Waves and Mean Flows

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The complete theory of wave-mean interactions, abridged

Abstract evolution equation

$$U(t): \quad \frac{\partial U}{\partial t} + \mathcal{L}(U) + \mathcal{B}(U, U) = 0$$

Averaging operator, defines mean and disturbance (eddy) fields

$$U = \overline{U} + U': \quad \overline{\alpha U + \beta V} = \alpha \overline{U} + \beta \overline{V} \quad \text{and} \quad \boxed{\overline{U'} = 0}$$

Evolution of mean field is coupled to eddy field; turbulence closure problem

$$\frac{\partial \overline{U}}{\partial t} + \mathcal{L}(\overline{U}) + \mathcal{B}(\overline{U}, \overline{U}) = -\overline{\mathcal{B}(U', U')}$$

But for small-amplitude waves this term can be evaluated from linear theory!

Small-amplitude waves, aka linear or quasi-linear eddies

Small wave amplitude $\boxed{a \ll 1}$

$$U = U_0 + aU_1 + a^2 U_2 + \cdots \qquad U_0' = 0 \qquad \overline{U_1} = 0$$

Basic flow assumed known
$$O(1): \quad \frac{\partial U_0}{\partial t} + \mathcal{L}(U_0) + \mathcal{B}(U_0, U_0) = 0$$

Linear waves
$$O(a): \quad \frac{\partial U_1'}{\partial t} + \mathcal{L}(U_1') + \mathcal{B}(U_0, U_1') + \mathcal{B}(U_1', U_0) = 0$$

Nonlinear mean-flow response
$$O(\overline{a^2}): \quad \frac{\partial \overline{U}_2}{\partial t} + \mathcal{L}(\overline{U}_2) + \mathcal{B}(U_0, \overline{U}_2) + \mathcal{B}(\overline{U}_2, U_0) = -\overline{\mathcal{B}(U_1', U_1')}$$

Key linear operator:
$$\boxed{\frac{\partial(\cdot)}{\partial t} + \mathcal{L}(\cdot) + \mathcal{B}(U_0, \cdot) + \mathcal{B}(\cdot, U_0)}$$

Resonant forcing may lead to unbounded mean-flow growth as O(a^2 t)

Such strong interactions can break the perturbation expansion and lead to the most interesting dynamics!

Strong interactions caused by Stationary Waves

Stationary waves often relevant in practice

$$\frac{\partial}{\partial t}\left(\overline{\mathcal{B}(U_1', U_1')}\right) \approx 0$$

$$O(\overline{a^2}): \quad \frac{\partial \overline{U}_2}{\partial t} + \mathcal{L}(\overline{U}_2) + \mathcal{B}(U_0, \overline{U}_2) + \mathcal{B}(\overline{U}_2, U_0) = \boxed{-\overline{\mathcal{B}(U_1', U_1')}}$$

projects onto steady modes!

Strong interactions naturally associated with steady, zero-frequency modes of the key linear operator

So, which part of the linear flow dynamics is slow?

Classical answer: Zonal averaging and jets

Spatially periodic flow in x (longitude, azimuthal angle in tokamak) $U(x+L) = U(x)$

Zonal averaging in x induces zonal mean-flow symmetry

$$\frac{\partial \overline{U}}{\partial x} = 0$$

Unforced linear zonal mean flow equation
Fast pressure force has dropped out!

$$\boxed{\frac{\partial \overline{u}}{\partial t} = 0}$$

$$\boxed{\frac{\partial \overline{p}}{\partial x} = 0}$$

Zonal jets exhibit slow linear dynamics

Makes obvious the importance of zonal jets for strong wave-mean interactions

Another answer: Slow vortex dynamics

Euler equations for compressible barotropic flow $$\frac{D\boldsymbol{u}}{Dt} + \frac{\boldsymbol{\nabla} p}{\rho} = 0, \quad \rho = f(p)$$

Vorticity equation $$\frac{D\boldsymbol{\nabla}\times\boldsymbol{u}}{Dt} + (\boldsymbol{\nabla}\times\boldsymbol{u})\,\boldsymbol{\nabla}\cdot\boldsymbol{u} = (\boldsymbol{\nabla}\times\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$$

Fast pressure force has dropped out!

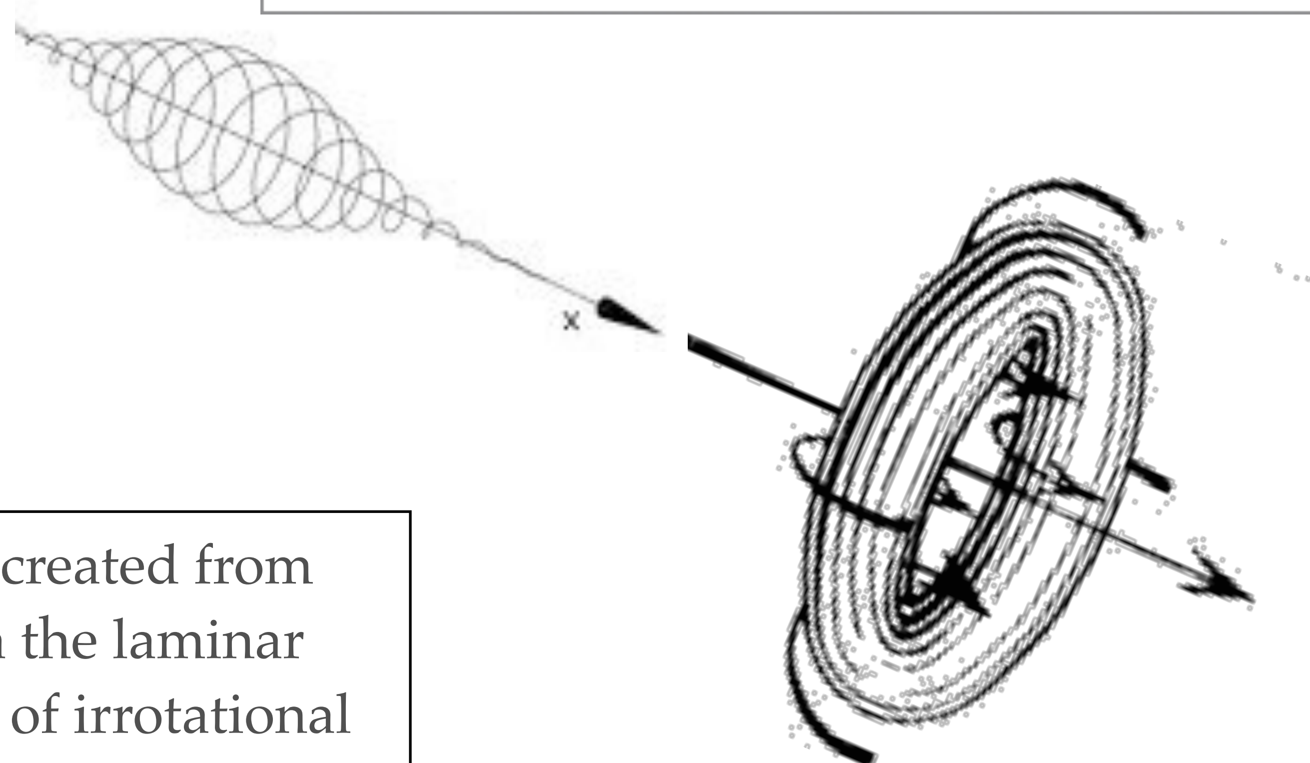
Unforced linear vorticity equation with zero basic flow $$\boxed{\frac{\partial\boldsymbol{\nabla}\times\boldsymbol{u}}{\partial t} = 0}$$

The vortical mode is also a natural candidate for a strong mean-flow response
No spatial symmetry needed (eg time-averaging ok)

Waves and vortices

a packet of sound waves dissipates and creates a vortex ring

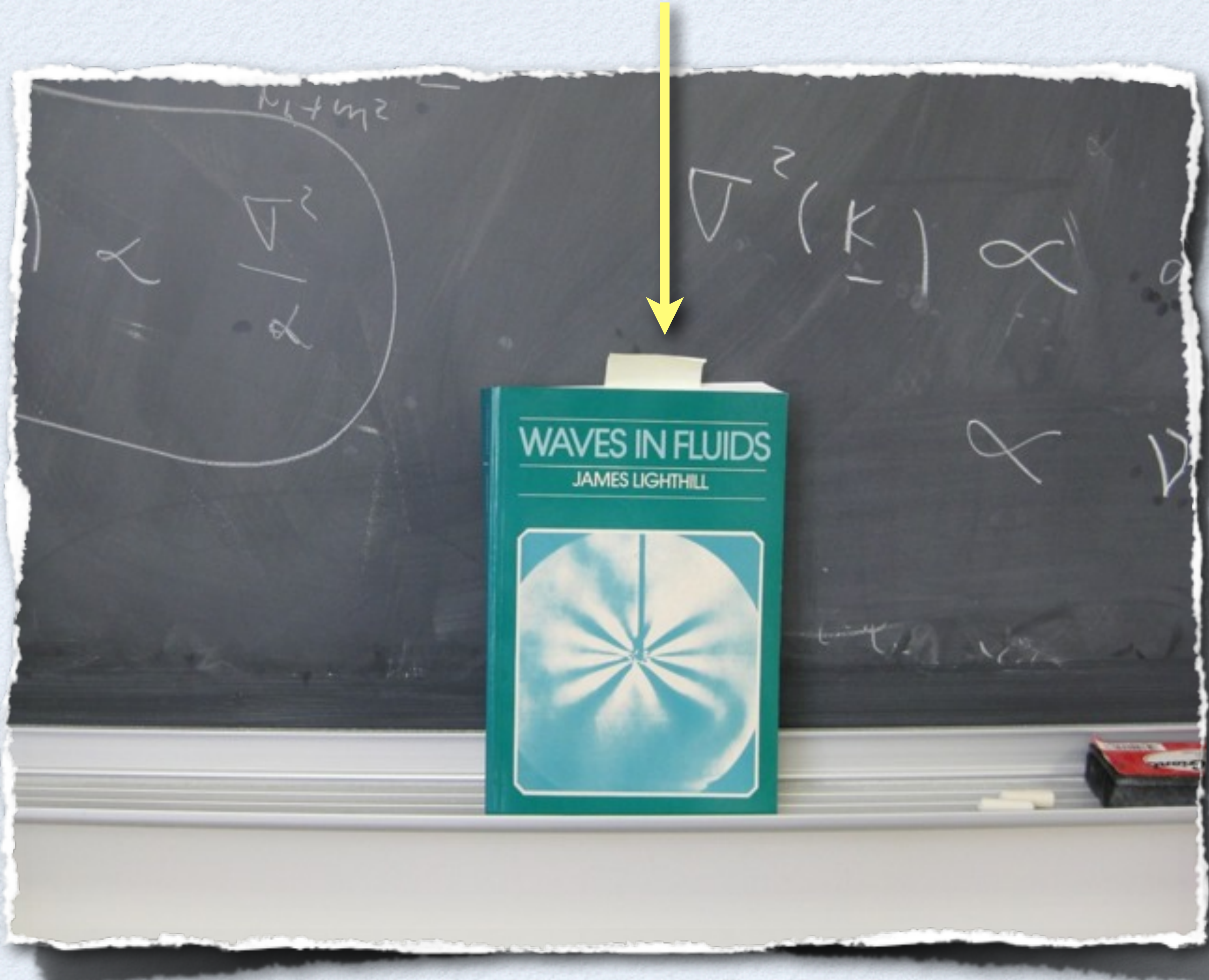
Wave-dissipation creates vorticity



vorticity is created from nothing via the laminar dissipation of irrotational sound waves!

Lighthill on Waves

Yellow marker for page 347: "steady streaming generated by wave attenuation"



Micro-mixing in a drop induced by sound waves

PROCEEDINGS OF THE ROYAL SOCIETY A

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Streaming by leaky surface acoustic waves

