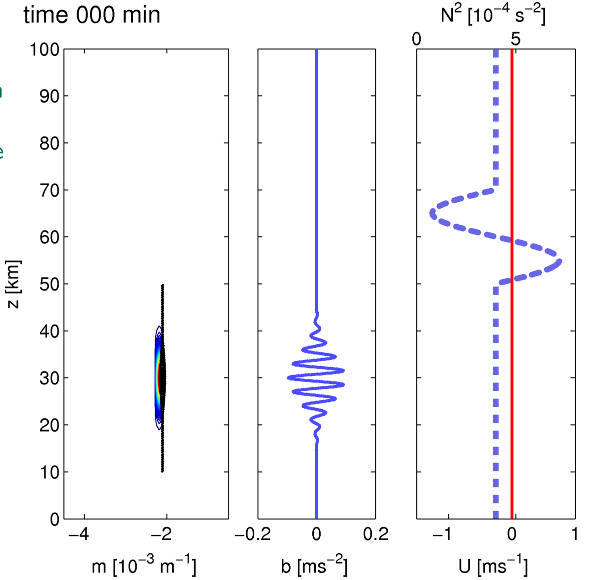
sinusoidal perturbation to mean buoyancy between 50 km and 70 km; $N = N_0 = 0.02\,\mathrm{s}^{-1}$ elsewhere

No initial mean flow

$$U(t_0) = 0$$

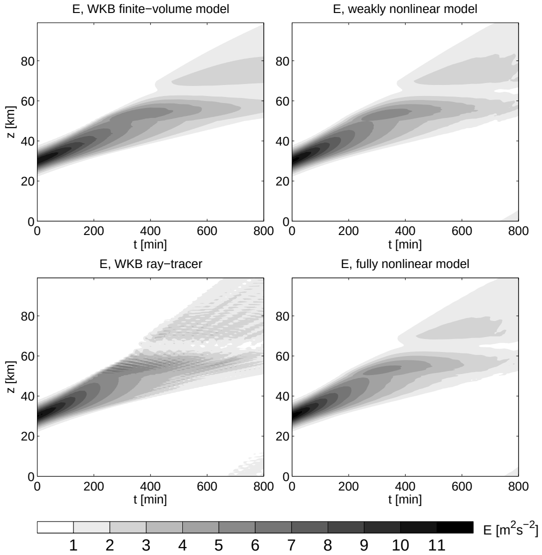
Waves:

$$k = 2\pi/(3\ \mathrm{km})$$
$$m_0 = -2\pi/(30\ \mathrm{km})$$
$$\omega = \Omega_+$$
$$a_0 = 0.5$$



Example 2: Refraction by a variable stratification

- Wave-energy density versus z and t from WKB and wave-resolving models
- Note E becomes small where N is small



Example 3: Modulationally unstable wave packet

Background:

Uniform stratification

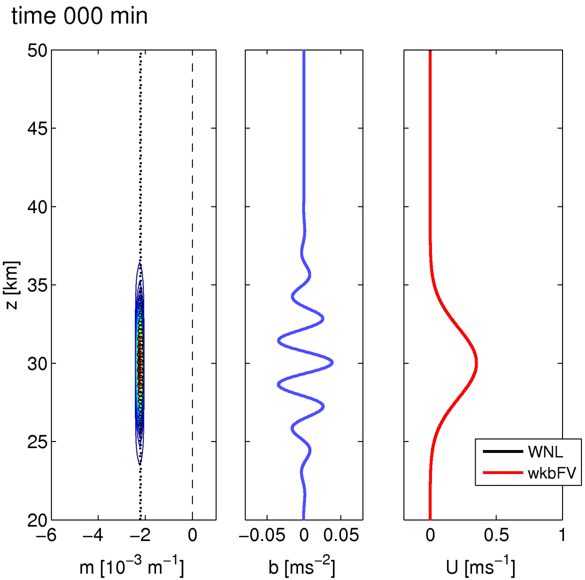
$$N = N_0 = 0.02 \text{ s}^{-1}$$

Initial U equal to pseudomomentum:

$$U(t_0) = \frac{k A_b^2(z)}{N_0^2 \hat{\omega}_0}$$

Waves:

$$k = 2\pi/(2 \text{ km})$$
$$m_0 = -2\pi/(2.9 \text{ km})$$
$$\omega = \Omega_+$$
$$a_0 = 0.21$$



