Waves in Shallow Water Magnetohydrodynamics
With Application to the Solar Tachocline

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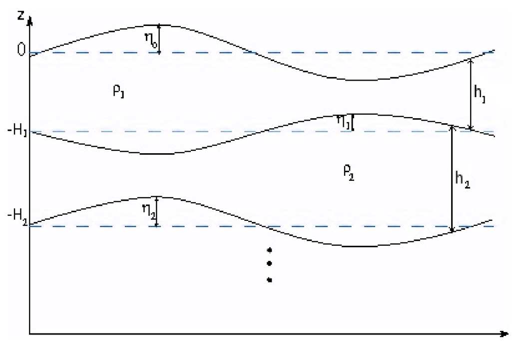
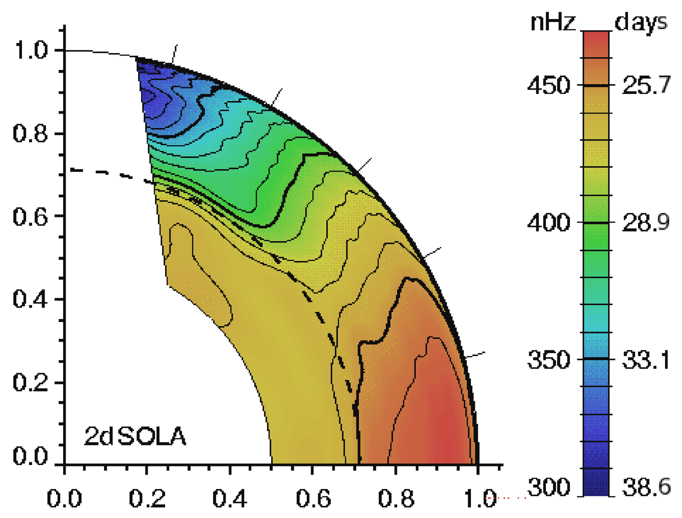
Introduction and motivation



Schematic for the n-layer model.

The dynamics of astrophysical objects are dominated by the fluid motions of electrically conducting media. In some cases, plasma is confined to a 'shallow' layer; the solar tachocline, a thin layer of velocity shear at the base of the convective zone in the Sun, is a prime example. Being at most 4% of the solar radius in thickness [2], and stably stratified, the tachocline lends itself to shallow water analysis.

The n-layer stacked shallow water system is a conceptual model for continuous stratification in fluid layers with a small aspect ratio of vertical to horizontal length scales. We consider small-amplitude long-wavelength perturbations to a motionless n-layer system. Waves would almost certainly be present in the tachocline, and could contribute to our understanding of mixing mechanisms, how energy is transferred, and perhaps even the lithium problem.



The rotation profile of the Sun. The inner two thirds by radius exhibits approximate solid-body rotation, whereas the outer convection zone rotates differentially. The tachocline is the transition layer between the two.

The shallow water approximation

Scaling arguments in the vertical yield a balance between the total pressure gradient and gravity (magnetohydrostatic balance). Integrating, one can find an expression for pressure, which can then be used to modify the horizontal momentum equation. Together with a shallow water induction equation and a modified conservation of mass, we have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -g\nabla\eta_0 + \frac{1}{\rho\mu_0}\mathbf{B}\cdot\nabla\mathbf{B} \qquad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B} \qquad (2)$$

$$\frac{\partial h}{\partial t} + \nabla\cdot(h\mathbf{u}) = 0, \qquad (3)$$

where $h = H + \eta_0$ is the depth of the layer. All vectors and differential operators are 2-dimensional (horizontal) and independent of height [1]. Velocity and magnetic field have a third, vertical component, but these are implied by solenoidal conditions and do not contribute to the dynamics of the system. These equations are complemented by the condition

$$\nabla\cdot(h\mathbf{B}) = 0. \qquad (4)$$

Figures for the solar tachocline

Property	Symbol	Value	Units
Density	ρ	210	kg m^{-3}
Gravity	g	54	m s^{-2}
Buoyancy frequency	N	8×10^{-5}	s^{-1}
Mean depth	H	0.03	$R_\odot$
Mean field strength	$\lvert\mathbf{B}\rvert$	0.2	T
Gravity wave speed	$\sqrt{gH}$	350	ms^{-1}
Alfvén wave speed	v_A	12	ms^{-1}