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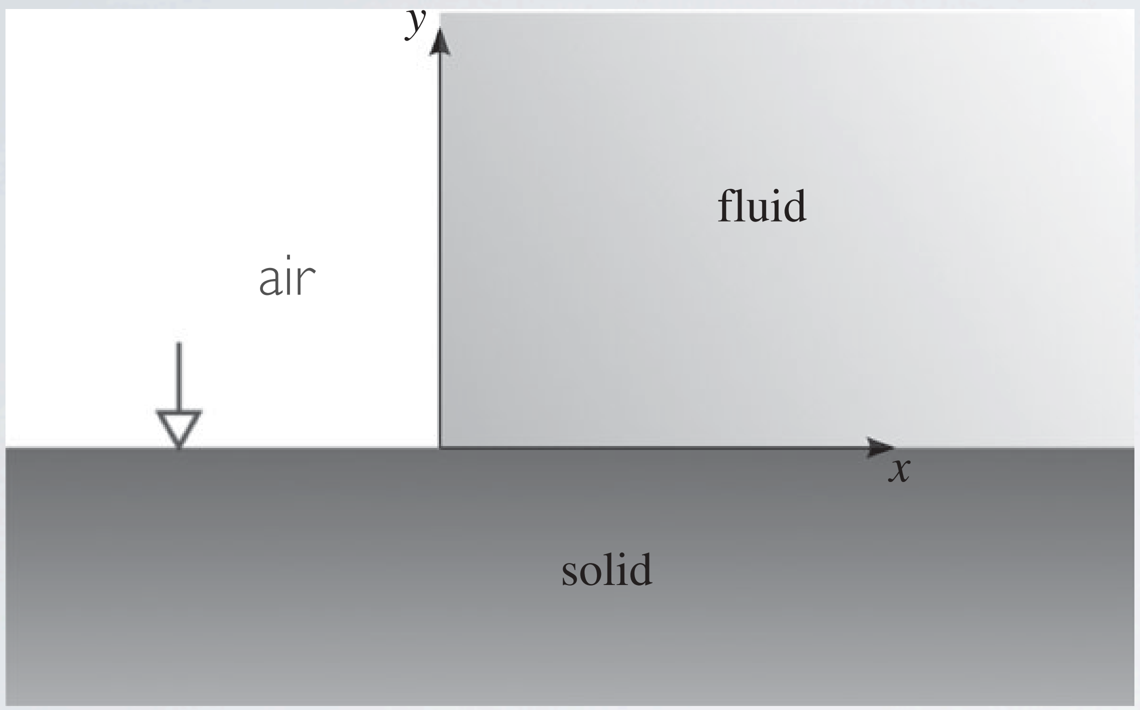
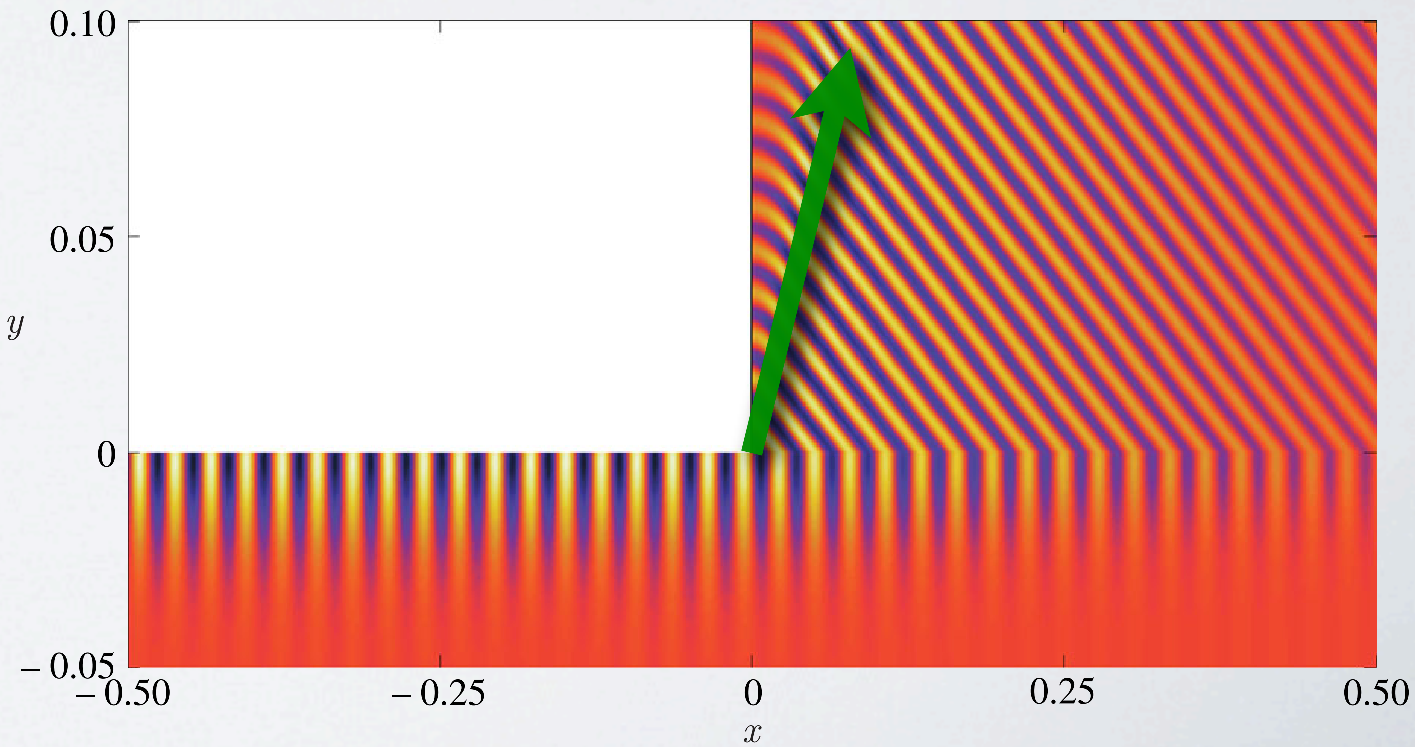
- ★ interested in mixing the drop using high frequency sound waves, a multi-dimensional version of acoustic streaming
- ★ acoustic streaming depends on presence of viscosity but is ***independent*** of its value; a textbook singular perturbation problem

MODEL BASED ON LEAKY WAVES



Waves in drop are excited by incoming surface wave from the left, generated by piezo-crystal.

Dissipating waves drive vortical mean flow, which saturates against viscous diffusion.



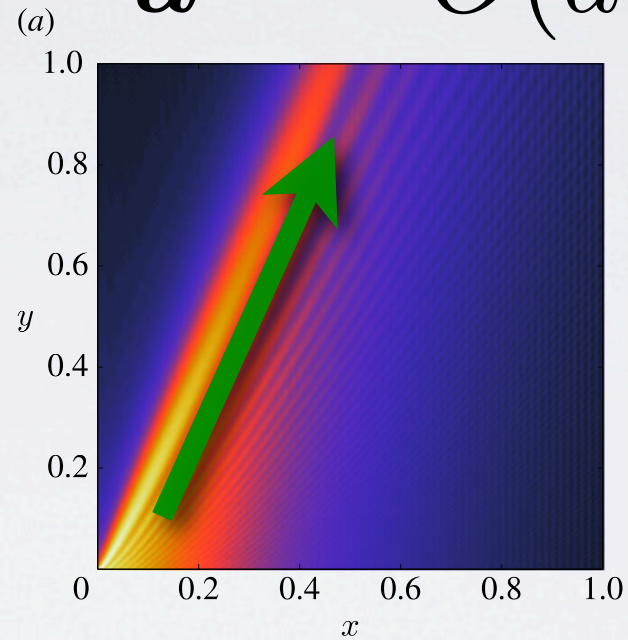
EULERIAN MEAN FLOW

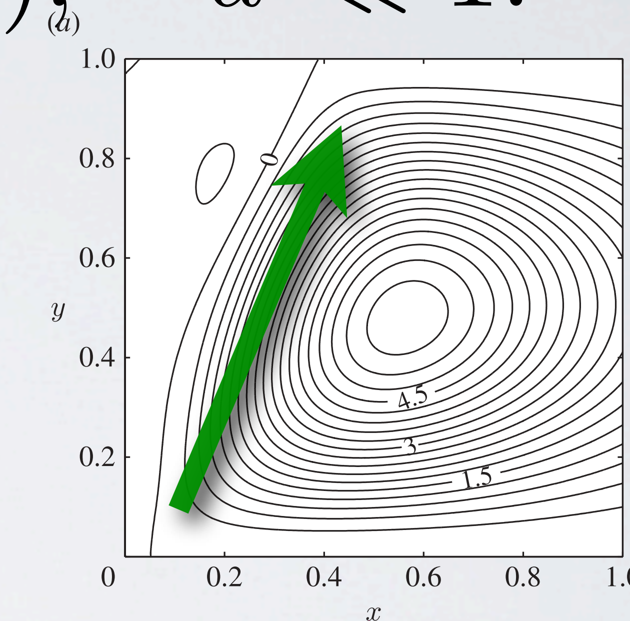
$$\boldsymbol{u} = \overline{\boldsymbol{u}}^E + \boldsymbol{u}'$$

$$\boldsymbol{u}' = O(a), \quad \overline{\boldsymbol{u}}^E = O(a^2), \quad a \ll 1.$$

Time-averaged mean flow
Find balance between wave driving and viscous dissipation at second order in wave amplitude

mean vorticity budget





wave dissipation

waves drive vortex roll

$$\nabla^2 \nabla \times \bar{\boldsymbol{u}}^{\mathrm{E}} = -\frac{\nu + \nu'}{\nu} \frac{\omega^2}{\rho_0 c^2} \nabla \times \overline{\rho_1 \boldsymbol{u}_1}$$

mean flow dissipation

linear wave field
B(u’,u’)

Only the dissipation ratio enters the equation for the steady mean flow!
Hence steady mean flow depends on presence of viscosity, but not its value.

Mutatis mutandis, streaming concepts can be applied to nonlinearly breaking waves

Peregrine (1998)

"Surf zone currents"

Surf Zone Currents 303

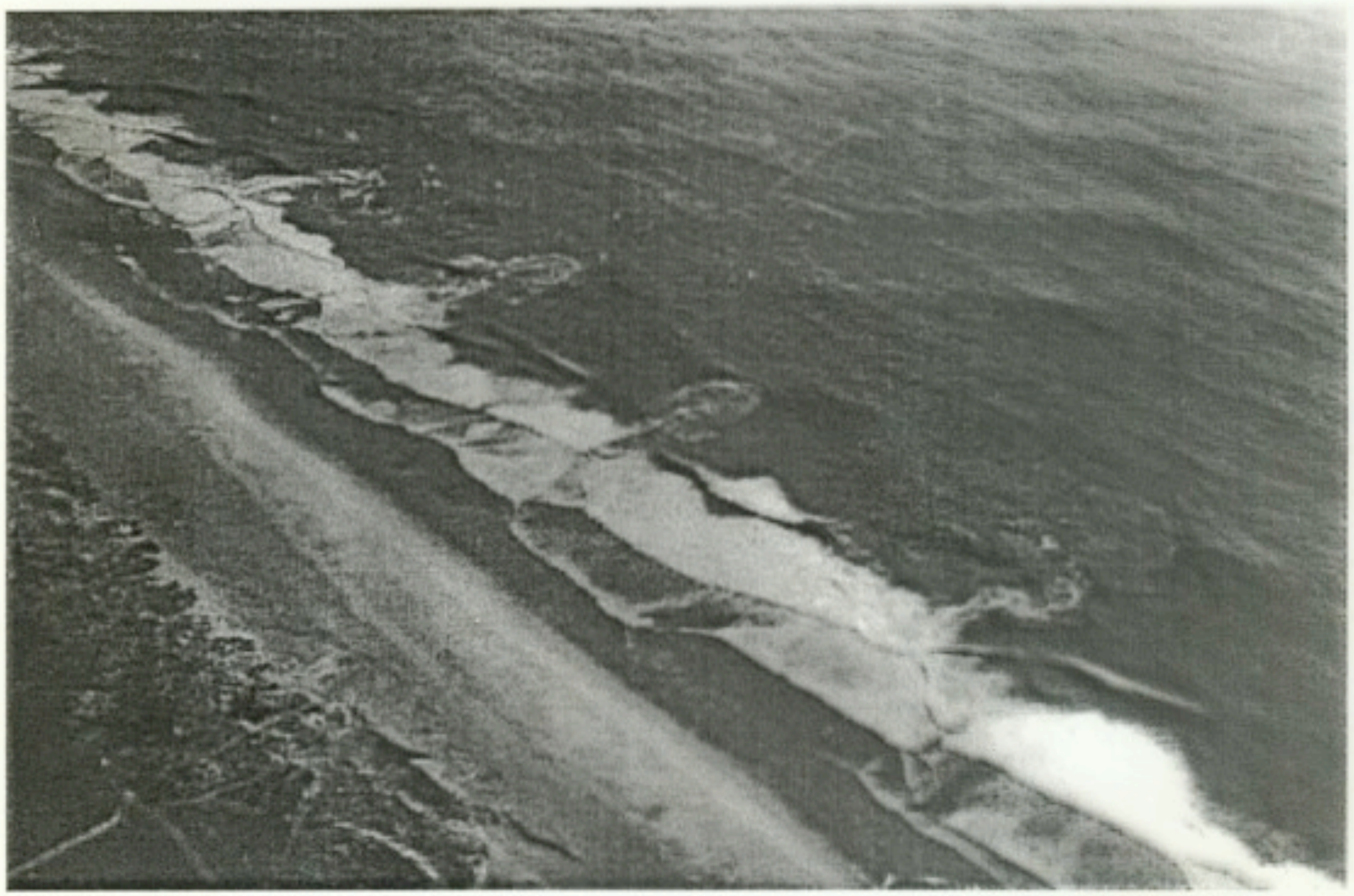


Figure 5. Rip currents, Rosarita Beach, Baja California, Mexico, October 1956. The wide round head of the currents indicates the existence of an eddy couple [Courtesy D.L. Inman.]

Breaking of obliquely incident surface waves drives vortical motions in the surf zone

Example: longshore currents and rip currents

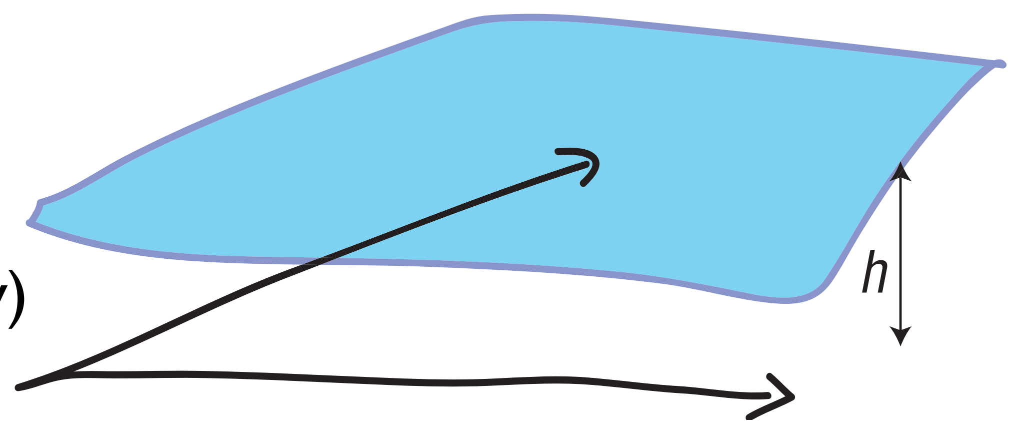
Classical theory by Longuet-Higgins 1970

Non-trivial longshore current structure on barred beaches (Buhler & Jacobson 2001)

Simplest model: shallow water equations

Single layer of hydrostatic incompressible fluid

(Topography ignored for simplicity)



depth h

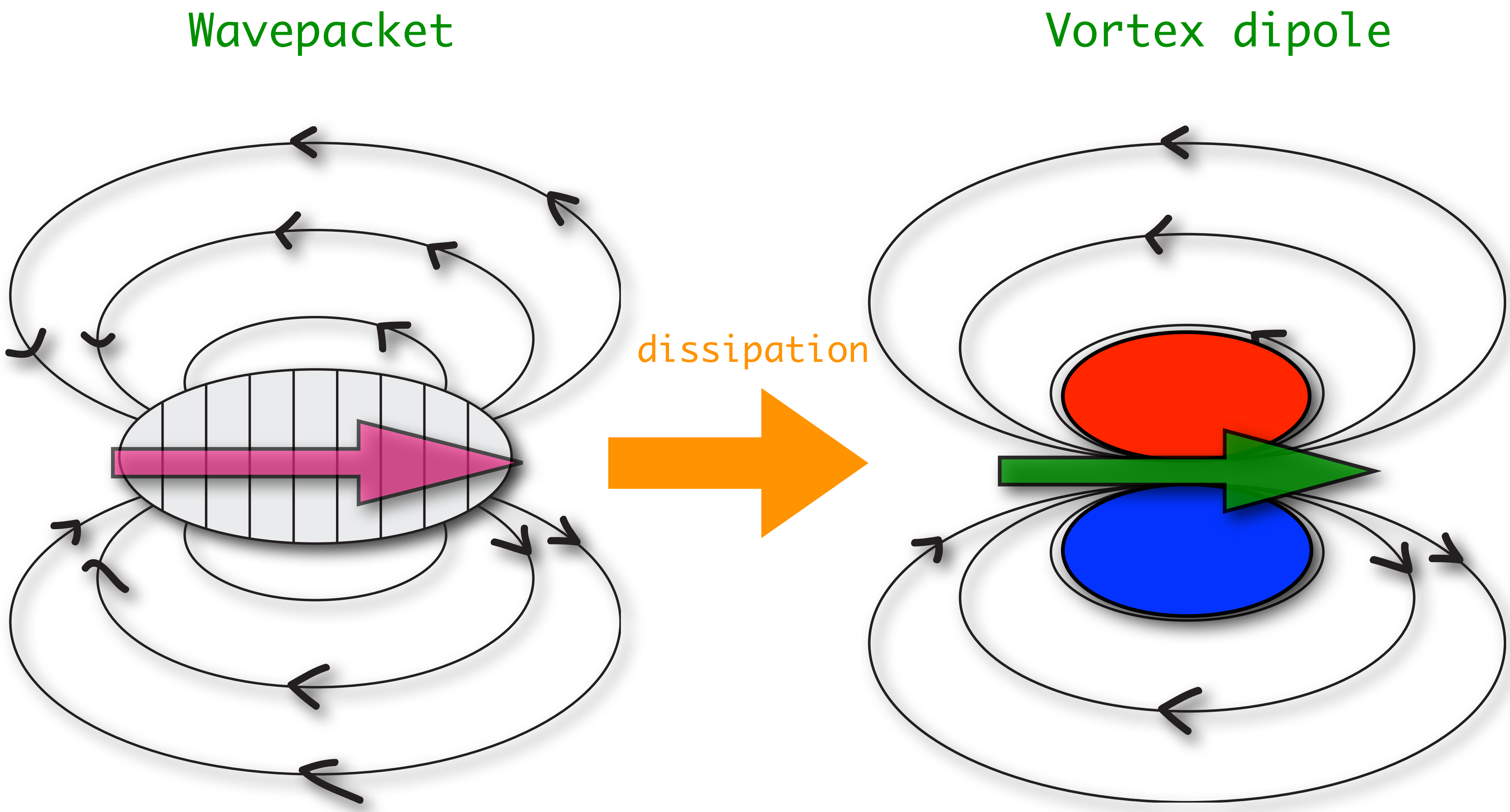
$$\boldsymbol{x} = (x, y) \qquad \boldsymbol{u} = (u, v) \qquad \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})$$

$$\frac{D\boldsymbol{u}}{Dt} + g\boldsymbol{\nabla} h = \boldsymbol{F} \qquad \frac{\mathrm{D}h}{\mathrm{D}t} + h\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

