Symmetry Breaking, Zonostrophic Bifurcation, and Beyond*

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Abstract

Recent results on the zonostrophic bifurcation are described. A brief description of the motivations from a plasma-physics perspective is given. The S3T/CE2 formalism is reviewed. The zonostrophic instability is cast as a problem of spontaneous symmetry breaking of the statistical homogeneity of turbulence. The results of Srinivasan and Young are recovered and generalized. The intimate relationship between zonostrophic instability and modulational instability is described. It is shown that zonostrophic instability is one example of a pattern-formation phenomenon. That insight is exploited to discuss the fate of the bifurcated zonal flows. Above threshold, a continuous band of zonal-flow wave numbers is allowed; however, only a restricted band is stable. The implication for the observed phenomenon of merging jets is discussed. The stability region is calculated numerically for the CE2 closure of the barotropic vorticity equation. Some future lines of research are indicated.

Much of this work was done in a great collaboration with Jeff Parker.

- **J. B. Parker and J. A. Krommes**, "Zonal flow as pattern formation: Merging jets and the ultimate jet length scale," Phys. Plasmas **20**, 100703 (2013). *(4-page Letter.)*
- **J. B. Parker and J. A. Krommes**, "Generation of zonal flows through symmetry breaking of statistical homogeneity," New J. Phys. (2014), in press. *(28 pages; lots of details.)*
- **J. B. Parker and J. A. Krommes**, "Zonal flow as pattern formation," in *Zonal Jets*, edited by B. Galperin and P. Read (Cambridge U. Press, Cambridge, 2014), Chap. V.2.4 (submitted). *(Partly pedagogical, but also some new results on modulational instability.)*
- **J. A. Krommes and J. B. Parker**, "Statistical closures and zonal flows," in *Zonal Jets*, edited by B. Galperin and P. Read (Cambridge U. Press, Cambridge, 2014), Chap. V.2.4 (submitted). *(Introductory review-type article.)*
- **J. B. Parker**, *Zonal Flows and Turbulence*, PhD dissertation (Princeton U.; expected May, 2014). *(Describes most of our current understanding.)*

Various posters are relevant to this talk.

- **Marston & Tobias** — "Direct statistical simulation of astrophysical flows"
- **Meyer** — "2nd order closure form cumulant expansion for boundary layer turbulence"
- **Nardini** — "Stochastic averaging, jet formation, and bistability in turbulent planetary atmospheres"
- **Parker** — "Connection between zonostrophic instability and modulational instability"
- **Qi** — "Direct statistical simulation of flows by expansions in cumulants"

See also the later talks by

- **Bouchet** — "Abrupt transitions and large deviations in geophysical turbulent flows"
- **Marston** — "Multiscale approach to the direct statistical simulation of flows"

Zonal jets and flows are of interest in diverse physical contexts.

Zonal jets and flows have been seen in

- planetary atmospheres,
- geophysics,
- accretion disks, and
- fusion plasmas.

Some possible mechanisms include[a]

- turbulent cascade,
- **modulational instability**,
- mixing of potential vorticity,
- **statistical theories**.

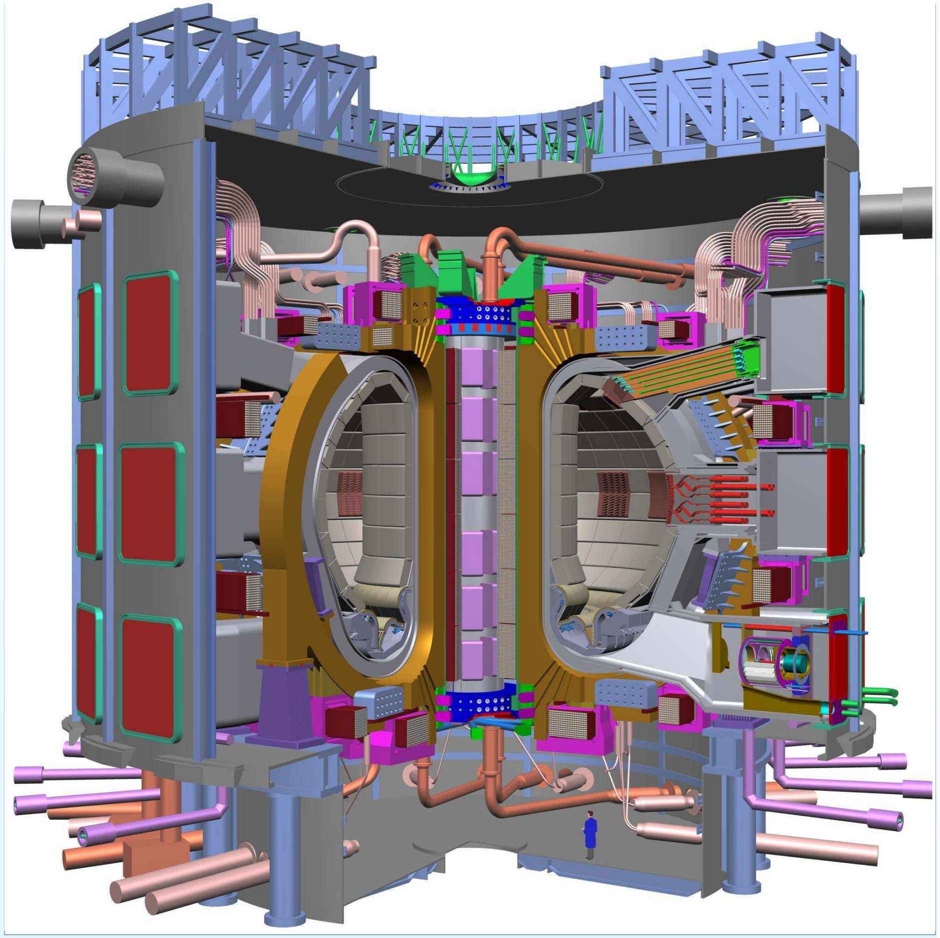

Fig. 1. The ITER fusion research device now under construction in France.