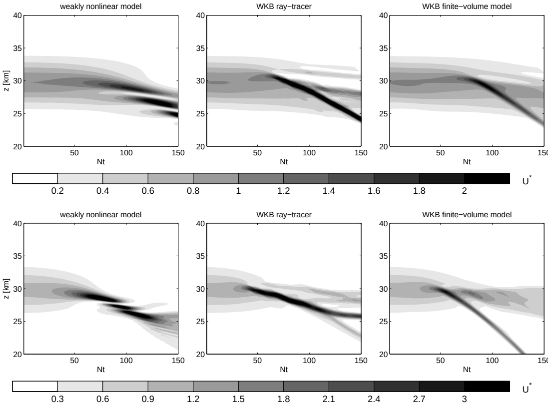
Example 3: Modulationally unstable wave packet

- Induced mean flow versus z and t in reference frame moving with c_{g0} (cf. Sutherland 2006):

WNL | WKB Ray-tracer | WKB FV

$a_0 = 0.21$
$\lambda_{z0} = 2.9$ km

$a_0 = 0.42$
$\lambda_{z0} = 5$ km



- Focusing and deceleration of wave packet captured by WKB models
- Fine spatial structure of mean flow not captured

Example 4: Wave packet reflected by a shear layer

Background:

Uniform stratification

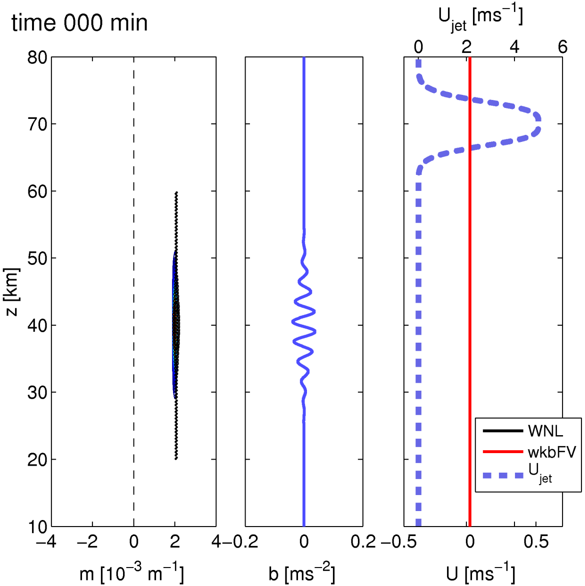
$$N = N_0 = 0.02 \text{ s}^{-1}$$

Jet centred at 70 km

$$U(t_0) = -U_{00} \operatorname{sech}\left[\frac{(z-z_1)^2}{\Sigma_U^2}\right]$$

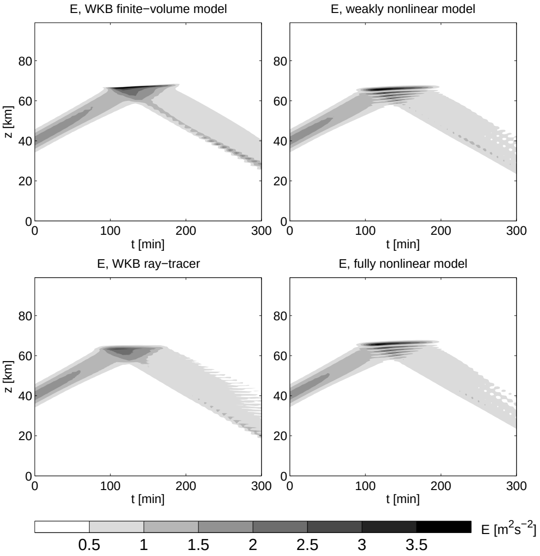
Waves:

$$k = 2\pi/(3 \text{ km})$$
$$m_0 = 2\pi/(3 \text{ km})$$
$$\omega = \Omega_-$$
$$a_0 = 0.2$$



Example 4: Wave packet reflected by a shear layer

- Wave-energy density versus z and t from WKB and wave-resolving models
- Standing-wave pattern below reflecting level absent in WKB simulations