Potential vorticity $q = \dfrac{\boldsymbol{\nabla} \times \boldsymbol{u}}{h}$ such that $\dfrac{Dq}{Dt} = \dfrac{\boldsymbol{\nabla} \times \boldsymbol{F}}{h}$

PV can be changed by the curl of a mean force F due to dissipating or breaking waves!

Key fact: non-uniform wave breaking creates vortex dipoles, as in acoustic streaming



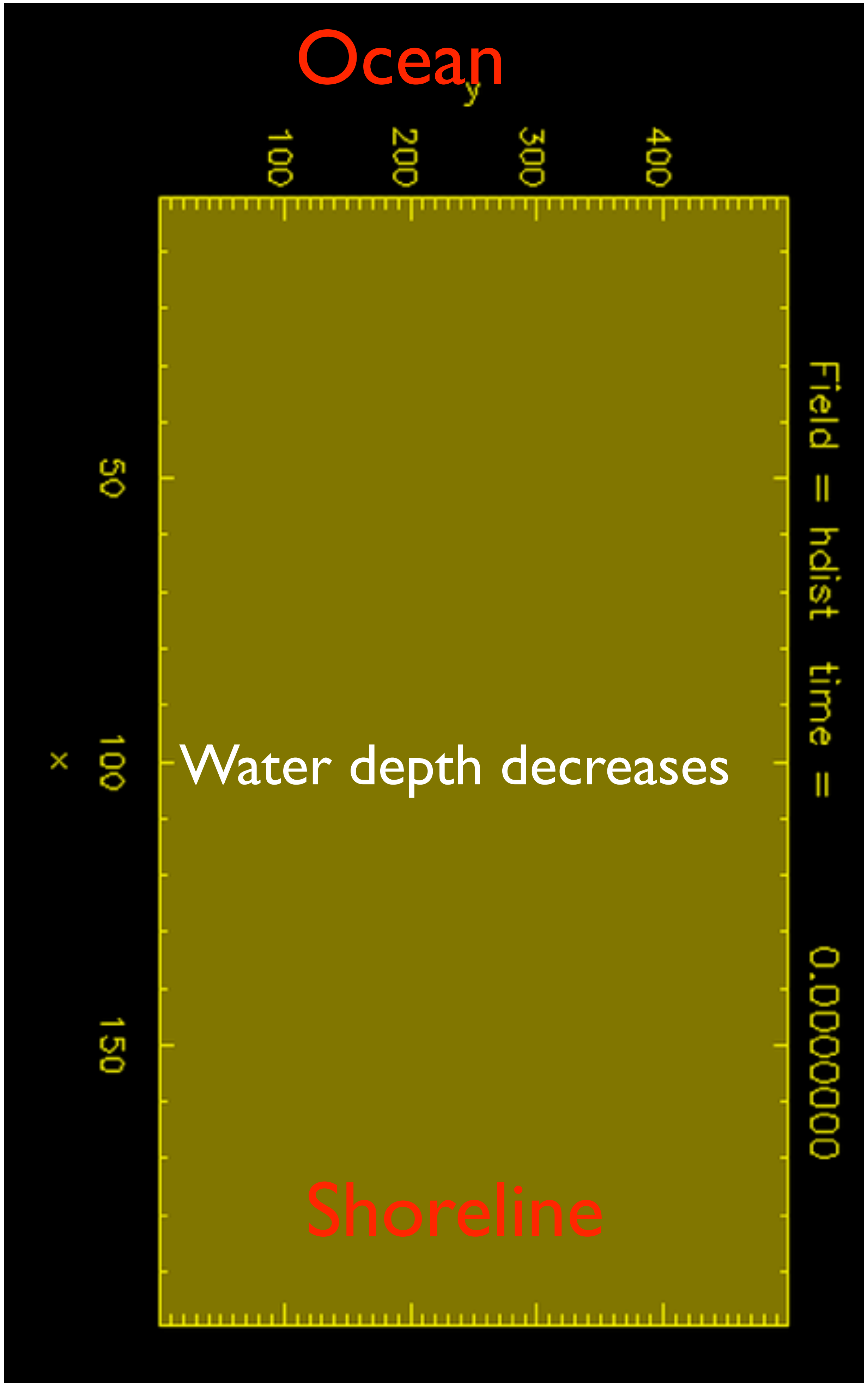
Important for wave-mean interactions
e.g. Bühler 2000, Bühler & McIntyre, 2003, 2005

Role of wave-driven vortices in the surf zone?

Numerical example: breaking shallow water waves

time = O(100)

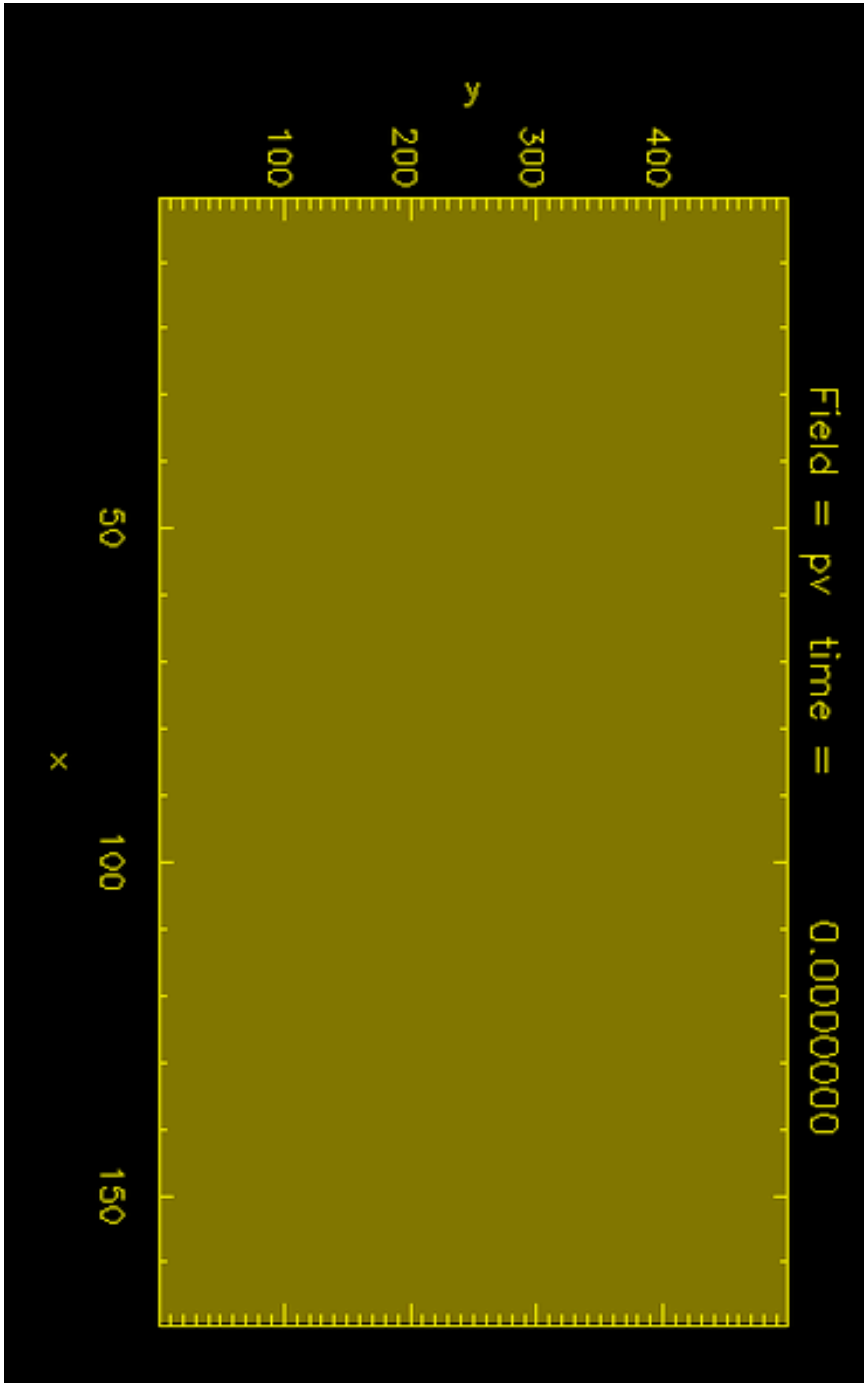
Waves are generated, refracted by decreasing water depth, finally decay due to shock formation and dissipative wave breaking



Potential vorticity - mind the rip current!

$$\mathrm{pv} = \boldsymbol{\nabla} \times \boldsymbol{u}/h$$

time = O(1000)



No special theory was needed

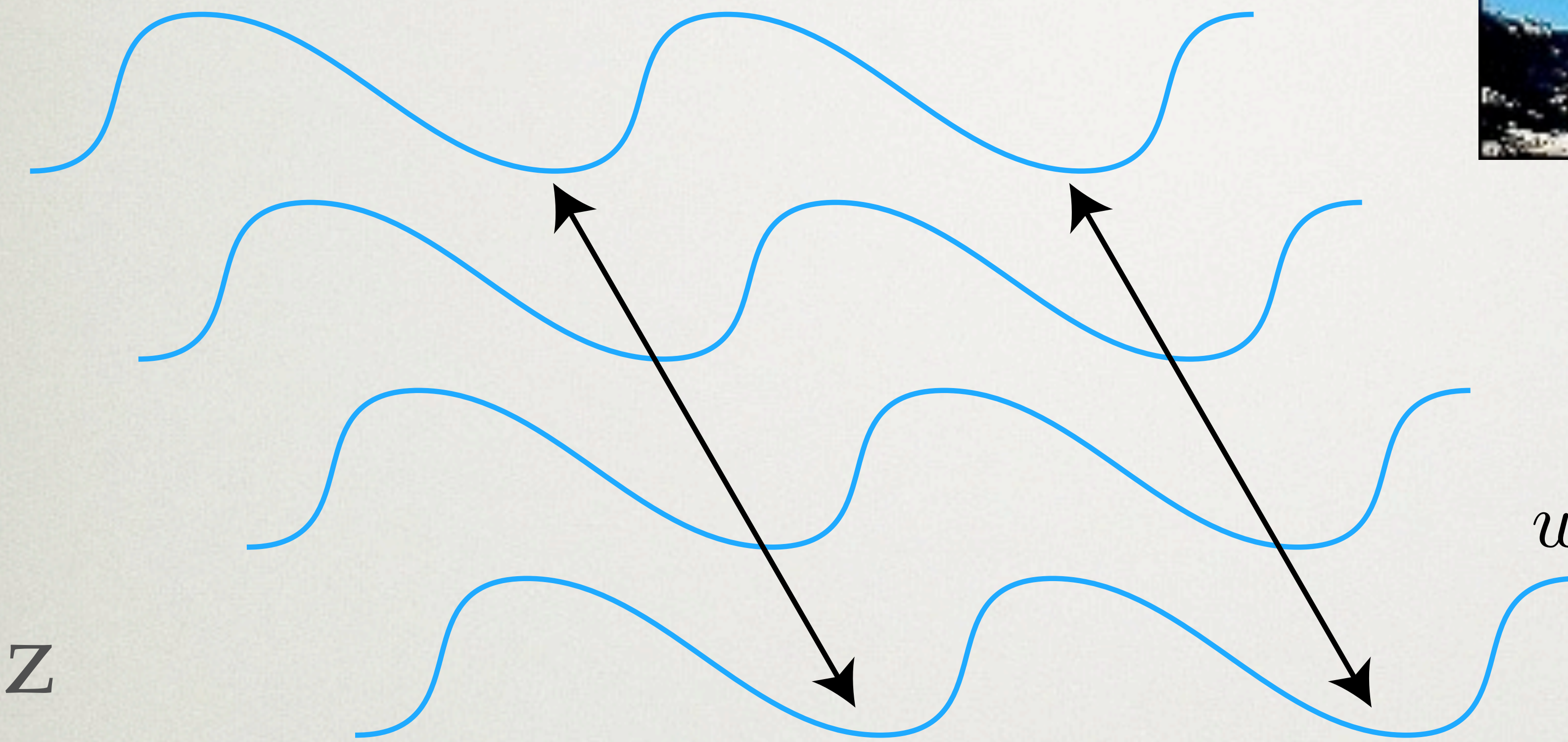
This was because the waves were irrotational...

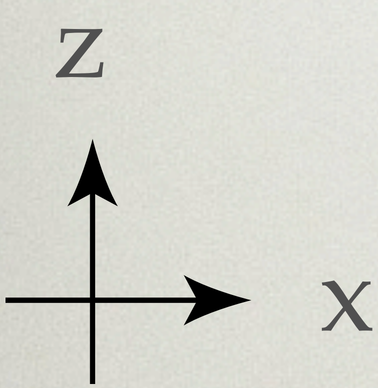
$$\boldsymbol{\nabla} \times \boldsymbol{u}' = 0$$

..don't need to look far for this to fail. Deeper...