[a] See, for example, **Bakas & Ioannou**, "On the mechanism underlying the spontaneous emergence of barotropic zonal jets," J. Atmos. Sci. **70**, 2251–71 (2013).

The Charney–Hasegawa–Mima equation is a historically important paradigm for plasma microturbulence.

$$\frac{\partial}{\partial t}\Big(\underbrace{\nabla_\perp^2\phi}_{\text{vorticity}} - \underbrace{\phi}_{\substack{\text{parallel}\\ \text{electron}\\ \text{response}}}\Big) + \underbrace{V_*\frac{\partial\phi}{\partial x}}_{\substack{\boldsymbol{E}\times\boldsymbol{B}\text{ advection}\\ \text{of background}\\ \text{density profile } \langle n_i\rangle(y)}}$$

$$+ \underbrace{\boldsymbol{V}_E\cdot\boldsymbol{\nabla}}_{\substack{\boldsymbol{E}\times\boldsymbol{B}\text{ advection}\\ \text{of fluctuations in}\\ \text{ion gyrocenter density}}} (\nabla_\perp^2\phi - \cancel{\phi}) = 0. \qquad (1)$$

(Geophysical coordinate system; $\boldsymbol{V}_E \doteq \widehat{\boldsymbol{z}} \times \boldsymbol{\nabla}\phi$.) This is derived from

$$\partial_t n_i^G + \boldsymbol{\nabla}\cdot(\boldsymbol{V}_E\, n_i^G) = 0 \qquad \text{(continuity eq'n for ion gyrocenters)}, \qquad (2a)$$

$$-\nabla_\perp^2\phi = n_i^G - n_e^G \qquad \text{(gyrokinetic Poisson equation)}, \qquad (2b)$$

$$\delta n_e^G = e\delta\phi/T_e \qquad \text{(adiabatic electron response)}. \qquad (2c)$$

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In dimensional units,

$$\nabla_\perp^2\phi - L_{\mathrm{d}}^{-2}\phi$$

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The modified Hasegawa–Mima equation is more physically accurate.

$$\partial_t(\nabla_\perp^2\phi - \alpha\phi) + \alpha V_* \partial_x \phi + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}(\nabla_\perp^2\phi - \alpha\phi) = 0. \quad (3)$$

Here the electron response has been modified to

$$\delta n_e^G = \begin{cases} e\delta\phi/T_e & \text{(non-zonal modes)}, \\ 0 & \text{(zonal modes)}^4, \end{cases} \quad (4a)$$

$$= \alpha(e\delta\phi/T_e). \quad (4b)$$

Thus

$$\alpha_{\mathrm{mHM}} = \begin{cases} 1 & (k_\parallel \neq 0), \\ 0 & (k_\parallel = 0). \end{cases} \quad (5)$$

One can treat various important models by just changing α:

$$\alpha_{\mathrm{CHM}} = 1; \quad \alpha_{\mathrm{mHM}} = 1 \text{ or } 0; \quad \alpha_{\mathrm{2DNS}} = 0.$$

[4] The necessity for this form of zonal response was first pointed out by G. Hammett (1993).

The HME has no drive or damping. Generalize to the Hasegawa–Wakatani system.

The (modified) Hasegawa–Wakatani system is

$$\partial_t \varpi + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} \varpi = \alpha D_{\parallel}(\phi - n) + \text{(dissipation)}, \tag{6a}$$

$$\partial_t n + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} n = \alpha D_{\parallel}(\phi - n) + V_* \, \partial_x \phi + \text{(dissipation)}. \tag{6b}$$

- This system is a paradigm for collisional fluctuations in the edge of tokamaks.
- It contains linear instability and damping.
- Its saturated states can contain a mixture of interacting zonal flows and turbulence [see the simulations of Numata et al. (2007)].
- It was then studied by **Farrell & Ioannou**, “A stochastic structural stability theory model of the drift wave–zonal flow system,” Phys. Plasmas **16**, 112903 (2009).

The work of Farrell & Ioannou (2009) was an important cross-over paper.

- “In this work a comprehensive theory for the interaction of jets with turbulence, [SSST], is applied to the problem of understanding the formation and maintenance of the zonal jets that are crucial for enhancing plasma confinement in fusion devices.”
- “Multiple DW–ZF regimes are predicted to exist in parameter space including a regime of steady zonal flows as well as regimes of periodic, quasiperiodic, and chaotic bursting or “sawtooth” behavior.”
- “These regimes provide opportunity for placing and manipulating confinement devices to be in a desired dynamical state between high and low confinements.”
- Irrational exuberance?

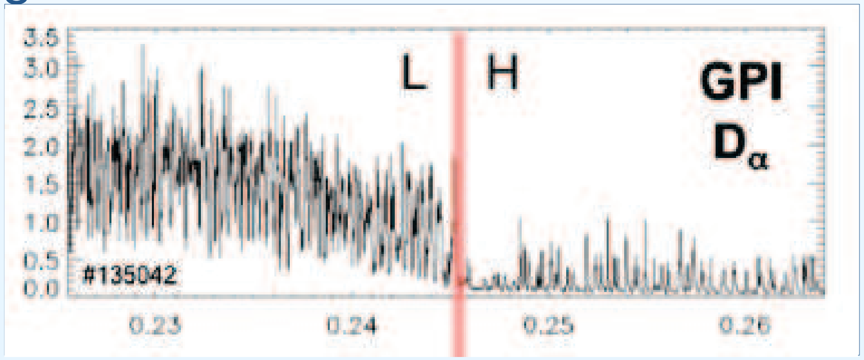

Fig. 2. Transition between the low (L) mode and the high (H) mode. From Zweben, PoP **17**,102502 (2010).