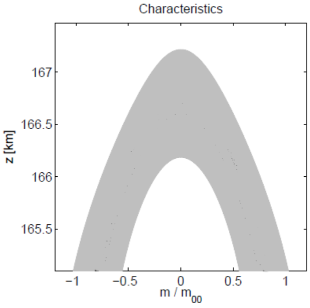


Example 4a: Wave train reflected by a shear layer

- Can find analytic solution to purely linear case of reflection of a steady wave train by a jet without feedback on the mean flow.
- Consider height $z = z_0$ far from reflecting level, where $U(z_0) = 0$ and let $\mathcal{N} = \mathcal{N}_0$ between the positive wavenumbers $m = m_{10}$ and $m = m_{20}$.
- In steady case, ω constant along a ray, so $\mathcal{N}$ must equal $\mathcal{N}_0$ everywhere between the two characteristic curves $m_1(z)$ and $m_2(z)$ defined by

$$kU(z) - \frac{kN_0}{\sqrt{k^2 + m_j^2(z)}} = \omega_j$$

- Energy density as a function of z is then



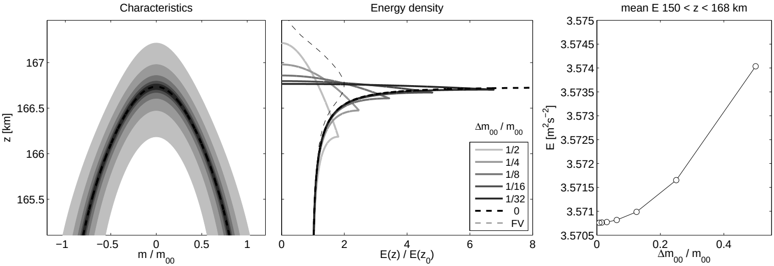
$$E(z) = 2\mathcal{N}_0 \begin{cases} \displaystyle\int_{m_1(z)}^{m_2(z)} \hat{\omega}(m)\mathrm{d}m, & z < z_1^r \\ \displaystyle\int_{0}^{m_2(z)} \hat{\omega}(m)\mathrm{d}m, & z_1^r < z < z_2^r \\ 0 & z > z_2^r \end{cases}$$

where z_j^r is turning point (reflecting level) of characteristic $m_j(z)$.

Example 4a: Wave train reflected by a shear layer

- The integral may be evaluated exactly:

$$\int \hat{\omega}(m)\mathrm{d}m = N \int \frac{1}{\sqrt{1+\frac{m^2}{k^2}}}\mathrm{d}m = Nk \log\left|\frac{m}{k} + \sqrt{1+\frac{m^2}{k^2}}\right| + \text{constant}$$



- In the limit $m_{10}, m_{20} \to m_{00}$, result tends towards conventional ray-tracing result obtained from $\omega = \text{constant}$ and $c_g \mathcal{A} = \text{constant}$ along a ray:

$$\frac{E_{conv}(z)}{E(z_0)} = \frac{m_{00}}{m_0(z)}\left[\frac{k^2 + m_0^2(z)}{k^2 + m_{00}^2}\right]$$

Summary

- Phase-space WKB equivalent to conventional WKB when $\mathcal{A}_\alpha$ and m_α differentiable and single valued
- Solution does not develop singularities (caustics) when none exist in initial conditions
- Compares well with wave-resolving simulations even in some cases where WKB assumptions violated (reflection, modulational instability)
- Two numerical implementations:
  - robust "Eulerian" finite-volume method
  - efficient (but home-made) "Lagrangian" ray-tracer

Summary

- Phase-space WKB equivalent to conventional WKB when $\mathcal{A}_\alpha$ and m_α differentiable and single valued
- Solution does not develop singularities (caustics) when none exist in initial conditions
- Compares well with wave-resolving simulations even in some cases where WKB assumptions violated (reflection, modulational instability)
- Two numerical implementations:
  - robust "Eulerian" finite-volume method
  - efficient (but home-made) "Lagrangian" ray-tracer
- Ongoing work:
  - experiments with more complicated initial wave fields (e.g. superposition of several wave packets)
  - implementing phase-space WKB in anelastic model where gravity waves increase in amplitude with height
  - parameterization of gravity wave drag due to breaking of waves
  - extension to 2 and 3 spatial dimensions, couple to GCM