Internal gravity waves

undulating material
stratification surfaces
(isentropes/isopycnals)
surfaces are flat at rest





linear particle trajectories

$$w' \propto \exp(i[kx + mz - \hat{\omega}t])$$

$$f^2 \leq \hat{\omega}^2 \leq N^2$$

scale-free dispersion relation

$$\hat{\omega}^2 = (N^2 - f^2)\,\frac{k^2}{k^2 + m^2} + f^2$$

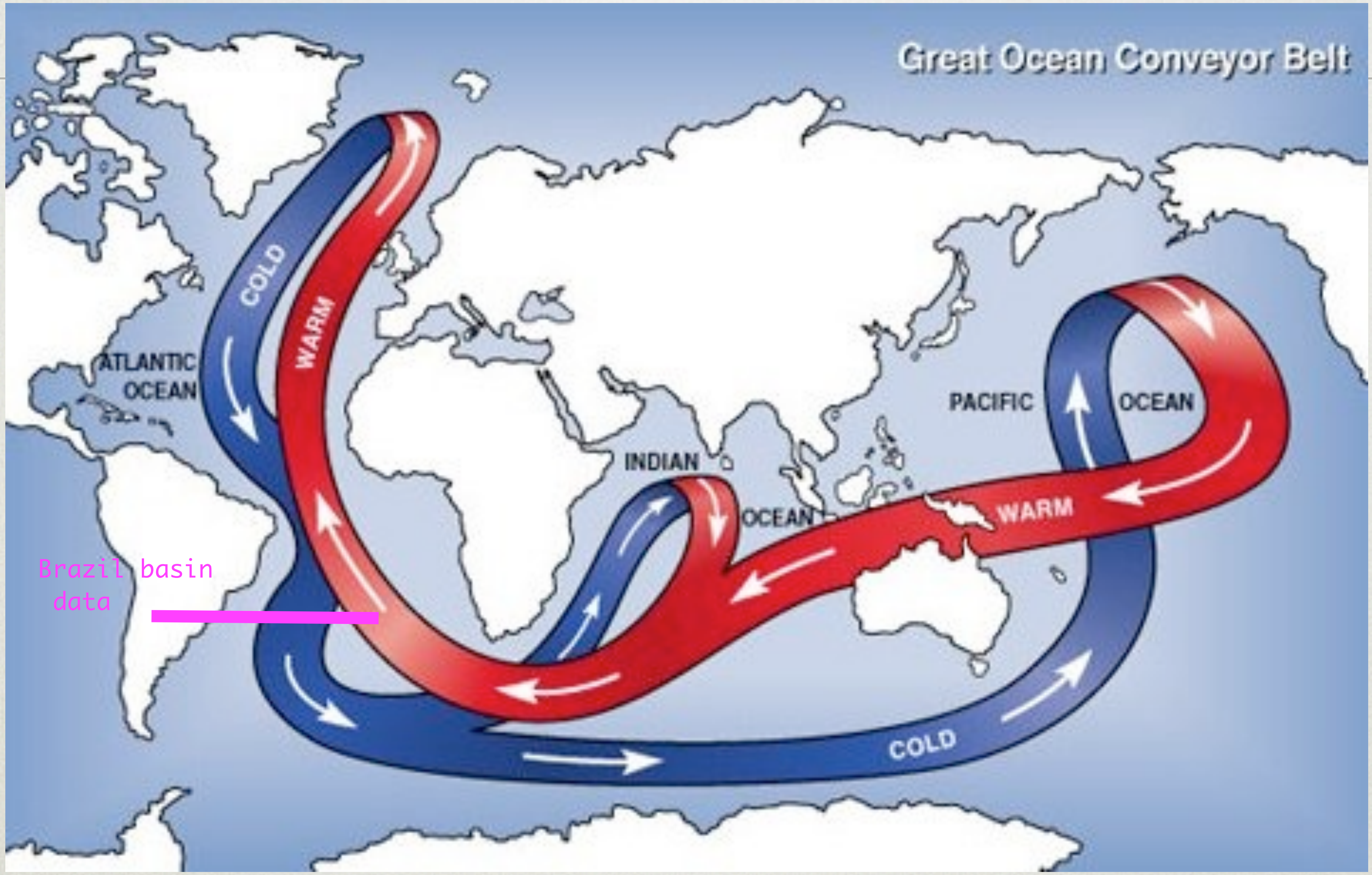
Momentum flux $\overline{u'w'} < 0$ (*unlike* surface waves)

$$\boldsymbol{\nabla} \times \boldsymbol{u} \neq 0, \quad q = \frac{\boldsymbol{\nabla} \times \boldsymbol{u} \cdot \boldsymbol{\nabla}\theta}{\rho} = 0$$

Waves contribute to vertical angular-momentum transport.

Breaking waves contribute to turbulent vertical diffusion.

Dissipating internal waves lubricate the ocean circulation

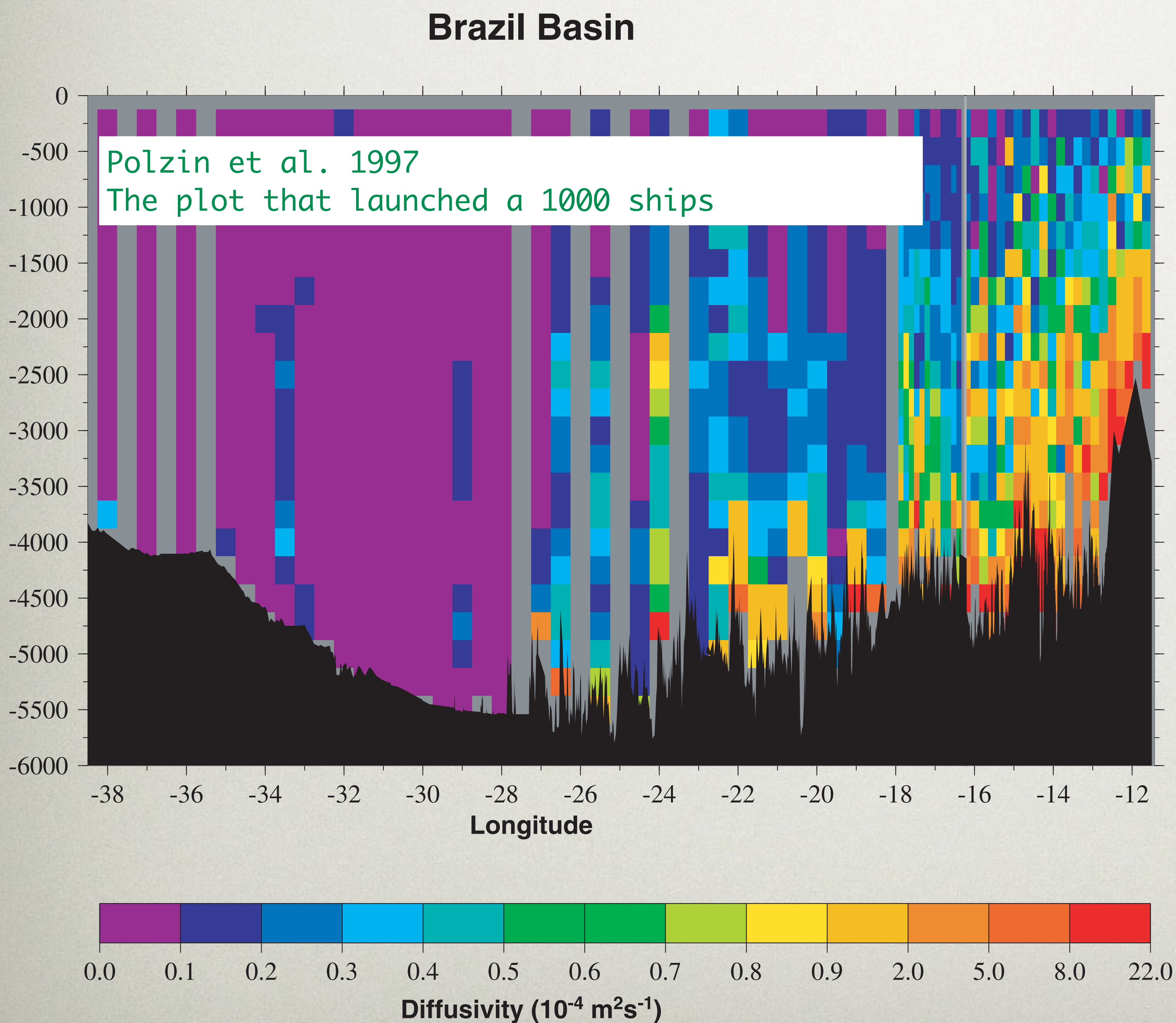


Narrow regions of cold downwelling (plumes) and
wider regions of warm upwelling (diffusion)

Small-scale internal gravity waves are believed to play a significant role here:
wave-breaking lubricates the ocean circulation

A substantial fraction of oceanic internal wave energy is of heavenly origin....

Microstructure measurements



Clear evidence of enhanced turbulence above rough topography

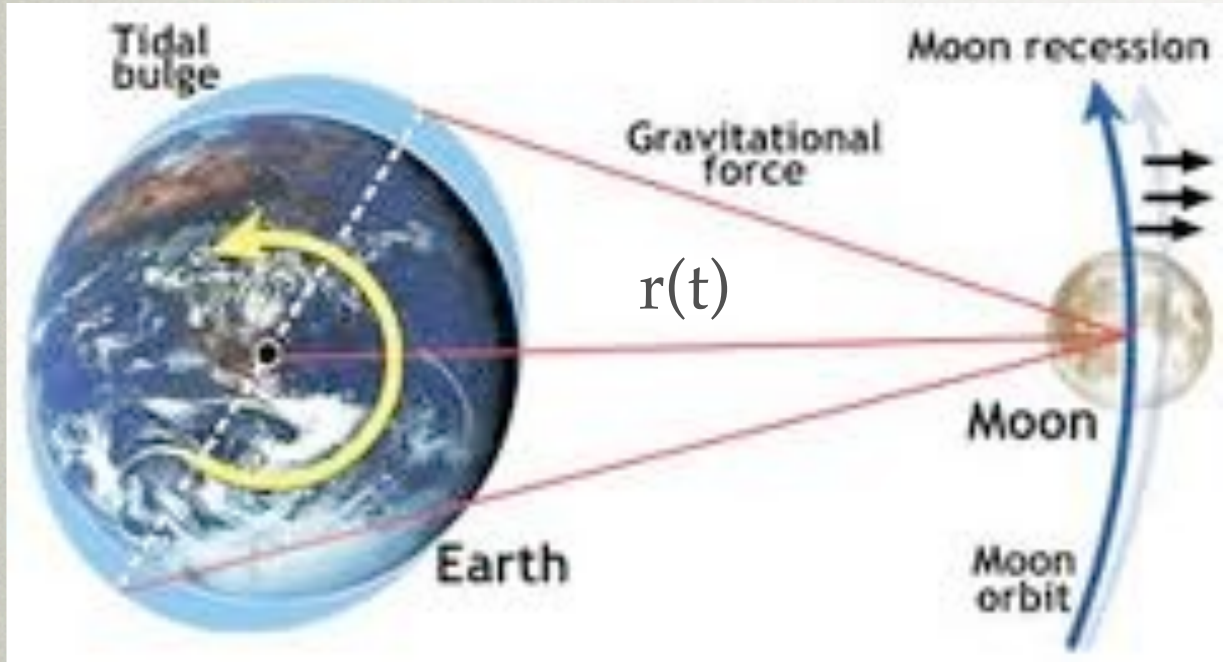
Points clearly to importance of both internal waves and of topography