What is Stochastic Structural Stability Theory (SSST or S3T)?

S3T is a particular stochastic model.

Consider the stochastic PDE

$$\partial_t \widetilde{\psi}(\boldsymbol{x}, t) = L\widetilde{\psi} + \tfrac{1}{2} N \widetilde{\psi}\widetilde{\psi} + \underbrace{\widetilde{f}_{\text{ext}}(\boldsymbol{x}, t)}_{\text{random forcing}}. \tag{7}$$

Decompose $\widetilde{\psi} = \langle \psi \rangle + \delta\psi$. Then

$$\partial_t \langle \psi \rangle = L\langle \psi \rangle + \tfrac{1}{2} N \langle \psi \rangle \langle \psi \rangle + \underbrace{\tfrac{1}{2} N \langle \delta\psi\, \delta\psi \rangle}_{\text{Reynolds stress}}, \tag{8a}$$

$$\partial_t \delta\psi = \underbrace{L\delta\psi}_{\substack{\text{linear waves} \\ \text{\& instability}}} + \underbrace{N\langle \psi \rangle \delta\psi}_{\substack{\text{coupling between} \\ \text{mean \& fluctuations}}} + \underbrace{\tfrac{1}{2} N(\delta\psi\, \delta\psi - \langle \delta\psi\, \delta\psi \rangle)}_{\text{eddy–eddy interactions}} + \delta f_{\text{ext}}. \tag{8b}$$

- When doing stochastic modeling, the choice of ensemble is important:
  - homogeneous: high degree of symmetry $\Rightarrow \langle \psi \rangle = 0$;
  - inhomogeneous: less symmetry $\Rightarrow$ possible mean field $\langle \psi(\boldsymbol{X}) \rangle$.

In a homogeneous ensemble, zonal flows must be treated in mean square.

- One realization of a ZF is inhomogeneous: $\widetilde{u}(y,t)$.
- But in a translationally invariant background and with random i.c.'s, the ensemble is homogeneous.
- Then standard homogeneous closures can be applied (e.g., DIA-based Markovian or TFM).

Historically,

- **Diamond et al.** (1998) proposed a "wave" kinetic algorithm for calculation of the ZF growth rate.
- In the wave kinetic approach for a homogeneous ensemble, everything in sight is treated as random and described by a covariance C:

$$\partial_t C_{\rm turb} = \cdots\,, \tag{9a}$$

$$\partial_t C_{\rm ZF} = \text{Reynolds stress}[C_{\rm turb}] + \cdots \tag{9b}$$

$$= 2\gamma_{\rm ZF} C_{\rm ZF} + \cdots\,. \tag{9c}$$

A wave kinetic formalism can be derived by systematic expansion in small q/k.

- **Krommes & Kim** (2000) derived such a wave kinetic algorithm (for the modified HME) by expanding the test-field closure in $\epsilon \doteq q/k \ll 1$.
  - q: ZF wave number; k: turbulence wave number.
  - **Krommes & Kim**, "Interactions of disparate scales in drift-wave turbulence," Phys. Rev. E 62, 8508–39 (2000).
- This expansion procedure was the anisotropic version of Kraichnan's 1976 calculation of eddy viscosity in 2D Navier–Stokes turbulence.
- Recently we have revisited this calculation; we have new results about the relationship between eddy viscosity and modulational instability.

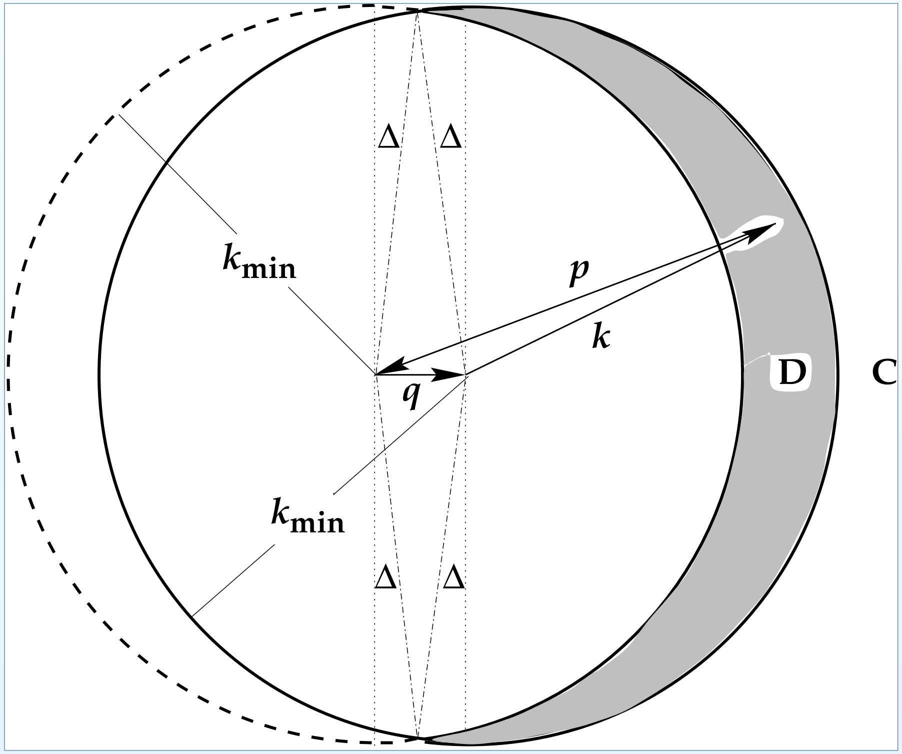

Fig. 4. Integration domains C and D for all turbulence wave vectors $\boldsymbol{k}$ and $\boldsymbol{p}$ that sum to zonal wave vector $\boldsymbol{q}$.