Energy conservation

The single layer (indeed, the n-layer) system can be shown to conserve energy. Equations (1-3) satisfy the conservation law

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho h|\mathbf{u}|^2 + \frac{1}{2}\rho g\eta_0^2 + \frac{1}{2\mu_0}h|\mathbf{B}|^2\right) + \nabla\cdot\left(\frac{1}{2}\rho h|\mathbf{u}|^2\mathbf{u} + \rho g h\eta_0\mathbf{u} + \mathbf{S}\right) = 0. \qquad (5)$$

The terms in the time derivative are the kinetic, gravitational potential, and magnetic energies present in the system. The terms in the divergence comprise the energy flux; both kinetic and potential energies are carried around with the flow. The final term is the shallow water Poynting flux

$$\mathbf{S} = \frac{1}{\mu_0}h\left(\frac{1}{2}|\mathbf{B}|^2\mathbf{u} - (\mathbf{u}\cdot\mathbf{B})\mathbf{B}\right). \qquad (6)$$

This describes the transport of magnetic energy. It differs from the 3-D MHD Poynting vector in the factor 1/2, which arises due to the necessary absorption of the magnetic pressure into the total pressure term in the derivation of (1).

Wave motion in the n-layer model

We perturb a motionless basic state with a horizontal magnetic field, which can differ in each layer. Upon substitution of $\mathbf{u} = \mathbf{u}'$ and $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'$, the linear analogue of the conservation law (5) can be found. We then substitute the ansatz

$$\mathbf{u}' = \hat{\mathbf{u}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad (7)$$

and take the spatial and temporal average. The result indicates that ω is real, implying that in a stably stratified system, a motionless basic state is stable to infinitesimal disturbances. This allows us to look for small wave-like disturbances without worrying about any lurking instabilities!

Equations (1-3) can be combined for each layer. After substituting according to (7), we arrive at

$$(c^2 - v_{A_j}^2)(\hat\eta_j - \hat\eta_{j-1}) + \frac{1}{\rho_j}H_j\sum_{i=1}^{j}\rho_i g_{i-1}^r\hat\eta_{i-1} + \frac{1}{\rho_j}H_jP = 0. \qquad (8)$$

Using this equation, we can find dispersion relations, and equations describing interesting properties of supported waves modes and their structure. The phase speed is given by c, and v_{A_j} is the Alfvén speed, given by

$$v_{A_j} = \frac{|\mathbf{B}_{0_j}|}{\sqrt{\mu_0\rho_j}}\cos\theta_j, \qquad (9)$$

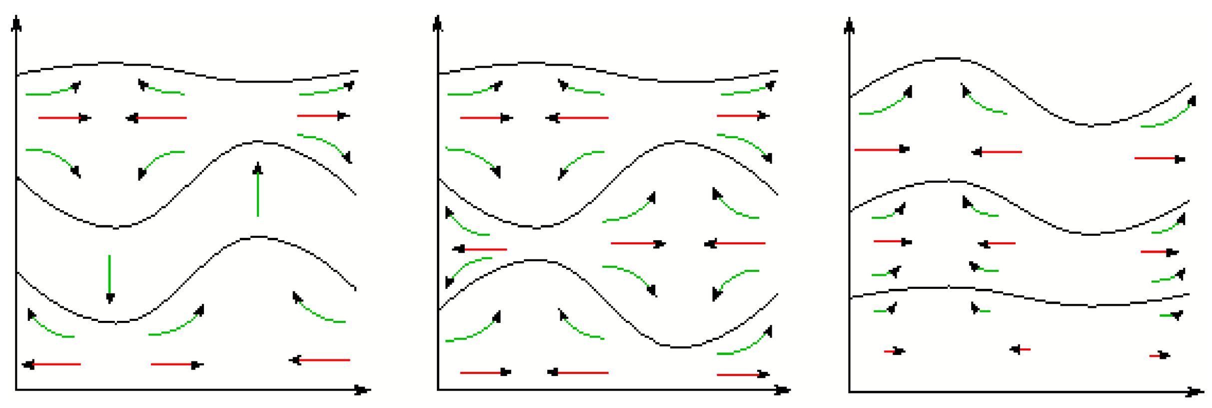
where θ_j is the angle between the wavevector $\mathbf{k}$ and the ambient magnetic field.