Smoking gun: internal waves generated by the lunar tide

Internal tides

Want to study those

Lunar recession and Energy dissipation

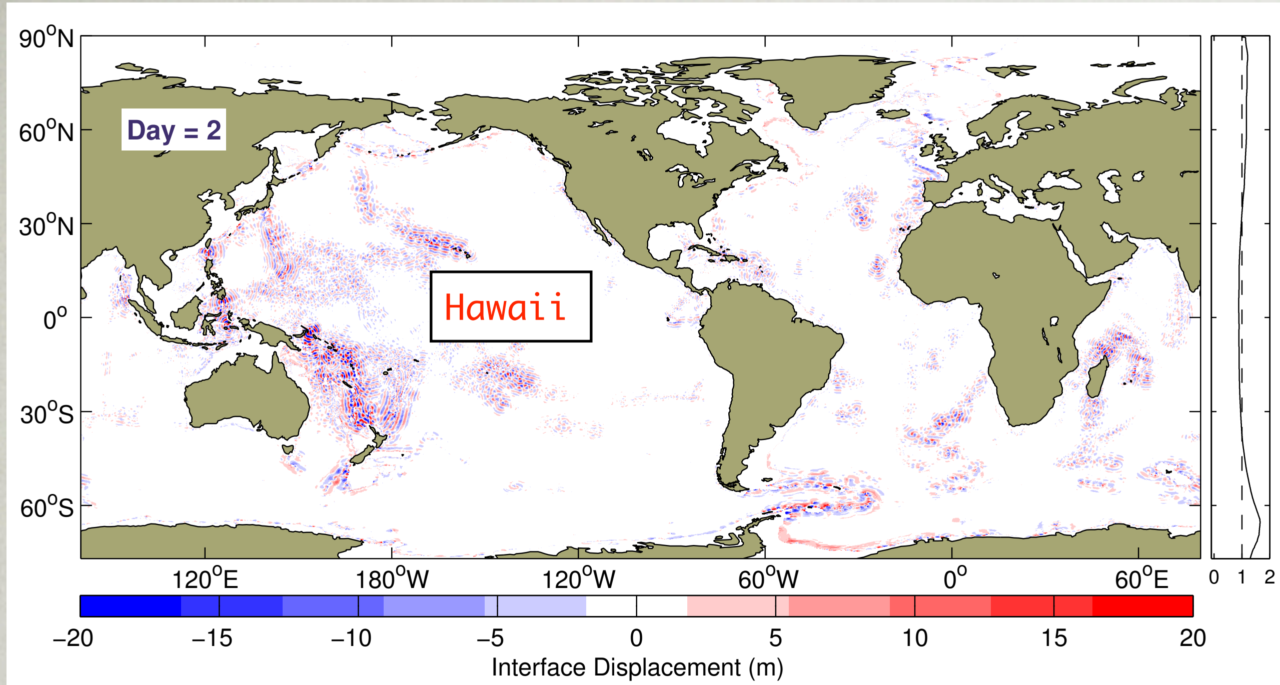


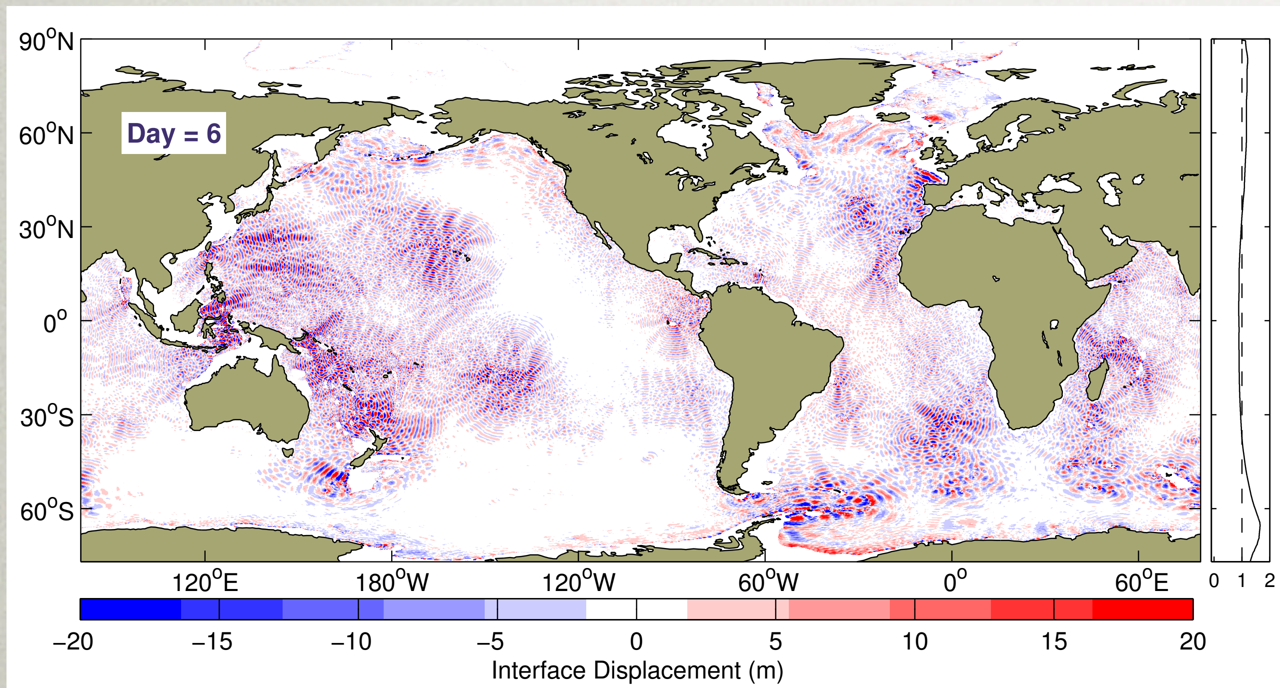
$$\frac{dr}{dt} = 3.8 \frac{\text{cm}}{\text{year}}$$

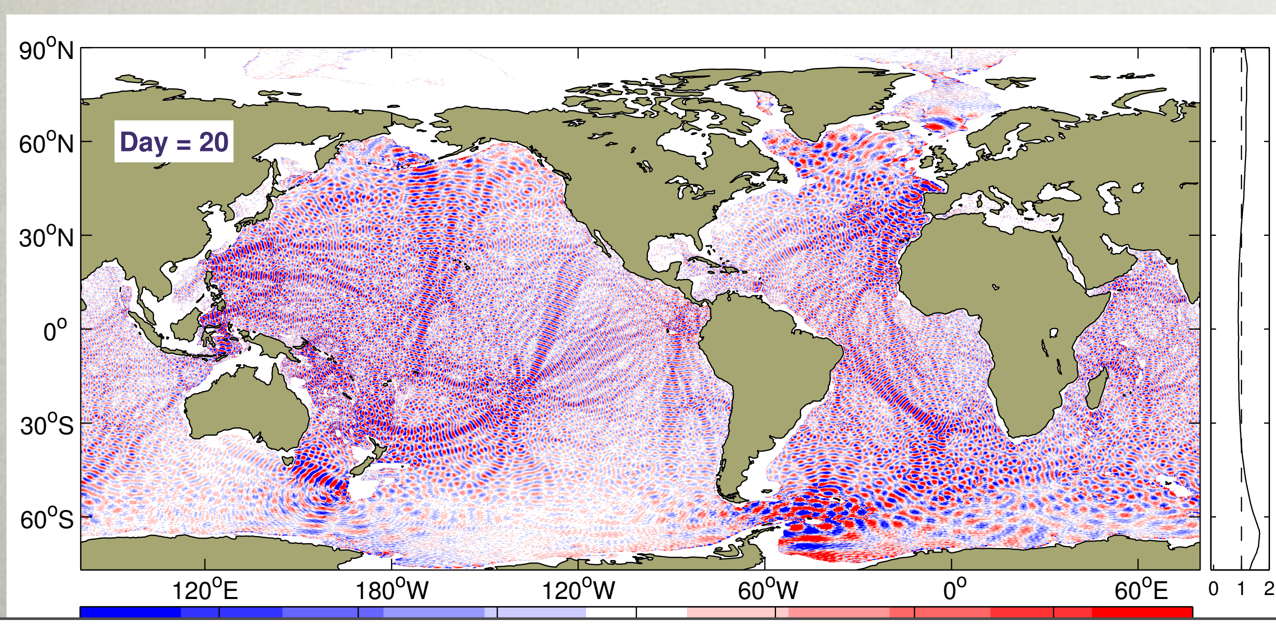
Moon gains angular momentum and orbital energy
Earth loses both, plus extra energy dissipation

$$\Rightarrow \quad \frac{dE}{dt} = -3.2 * 10^{12}\,\text{Watts} = -3.2\,\text{TW}$$

About 2 TW of that manifests itself as small-scale turbulent dissipation in the ocean
Part of it is found in internal tides and their interaction with topography







Internal tides spreading through the ocean

Simmons, Hallberg, Arbic 2004
simple two-layer model

Related work with
Miranda Holmes-Cerfon:

How long can the internal tide survive in real ocean?

Under consideration for publication in J. Fluid Mech. 1

Decay of an internal tide due to random topography in the ocean

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(Received 5 February 2011)

Nonlinear interactions with the mean flow

Under consideration for publication in J. Fluid Mech. 1

Forcing of oceanic mean flows by dissipating internal waves

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(Received 17 January 2012)

Internal waves have strong
horizontal vorticity
Requires different flavour of theory

Governing Equations

Rotating Boussinesq equations on an f-plane

Velocity $\boldsymbol{u} = (u, v, w)$ Coriolis vector $\boldsymbol{f} = f\widehat{\boldsymbol{z}}$

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f} \times \boldsymbol{u} + \boldsymbol{\nabla} P = b\widehat{\boldsymbol{z}} - \boldsymbol{\nabla}\phi$$

tidal potential

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

The tidal potential can be eliminated by using a reference frame moving back and forth with the barotropic tide of the ocean

$$\frac{D(b + N^2 z)}{Dt} = \frac{Db}{Dt} + N^2 w = 0$$

stratification and buoyancy

No-normal-flow boundary conditions at ocean top and bottom

Linear equations

$$\boldsymbol{u} = a\,\boldsymbol{u}' + O(a^2) \quad \text{where} \quad a \ll 1$$

$$(\partial_{tt} + N^2)(w'_{xx} + w'_{yy}) + (\partial_{tt} + f^2)w'_{zz} = 0$$

$$w'(x, y, z, t) = \Re\left(e^{-i\omega t}\, w(x, y, z)\right)$$

time-periodic wave motion

$$w_{xx} + w_{yy} - \underbrace{\frac{\omega^2 - f^2}{N^2 - \omega^2}}_{=\mu^2}\, w_{zz} = 0$$

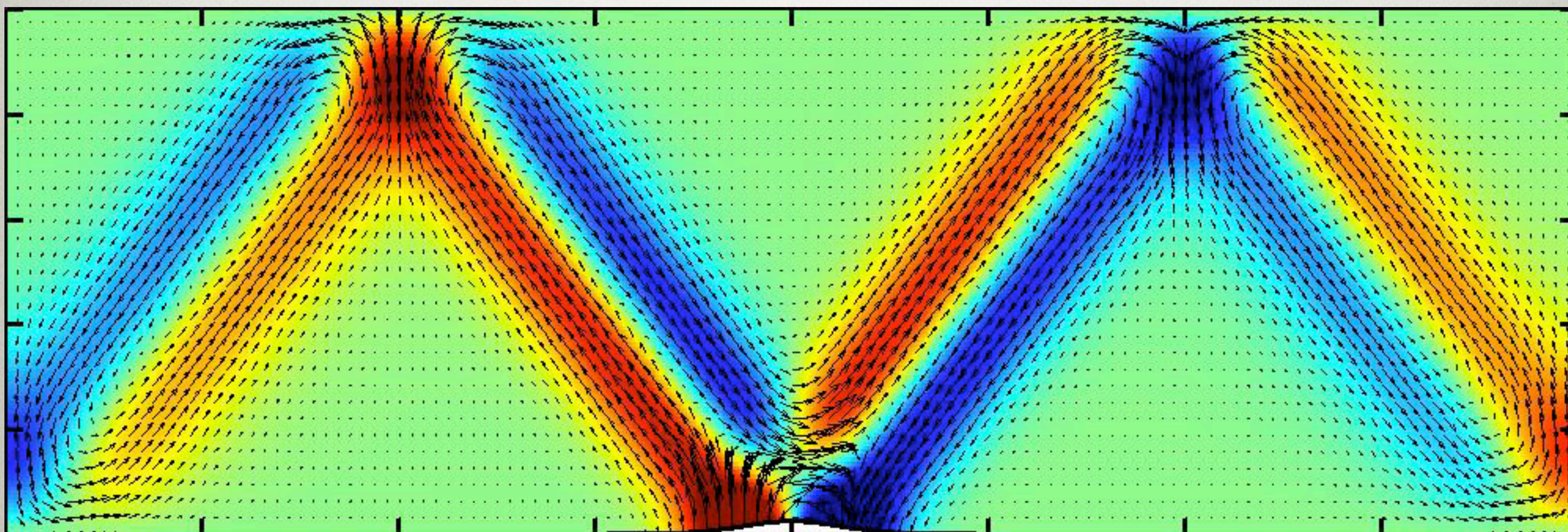
Spatial wave structure governed by a hyperbolic equation ..

$\mu = 0.075$ for lunar semi-diurnal tide at Hawaii (M2)

Internal Tide Generation (Two spatial dimensions)

color: vertical velocity
arrows: velocity

Solution for compact topography computed using a Green's function for unbounded channel

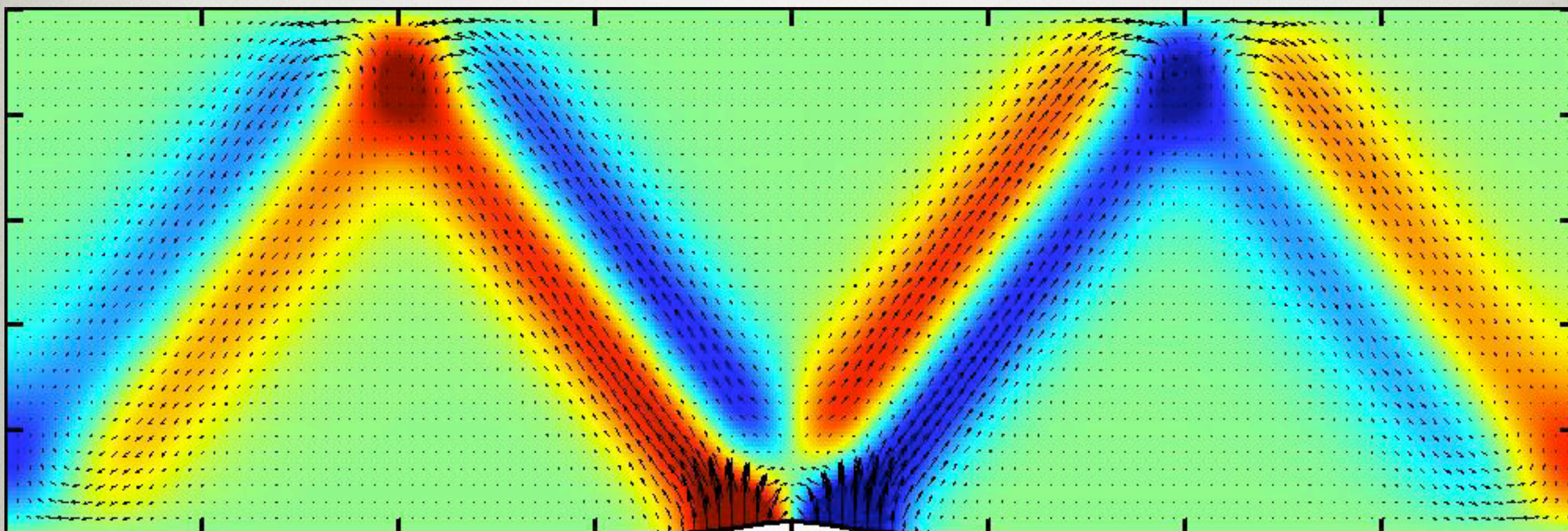


Mathematical model with ocean at rest, bottom topography moving back and forth with excursion amplitude 100-200 metres (exaggerated in plot)

Wave Energy flux does not tell..

color: vertical velocity
arrows: energy flux

Time-averaged wave energy flux can be used to diagnose energy conversion (approx. 1.5 TW in global ocean), but does not explain where interactions with mean flow take place



Wave energy $E = \frac{1}{2}\left(|\boldsymbol{u}'|^2 + \frac{b'^2}{N^2}\right)$ Energy conservation law $\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot(P'\boldsymbol{u}') = 0$

Eulerian Wave-Mean Interaction Theory

Phase-averaged mean flow + Reynolds decomposition

$$\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}' \quad \Rightarrow \quad \overline{\boldsymbol{u}'} = 0$$