CE2/S3T deal with an inhomogeneous ensemble.

$$\partial_t\langle\psi\rangle = L\langle\psi\rangle + \tfrac{1}{2}N\langle\psi\rangle\langle\psi\rangle + \underbrace{\tfrac{1}{2}N\langle\delta\psi\,\delta\psi\rangle}_{\text{Reynolds stress}}, \tag{10a}$$

$$\partial_t\delta\psi = \underbrace{L\delta\psi}_{\substack{\text{linear waves}\\ \text{\& instability}}} + \underbrace{N\langle\psi\rangle\delta\psi}_{\substack{\text{coupling between}\\ \text{mean \& fluctuations}}} + \underbrace{\tfrac{1}{2}N(\delta\psi\,\delta\psi - \langle\delta\psi\,\delta\psi\rangle)}_{\text{eddy–eddy interactions}} + \delta f_{\text{ext}}. \tag{10b}$$

- CE2: neglect eddy–eddy; $\delta f_{\text{ext}} =$ white noise.
- S3T:
  - eddy–eddy = $\delta f_{\text{int}} - \eta\,\delta\psi$, pick η to conserve energy; or
  - model eddy–eddy by δf_{int}; $\delta f = \delta f_{\text{int}} + \delta f_{\text{ext}} =$ white noise.

$$\partial_t\langle\psi\rangle = L\langle\psi\rangle + \tfrac{1}{2}N\langle\psi\rangle\langle\psi\rangle + \tfrac{1}{2}N\langle\delta\psi\,\delta\psi\rangle, \tag{11a}$$

$$\partial_t\delta\psi = L\delta\psi + N\langle\psi\rangle\delta\psi + \delta f. \tag{11b}$$

White $\delta f \Rightarrow$ exact covariance equation. CE2/S3T are *realizable*.

SSST or CE2 allow one to consider a single ZF realization.

- **Farrell & Ioannou**, "Structural stability of turbulent jets," J. Atmos. Sci. 60, 2101–18 (2003).
- SSST (or S3T) uses zonal averaging to find an equation for the mean ZF.
- Fluctuations are stirred up by random white-noise forcing, so they can be treated by an exact covariance equation for $C(\boldsymbol{x}, \boldsymbol{x}') = C(\boldsymbol{x} - \boldsymbol{x}' \mid \frac{1}{2}(\boldsymbol{x} + \boldsymbol{x}')) \to C_{\boldsymbol{k}}(\boldsymbol{X})$.

Thus the structure of the S3T/CE2 system is

$$\partial_t C_{\boldsymbol{k}}(Y, t) = \underbrace{2\gamma_{\mathrm{lin},\boldsymbol{k}} C_{\boldsymbol{k}}}_{\text{linear growth/damping}} + \underbrace{(NUC)_{\boldsymbol{k}}(Y, t)}_{\text{mean field–eddy interaction}} + \underbrace{2F_{\boldsymbol{k}}}_{\text{forcing}}, \tag{12a}$$

$$\partial_t U(Y, t) = -\mu U + \text{Reynolds stress}[C]. \tag{12b}$$

- This treats the interaction between the ZFs and the turbulence exactly.
- No inverse cascade here (so no zonal collapse at the Rhines scale).

In detail, the CE2 equations are nontrivial.

For the equivalent barotropic vorticity equation and with $\boldsymbol{x} \doteq \boldsymbol{x}_1 - \boldsymbol{x}_2$, $\boldsymbol{X} \doteq \frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$, the two-point correlation functions of vorticity (W) and stream function (C) obey (x = longitude, y = latitude)

$$\partial_t W(\boldsymbol{x} \mid Y, t) + (U_+ - U_-)\partial_x W - (\overline{U}'''_+ - \overline{U}''_-)(\overline{\nabla}^2 + \tfrac{1}{4}\partial_Y^2)\partial_x C$$
$$- [2\beta - (\overline{U}''_+ + \overline{U}''_-)]\partial_Y \partial_y \partial_x C = \underbrace{F(\boldsymbol{x}) - 2\mu W}_{\text{homog. equilib.}}, \quad (13a)$$

$$\partial_t \overline{I} U(Y, t) + \mu U = -\partial_Y \underbrace{[\partial_y \partial_x C(\mathbf{0} \mid Y, t)]}_{\text{Reynolds stress}}, \quad (13b)$$

where

$$U_\pm \doteq U(Y \pm \tfrac{1}{2}y), \quad \overline{U}''_\pm \doteq U''_\pm - L_{\mathrm{d}}^{-2} U_\pm, \quad (14a)$$

$$\overline{\nabla}^2 \doteq \nabla^2 - L_{\mathrm{d}}^{-2}, \quad \overline{I} \doteq 1 - L_{\mathrm{d}}^{-2}\partial_Y^{-2}, \quad (14b)$$

$$W(\boldsymbol{x} \mid Y, t) \doteq (\overline{\nabla}^2 + \partial_y \partial_Y + \tfrac{1}{4}\partial_Y^2)(\overline{\nabla}^2 - \partial_y \partial_Y + \tfrac{1}{4}\partial_Y^2) C(\boldsymbol{x} \mid Y, t). \quad (14c)$$

Srinivasan & Young made a detailed study of the zonostrophic instability using CE2.