The 3-layer model

Putting $n = 3$ in (8) and looking for non-trivial solutions yields the dispersion relation

$$\zeta_1^2\zeta_2^2\zeta_3^2 - \sum_{i\neq j\neq k}\beta_i\zeta_j^2\zeta_k^2 + \sum_{i\neq j\neq k}\alpha_{i,j}\beta_i\beta_j\zeta_k^2 - \alpha_{1,2}\alpha_{2,3}\beta_1\beta_2\beta_3 = 0, \qquad (10)$$

where $\zeta_j^2 = c^2 - v_{A_j}^2$, $\alpha_{i,j}$ is a non-dimensional reduced gravity, and β_j is a fractional layer depth. This relation is cubic in c^2, indicating 3 pairs of distinct wave modes supported in this system.

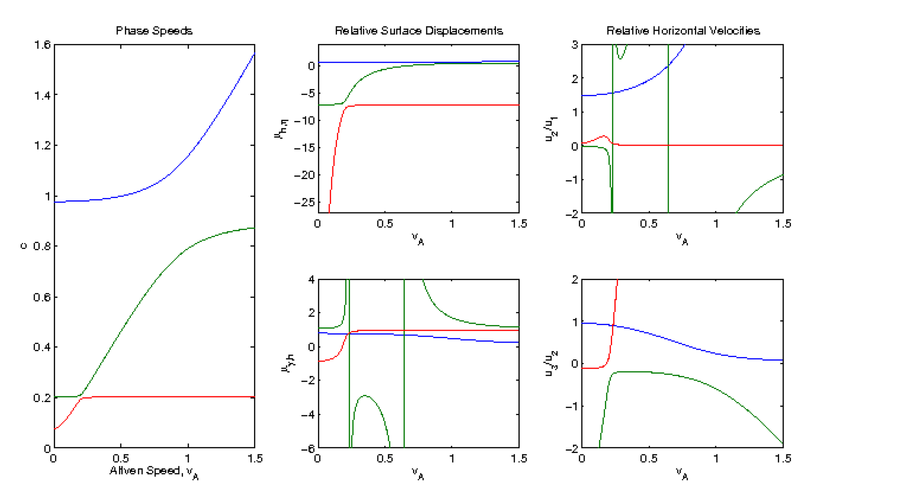


Illustrations of the modes in the 3-layer system. Red arrows represent horizontal fluid motion, and green arrows depict 3-dimensional flow.

The schematic to the left depicts the modes present when the magnetic field is structured in such a way that the v_{A_j} are equal in each layer. In this case, phase speeds increase in a hyperbolic way with increasing Alfvén speed, with wave structure remaining unchanged.

Interesting properties of the 3-layer system

- For certain stratification settings, a wave mode becomes completely independent of field strength.
- A strong field in the bottom layer can induce behaviour seen in the 2-layer model.
- Wave modes appear to 'swap' behaviour when the field is in the middle layer.
- The addition of a rigid lid not only suppresses free surface motion, but also one of the modes. Further, making the top layer thin and strongly magnetised induces a kind of elastic behaviour on the surface.



Plots of the 'behaviour swap' in the 3-layer model when magnetic field is confined to the middle layer. Each colour represents one of the 3 modes.

References

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