Uninviting already at leading order: $\overline{\boldsymbol{u}} = O(a^2)$

$$\frac{\partial \overline{\boldsymbol{u}}}{\partial t} + \boldsymbol{f} \times \overline{\boldsymbol{u}} + \boldsymbol{\nabla} \overline{P} - \overline{b}\widehat{\boldsymbol{z}} = -\boldsymbol{\nabla} \cdot (\underline{\overline{\boldsymbol{u}'\boldsymbol{u}'}})$$

$$\boldsymbol{\nabla} \cdot \overline{\boldsymbol{u}} = 0$$

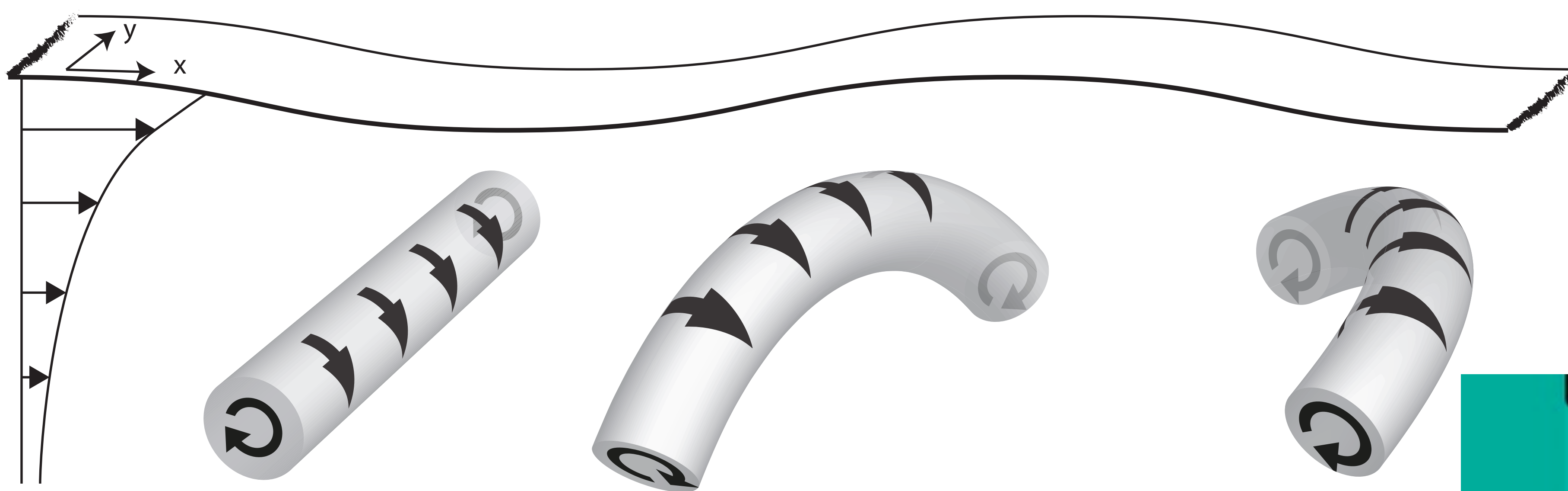
Many source terms, also complications at moving boundary: $\overline{\boldsymbol{u}} \cdot \boldsymbol{n} \neq 0$

Very hard to draw conclusions about $\overline{\boldsymbol{u}}_t$

$$\frac{\partial \overline{b}}{\partial t} + N^2 \overline{w} = -\boldsymbol{\nabla} \cdot (\underline{\overline{b'\boldsymbol{u}'}})$$

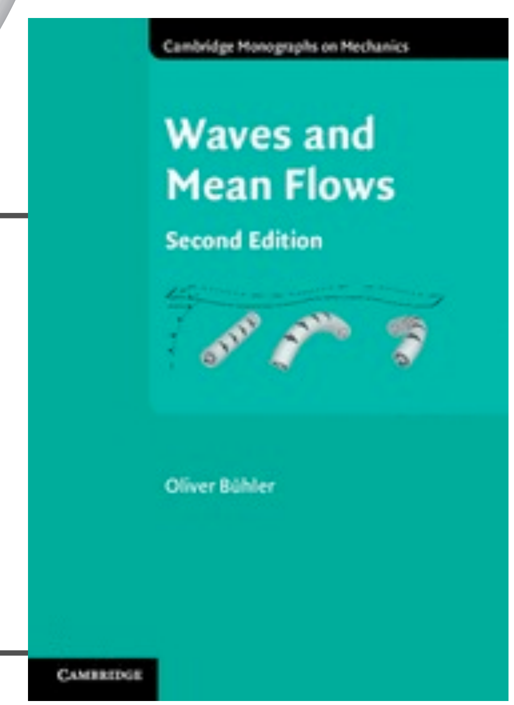
Lagrangian Wave-Mean Interaction Theory

Second, paperback edition: ~~Sept 2013, Dec 2013,~~ ~~March 2014~~ April 2014!



incompressible Lagrangian-mean flow is (10.99), i.e.,

$$\overline{\mathrm{D}}^L(\boldsymbol{\nabla}\times(\overline{\boldsymbol{u}}^L-\mathbf{p}))=[(\boldsymbol{\nabla}\times(\overline{\boldsymbol{u}}^L-\mathbf{p}))\cdot\boldsymbol{\nabla}]\overline{\boldsymbol{u}}^L.$$



Lagrangian Wave-Mean Interaction Theory

Obtain nice equations in vorticity form. At leading order:

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\overline{\boldsymbol{u}}^L + \boldsymbol{f}\,\boldsymbol{\nabla}\cdot\overline{\boldsymbol{u}}^L - f\frac{\partial\overline{\boldsymbol{u}}^L}{\partial z} - \boldsymbol{\nabla}\times(\bar{b}^L\widehat{\boldsymbol{z}}) = \frac{\partial}{\partial t}\boldsymbol{\nabla}\times\mathbf{p}$$

$$\frac{\partial\bar{b}^L}{\partial t} + N^2\overline{w}^L = 0 \qquad \boldsymbol{\nabla}\cdot\overline{\boldsymbol{u}}^L = \frac{1}{2}\frac{\partial}{\partial t}\sum_{i,j}\frac{\partial^2(\overline{\xi_i\xi_j})}{\partial x_i\partial x_j}$$

Pseudomomentum vector:

$$\mathbf{p} = -\overline{(\boldsymbol{\nabla}\boldsymbol{\xi}\cdot[\boldsymbol{u}' + \frac{1}{2}\boldsymbol{f}\times\boldsymbol{\xi}])} \qquad \frac{\partial\boldsymbol{\xi}}{\partial t} = \boldsymbol{u}'$$

$$\left(\mathbf{p} = \frac{\boldsymbol{k}}{\hat{\omega}}E \quad \text{for plane waves}\right)$$

Steady non-dissipating waves do not force the mean flow!

Add Wave Dissipation

Simple model for wave dissipation: buoyancy damping

$$\frac{\partial b'}{\partial t} + N^2 w' = -\alpha b' \quad \text{where} \quad \alpha > 0.$$

Leading-order mean flow equations for steady waves:

$$\frac{\partial \overline{b}^L}{\partial t} + N^2 \overline{w}^L = 0 \qquad \boldsymbol{\nabla} \cdot \overline{\boldsymbol{u}}^L = 0$$

$$\frac{\partial}{\partial t} \boldsymbol{\nabla} \times \overline{\boldsymbol{u}}^L - f \frac{\partial \overline{\boldsymbol{u}}^L}{\partial z} - \boldsymbol{\nabla} \times (\overline{b}^L \widehat{\boldsymbol{z}}) = \boldsymbol{\nabla} \times \boldsymbol{F}$$

Can compute effective mean force based on complex w(x,y,z):

$$\boldsymbol{F} = \frac{\alpha}{2\omega} \frac{N^2}{\omega^2 + \alpha^2} \Im(w^* \boldsymbol{\nabla} w)$$

$$w'(x, y, z, t) = \Re\left(e^{-i\omega t} w(x, y, z)\right)$$

Mean flow response

3d: dissipating tides cause a strong interaction

$$Q = \frac{1}{\rho}(\boldsymbol{\nabla} \times \boldsymbol{u} + \boldsymbol{f}) \cdot \boldsymbol{\nabla}\theta$$

GLM potential vorticity deviation at leading order

$$\overline{Q}^L = \widehat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times (\overline{\boldsymbol{u}}^L - \mathbf{p}) + \frac{f}{N^2}\overline{b}_z^L$$

dissipative force

Time evolution

$$\frac{\partial \overline{Q}^L}{\partial t} = \widehat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} \quad \Rightarrow \quad \overline{Q}^L(\boldsymbol{x}, t) = \overline{Q}^L(\boldsymbol{x}, 0) + t\,\widehat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F}$$

PV changes without bound due to wave-induced effective force, so strong interaction

Fundamental link between PV forcing and wave dissipation?

34 *Nicolas Grisouard and Oliver Bühler*

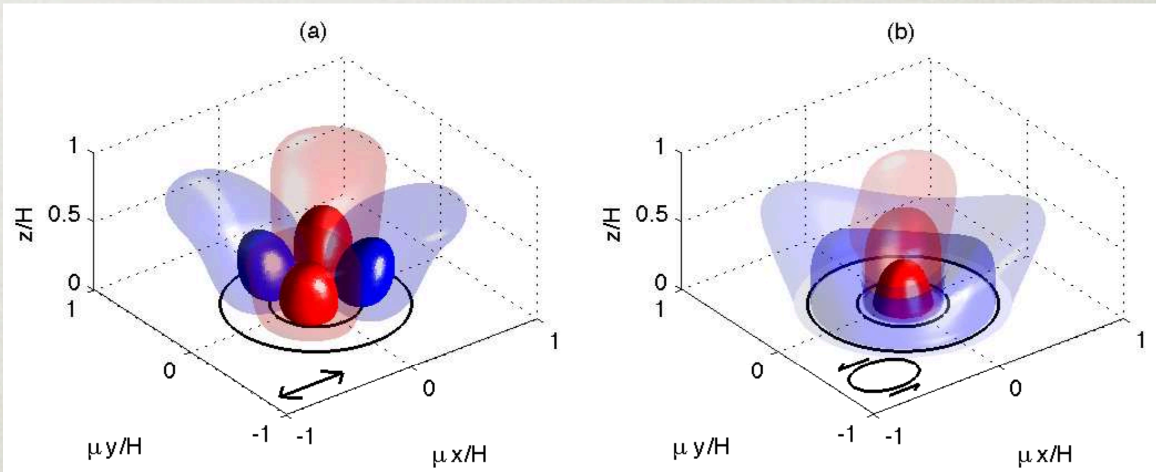
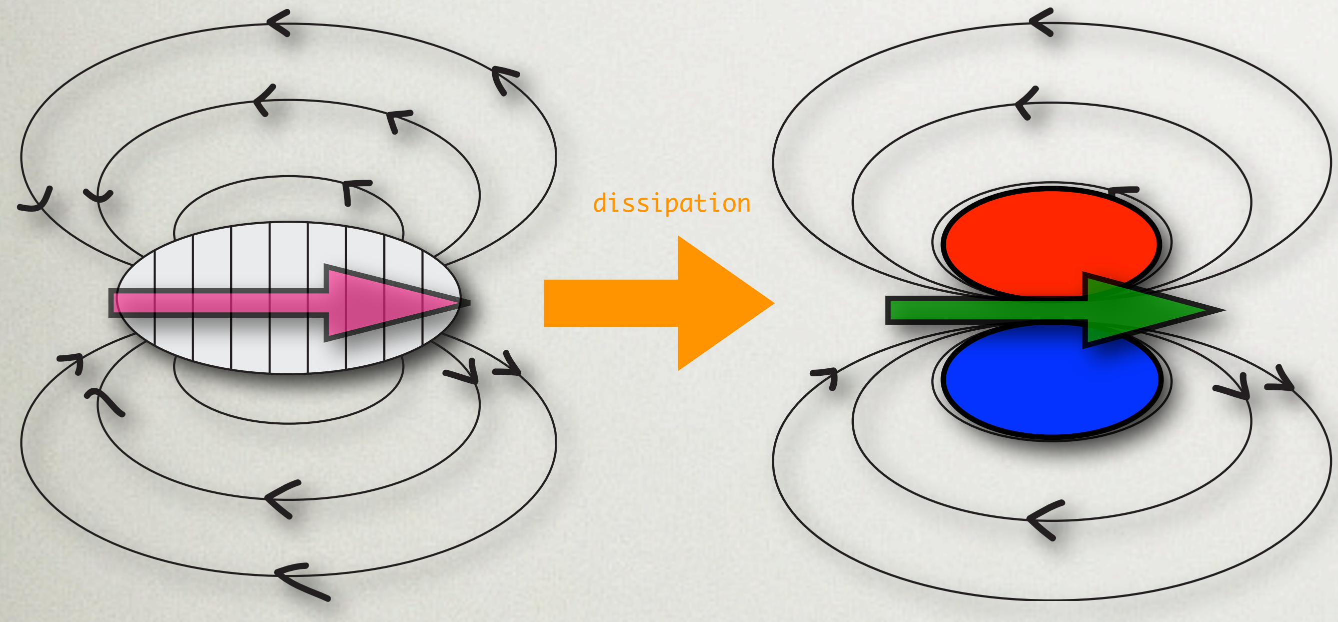


FIGURE 7. Two cases for which the linear dissipation rate per unit time α is increased from $\omega/10$ to $\omega/2$. (a) Caption as for Figure 3, with the exception that $\max[C_z L/(aU_0\omega)] \approx 5.9 \times 10^{-6}$. (b) Caption as for Figure 6(a), with the exception that $\max[C_z L/(aU_0\omega)] \approx 140 \times 10^{-6}$.

A rose by any other name: pseudomomentum and impulse, flip sides of the same coin

Dissipative pseudomomentum rule



Wavepacket

Vortex dipole

$$\boxed{\mathcal{P} + \mathcal{I} = \text{const}}$$

Effective mean force for PV is (minus) the dissipation density of pseudomomentum

2d examples

$$\mathcal{P} = \int \mathbf{p}\, dxdy$$

$$\mathcal{I} = \int (y, -x)\, q\, dxdy$$

$$q = \frac{\boldsymbol{\nabla} \times \boldsymbol{u}}{h}$$

Pseudomomentum is converted into vorticity impulse under dissipation

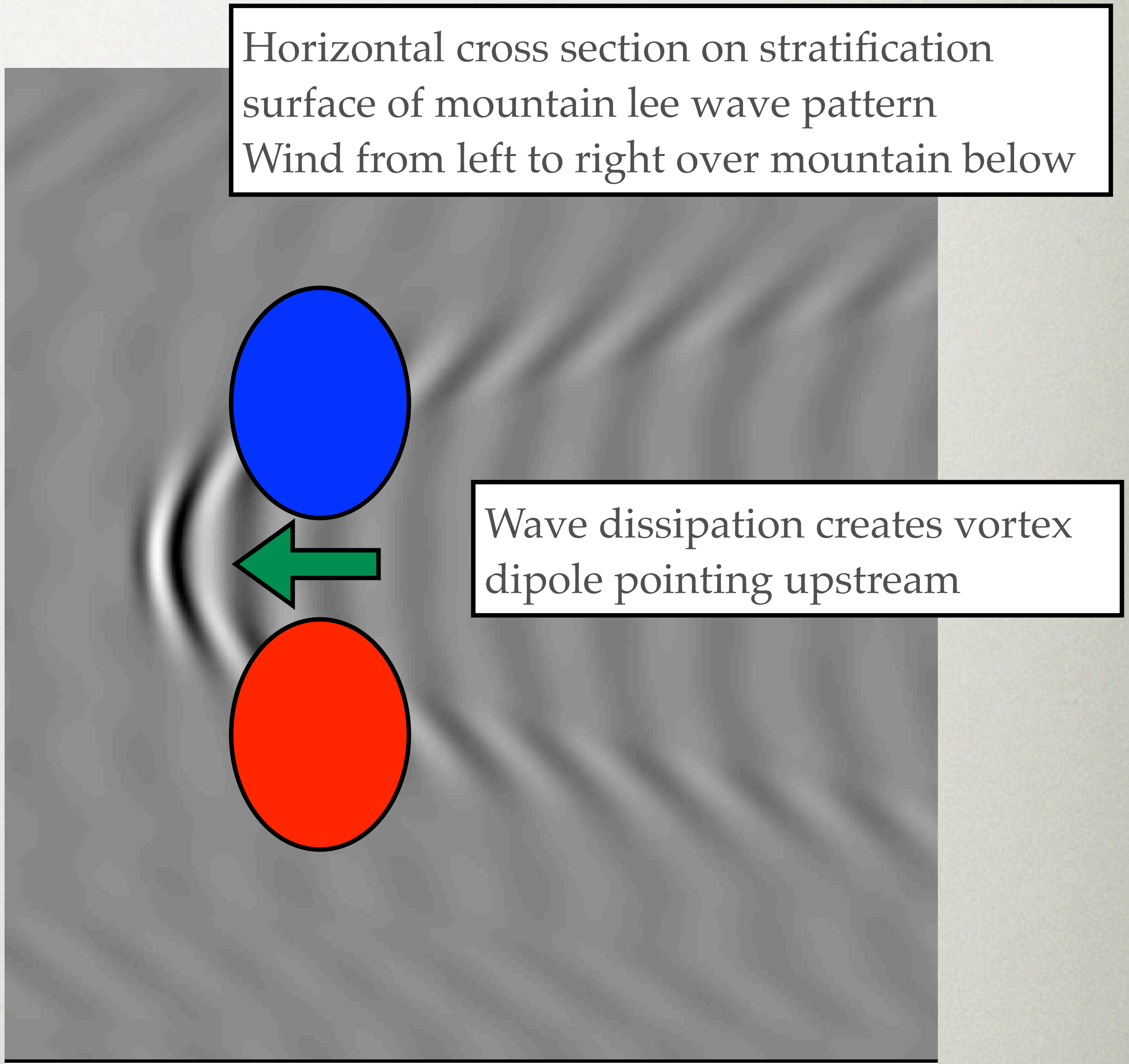
3d: potential vorticity response to localized wavetrain