- **Srinivisan & Young**, Zonostrophic instability, J. Atmos. Sci. 69, 1633–56 (2012). *(Analysis of the barotropic vorticity equation.)*
- This study determines the shape of the *neutral curve* (Fig. 5):

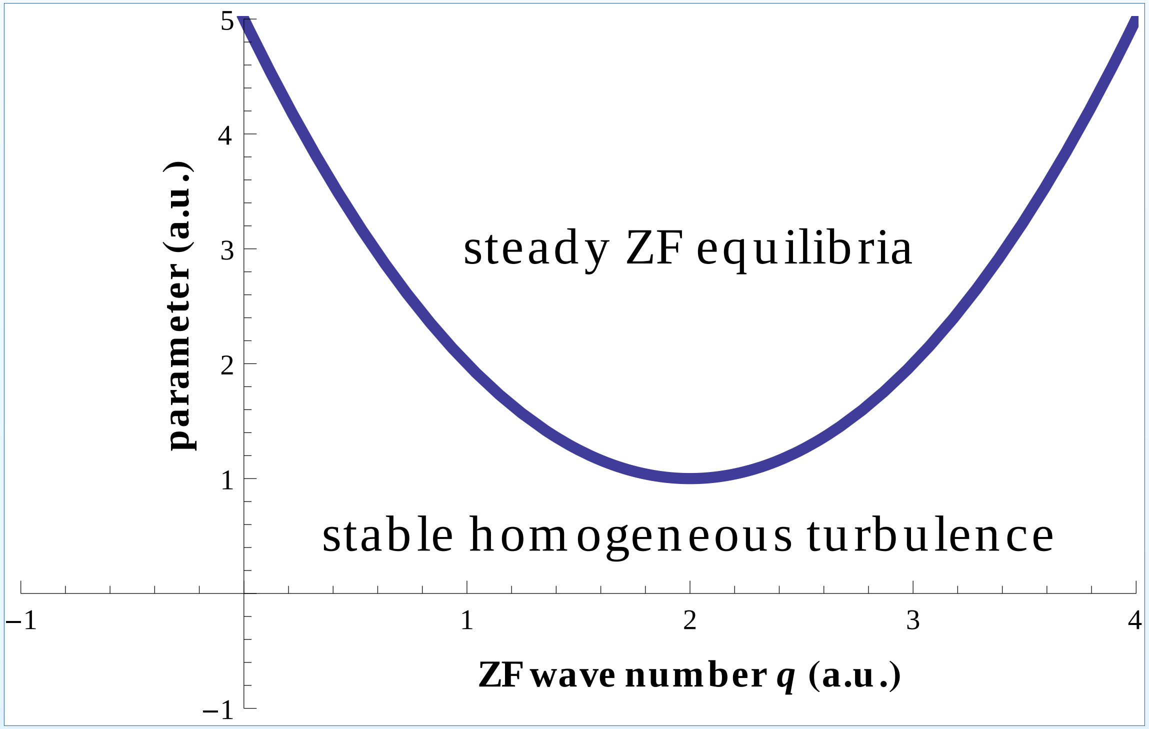

Fig. 5. Illustration of the neutral curve for zonostrophic instability.

What is the fate of the bifurcated zonal flow?

- The zonostrophic instability calculation for CE2 is tedious, but it is linear and deals with a single ZF wave number q.
- However, ZF equilibria are nonlinear and possess harmonic structure.

Strategies (applied to CE2):

- Classical bifurcation analysis: derive Ginzburg–Landau equation (valid just above threshold).
- Calculate bifurcated equilibria numerically.
- Examine the stability of the bifurcated equilibria.

Jeff Parker (PhD dissertation, Princeton U., expected May, 2014) has done all of that and more.

- The entire program can be viewed as an example of *pattern formation*; cf. the onset of convection rolls in a thin layer heated from below. See **Parker & Krommes** (2013; 2014a,b).

A combination of serious analytical and numerical work leads to a stability diagram for steady jet equilibria.