Pseudomomentum plus Impulse conservation law holds in 3d stratified flow at low Froude number

Impulse now based on 3d PV:

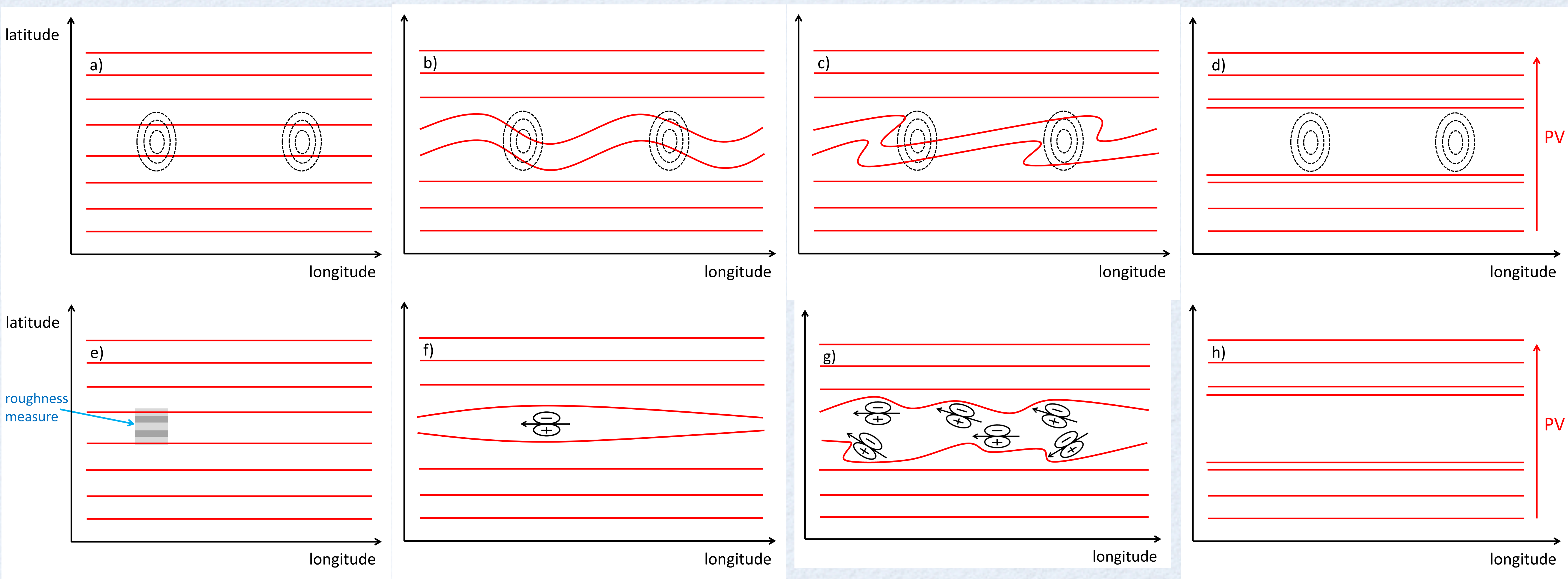
Rossby-Ertel PV

$$q = \frac{(\boldsymbol{\nabla} \times \boldsymbol{u} + \boldsymbol{f}) \cdot \boldsymbol{\nabla}\theta}{\rho}$$



Local pseudomomentum rule is also helpful for thinking about the zonally averaged problem

Atmospheric Wave Drag Under the Microscope (Cohen, Gerber, Bühler 2014)

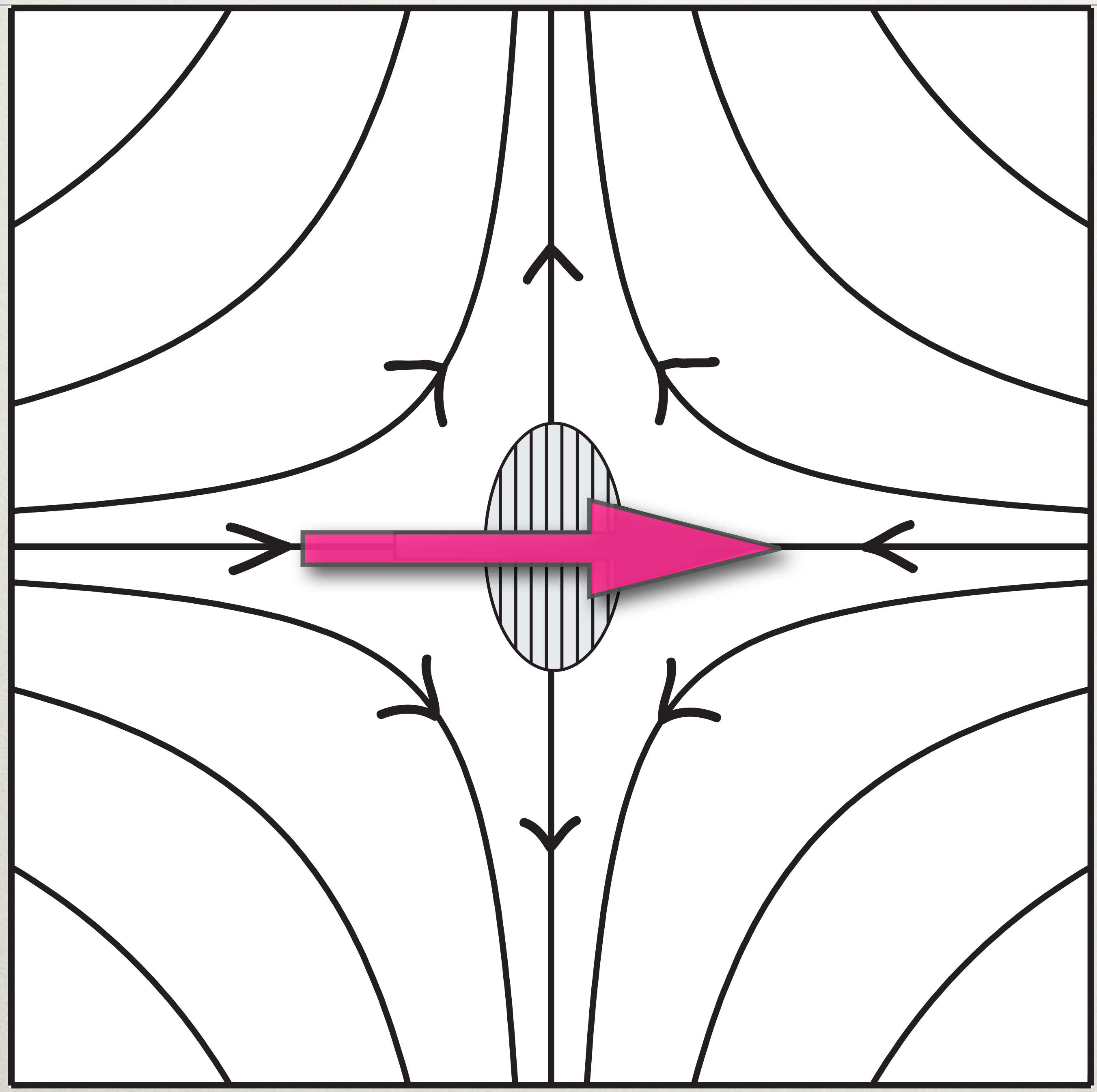


Can piece together global PV rearrangement from localized wave breaking events

Mean-flow refraction and pseudomomentum

$$\boxed{\mathbf{p} = \boldsymbol{k}\, A}$$

$$A = \frac{E}{\hat{\omega}}$$



Wavepacket is squeezed in x and stretched in y by basic strain flow. Wave action is constant

Ray tracing: Wavenumber vector **k** increases in size

Pseudomomentum **p** increases as well

Pseudomomentum changes, what about the vortex impulse?

Pseudomomentum plus impulse conservation law

McIntyre+B, 2005

Potential vorticity

$$\overline{q}^L = \overline{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}^L = \widehat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times (\overline{\boldsymbol{u}}^L - \mathbf{p}_H)$$

GLM theory used here

Impulse

$$\mathbf{I}(t) = \iiint \mathbf{i}\,(\boldsymbol{x}, t)\; \mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad \text{where} \qquad \mathbf{i} = (y, -x, 0)\,\overline{q}^L$$

skew linear moment of PV

Pseudomomentum

$$\mathbf{P}_H \equiv \iiint \mathbf{p}_H \; \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\frac{\mathrm{d}\mathbf{P}_H}{\mathrm{d}t} = -\iiint (\boldsymbol{\nabla}_H \overline{\boldsymbol{u}}^L) \cdot \mathbf{p}_H \; \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

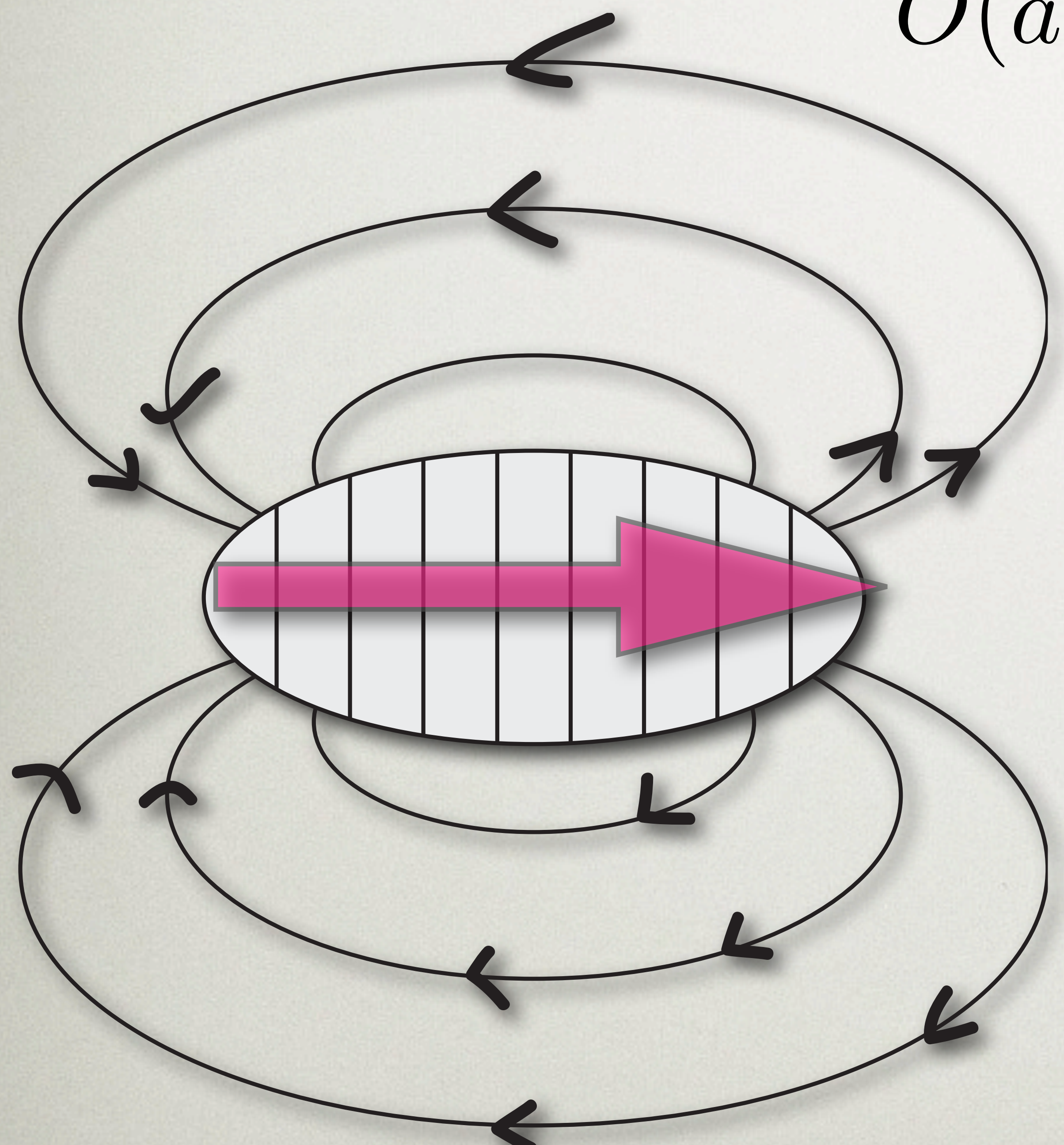
$$\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t} = \iiint (\boldsymbol{\nabla}_H \overline{\boldsymbol{u}}^L) \cdot \mathbf{p}_H \; \mathrm{d}x\mathrm{d}y\mathrm{d}z\ .$$

Refraction and dissipation terms included!!