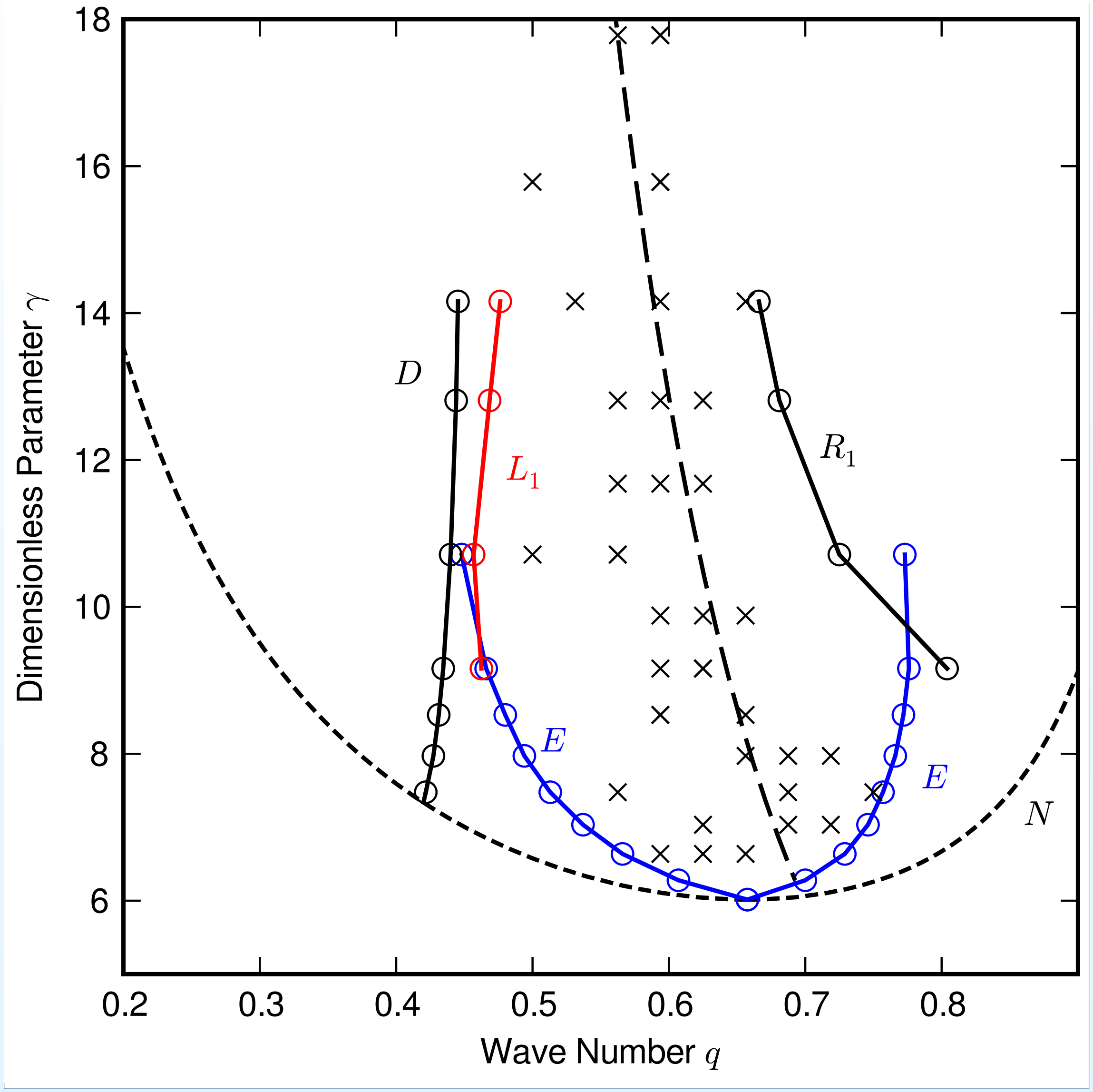

Fig. 6. Stability diagram for steady jet equilibria.

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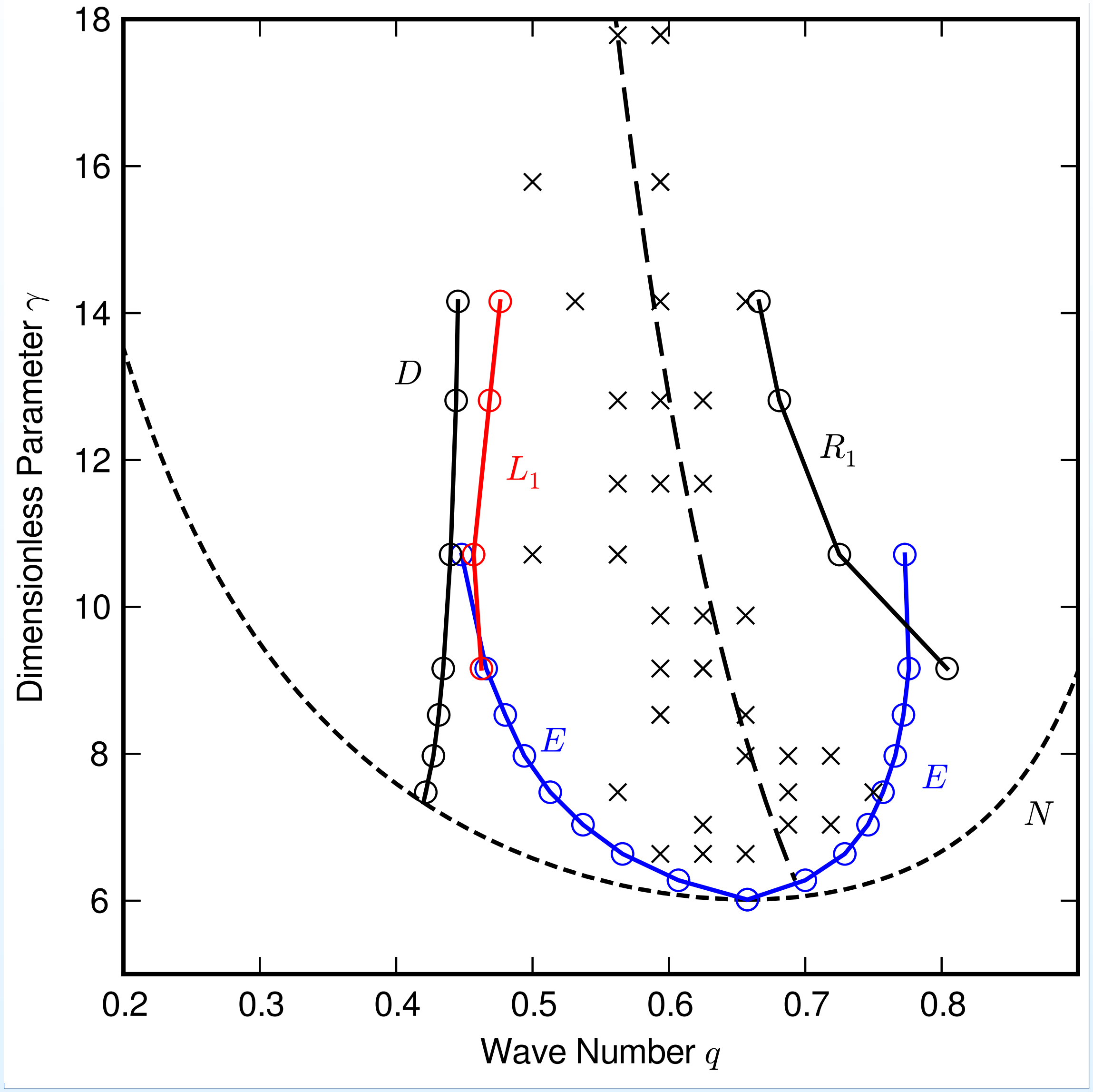

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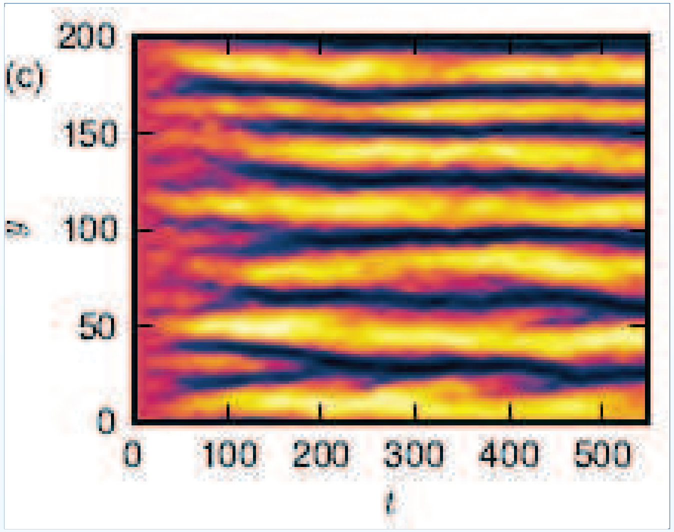

Fig. 7. Merging jets in simulation.

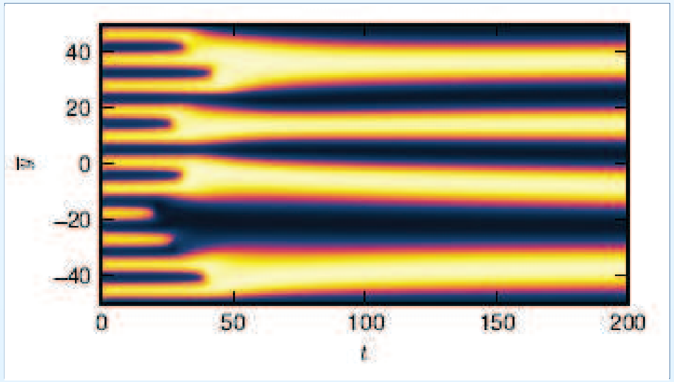

Fig. 8. Merging jets in the Ginzburg–Landau equation.

Modulational instability and zonostrophic instability are intimately related.

- It is often said that zonal flows arise by modulational instability.

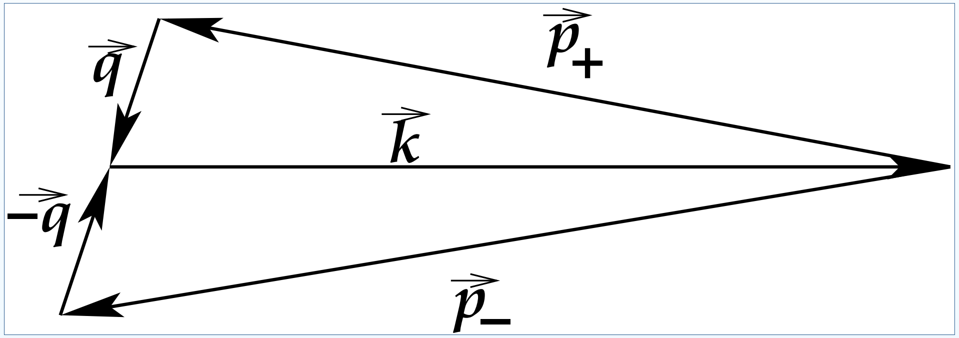

Fig. 9. Wave vectors involved in a MI calculation. $\boldsymbol{k}$: pump; $\boldsymbol{p}_\pm$: sidebands; $\boldsymbol{q}$: zonal.

  - Given a fixed pump at $\boldsymbol{k}$, modulational instability is an initial-value problem.
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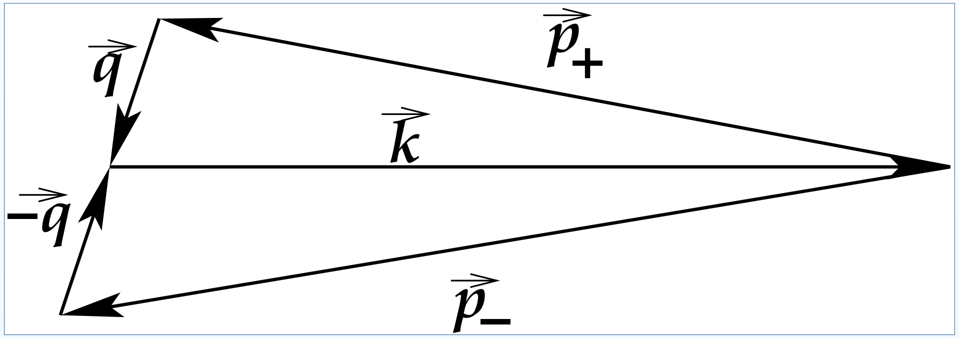

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  - Given a fixed pump at $\boldsymbol{k}$, modulational instability is an initial-value problem.
  - It has little relevance to *self-consistent* states of interacting ZFs and turbulence.
- The proper way to view modulational instability is in the context of zonostrophic instability.
- Parker has shown that one can recover the modulational instability dispersion relation *exactly* from the zonostrophic instability in CE2 if one chooses a particular (single-$\boldsymbol{k}$) spectrum for the background turbulence.