Holds with dissipation and with refraction!
How can this work?

$$\mathbf{P}_H + \mathbf{I} \;=\; \text{constant}$$

The missing link: Bretherton's return flow



$O(a^2)$ Large-scale dipolar return flow at second order in wave amplitude

Far-field mean velocity is non-divergent and decays with square of distance to wavepacket

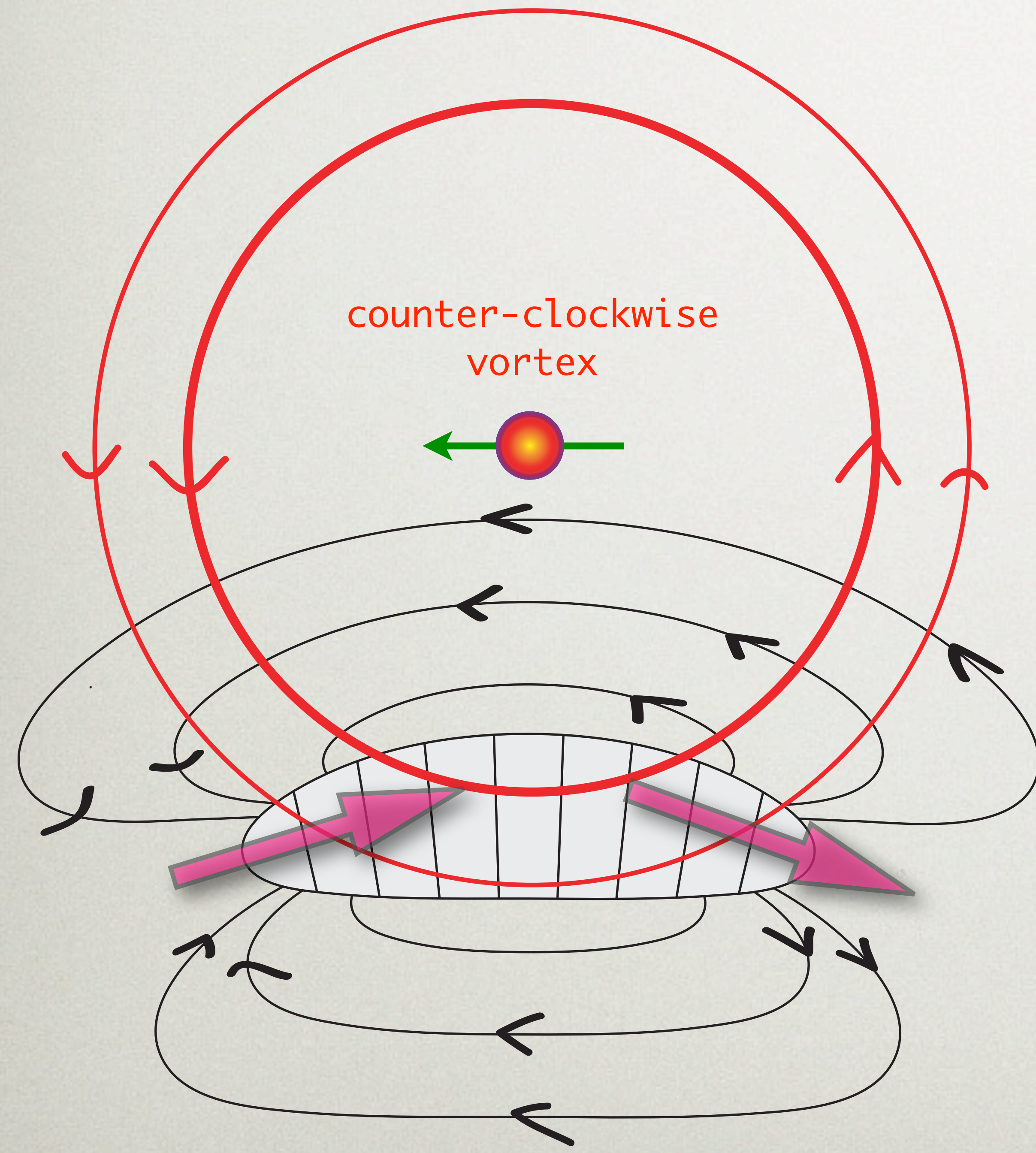
This O(a^2) Bretherton return flow can participate in wave-mean interactions and move O(1) vortices.

Can show that it contributes to vortex impulse dynamics!

Feynman:
"children on a slide"

Example 1: remote recoil

Wavepacket scattering by vortex



1) Wave-induced mean flow at O(a2) pushes O(1) vortex to the left

Impulse change well-defined and positive

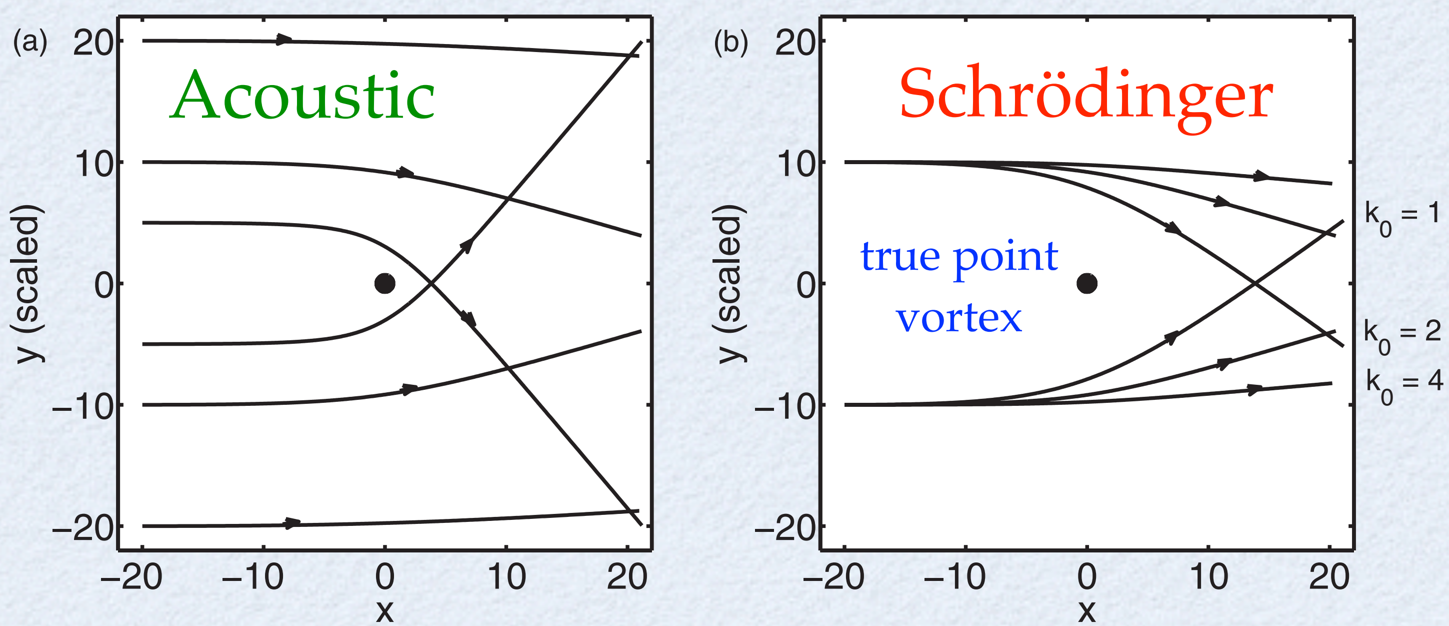
2) Pseudomomentum vector is changed by inhomogeneous vortex flow

Pseudomomentum change well-defined and negative

Equal and opposite recoil in acoustic system (B+McIntyre 2003)

Remote recoil also found in defocusing nonlinear Schrödinger equation (Guo & Bühler 2014)

027105-11 Y. Guo and O. Bühler Phys. Fluids **26**, 027105 (2014)



$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\nabla^2\psi + (V + U_0|\psi|^2)\psi$$

$$\psi = \sqrt{\rho}\,\exp(i\theta) \quad \text{and} \quad \boldsymbol{u} = \frac{\hbar}{m}\boldsymbol{\nabla}\theta$$

C. Validity of ray tracing during wave collapse

Ray tracing approximates linear wave theory under the assumption that the waves form a slowly varying wavetrain, so when ray tracing predicts a singular solution such as the formation of a wave

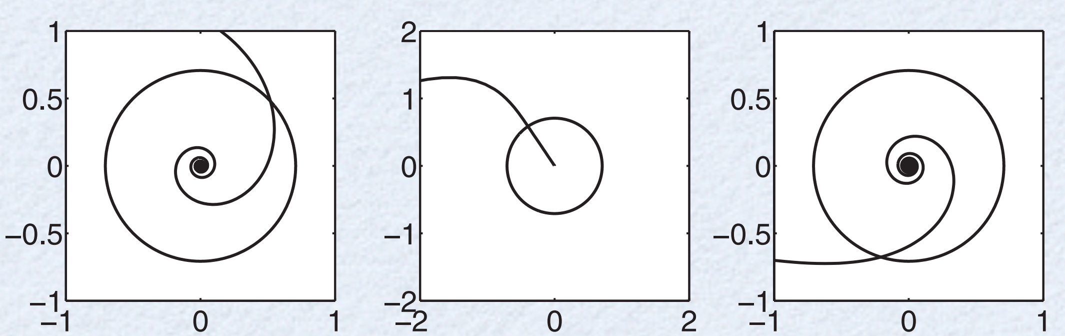


FIG. 6. A positive unit vortex is placed at (0, 0) and the circle $r = 1/\sqrt{2}$ broadly marks the vortex core region, in which $H \leq 0.07$. Three different rays are shown, with initial conditions corresponding to the three points in Fig. 5. All rays are started at $(x, y) = (-5, 0)$. Left (point A): retrograde collapsing ray with $M = -1.998$, $\omega = 0.7373$, and $\boldsymbol{k} = (0.6574, 0.3996)$. Middle (point B): non-rotating ray with $M = -1.000$, $\omega = 0.2913$, and $\boldsymbol{k} = (0.2626, 0.2)$. Right (point C): prograde collapsing ray with $M = 0$, $\omega = 0.515$, and $\boldsymbol{k} = (0.5, 0)$.

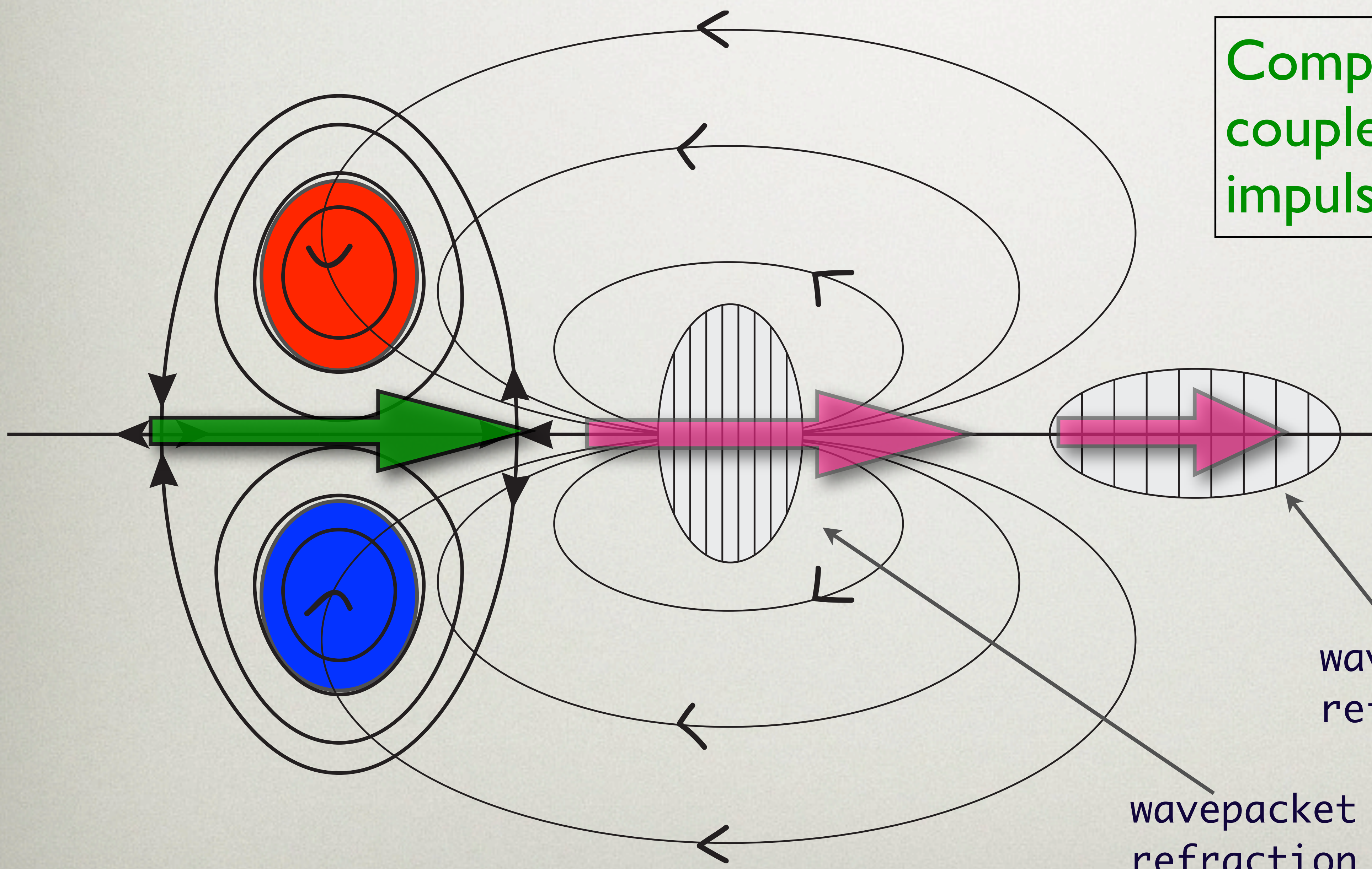
Scale-selective scattering angle and recoil force in Schrödinger equation

Also self-consistent collapsing wave rays all the way to the point vortex..

Example 2: Wave capture

Dipole straining increases wavepacket pseudomomentum.
Wavepacket return flow compresses vortex dipole and reduces impulse.
Both compensate, and the sum of P + I is conserved!

Compressing a vortex couple *reduces* its impulse $I \propto \Gamma d$



GI Taylor 1921: Diffusion by continuous movements

Effective particle diffusivity

$$D = \frac{1}{2}\frac{d}{dt}\mathbb{E}(X^2) = \int_0^\infty C(\tau)\, d\tau = \frac{1}{2}\hat{C}(0)$$

Strong particle dispersion by weakly dissipative random internal waves

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Simple stochastic models and direct nonlinear numerical simulations of three-dimensional internal waves are combined in order to understand the strong horizontal particle dispersion at second order in wave amplitude that arises when small-amplitude internal waves are exposed to weak dissipation. This is contrasted with the well-known results for perfectly inviscid internal waves, in which such dispersion arises only at fourth order in wave amplitude.

Lagrangian power spectrum at <u>zero frequency</u>