The growth rate for zonostrophic instability depends in an interesting way on L_d.

Srinivasan & Young (2012) $+$ Parker & Krommes (2014):

$$\gamma_{\boldsymbol{q}} \sim \int_0^\infty dk \int_0^{2\pi} d\phi \, A_q(k, \phi; \beta) W(k, \phi). \tag{15}$$

For example, consider an isotropic background $W(k)$. Then

$\underline{L_\mathrm{d} = \infty}$:

$$\gamma_{\boldsymbol{q}} \sim \begin{cases} \int_q^\infty dk \ \ldots \sim \beta^2 \to 0, \\ + \\ \int_0^q dk \ \ldots \ \text{(damping)}; \end{cases} \tag{16}$$

$\underline{L_\mathrm{d} \text{ finite}}$:

$$\gamma_{\boldsymbol{q}} \sim \begin{cases} \int_q^\infty dk \ \ldots \neq 0 \ \text{(even for } \beta = 0\text{)}, \\ + \\ \int_0^q dk \ \ldots \neq 0. \end{cases} \tag{17}$$

Some of these results have been known previously in other contexts.

From **Bakas & Ionannou (2013)**:

> "Previous studies have shown that shearing of isotropic eddies on an infinite domain and in the absence of dissipation and β does not produce any net momentum fluxes (Shepherd, 1985; Farrell, 1987; Holloway, 2010)."

- Only Holloway cited Kraichnan's seminal 1976 work on eddy viscosity — yet it is very relevant.
- **Parker & Krommes** (2014) showed that the growth rate $\gamma_{\boldsymbol{q}}$ for the zonostrophic instability is controlled by a certain factor $R_{\boldsymbol{k}}$ that is a measure of the portion of the physics devoted to perpendicular advection:

$$R_{\boldsymbol{k}} = \begin{cases} 1 & \text{(2D Navier–Stokes)}, \\ k^2/(\alpha_{\boldsymbol{k}} + k^2) & \text{(modified Hasegawa–Mima equation)}. \end{cases} \tag{18}$$

The isotropic eddy viscosity for 2D Navier–Stokes is interesting.

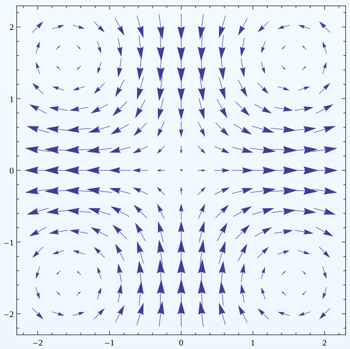
$$\mu(q \mid k_{\text{min}}) = \frac{\pi}{4}\int_{k_{\text{min}}}^{\infty} dk\, \theta_{qkk} \frac{\partial[k^2\mathcal{U}(k)]}{\partial k} \to -\frac{\pi}{4}\int_{k_{\text{min}}}^{\infty} dk\, \frac{\partial \theta_{qkk}}{\partial k}[k^2\mathcal{U}(k)]. \qquad (19)$$

"The integrand is a total derivative except for the k dependence of θ_{kkq}. This means that any addition to the spectrum $\mathcal{U}(k)$ for $k > k_{\text{min}}$ which vanishes at $k = k_{\text{min}}$ would add nothing to $\mu(q \mid k_{\text{min}})$ were it not for the k dependence of θ_{kkq}. . . .

"If θ_{kkq} is dominated by low-wavenumber straining, . . . , it is independent of k and the integrand of [Eq. (19)] is a total derivative. Thus any excitation, described by $\mathcal{U}(k)$, which is totally confined to $k > k_{\text{min}}$, gives zero contribution to the effective eddy viscosity exerted on $q \ll k$. This is a direct consequence of [analysis of a straining model] which says that low-wavenumber straining of the small scales gives a diffusion process in wavenumber with *no* average loss of kinetic energy. By conservation, there is then no net gain of kinetic energy by the straining scales. On the other hand, if k_{min} falls within the small-scale excitation, the diffusion of the excitation to smaller k occurs at wavenumbers $< k_{\text{min}}$ and is not counted in [Eq. (19)] which then includes only the outward diffusion. The latter *does* involve a net loss of kinetic energy by the small scales and thus gives rise to a negative contribution to the eddy viscosity."

The program to understand the significance of R_k involves a number of steps.

- Determine the form of the nonlinear spectral invariant that is conserved under long-wavelength straining (**Diamond & Smolyakov**, 1999; **Krommes & Kim**, 2000; **Krommes & Kolesnikov**, 2004).
- Derive a wave-number diffusion equation for the short-wavelength spectrum (**Krommes & Kim**, 2000).



- Study energy nonconservation and the rate of energy transfer $(\partial_{\boldsymbol{k}} \cdot \boldsymbol{\Gamma}_{\boldsymbol{k}})\mathcal{N}_{\boldsymbol{k}}$ into secondary flow and the large scales.
- Relate $\boldsymbol{\Gamma}_{\boldsymbol{k}}$ to the statistical description of random ray refraction.
- Prove that $\boldsymbol{\Gamma}_{\boldsymbol{k}} \propto \widehat{\boldsymbol{k}}\, k^{-1}(R_{\boldsymbol{k}}^2 \theta_{qkk})$. This reduces to Kraichnan's 2D Navier–Stokes result. We have generalized his discussion and simple model to the case of finite L_{d}.
- For the details, see **J. B. Parker and J. A. Krommes**, "Zonal flow as pattern formation," in *Zonal Jets*, edited by B. Galperin and P. Read (Cambridge U. Press, Cambridge, 2014), Chap. V.2.4 (submitted).

Bifurcation from homogeneous turbulence into steady ZFs is not the whole story.

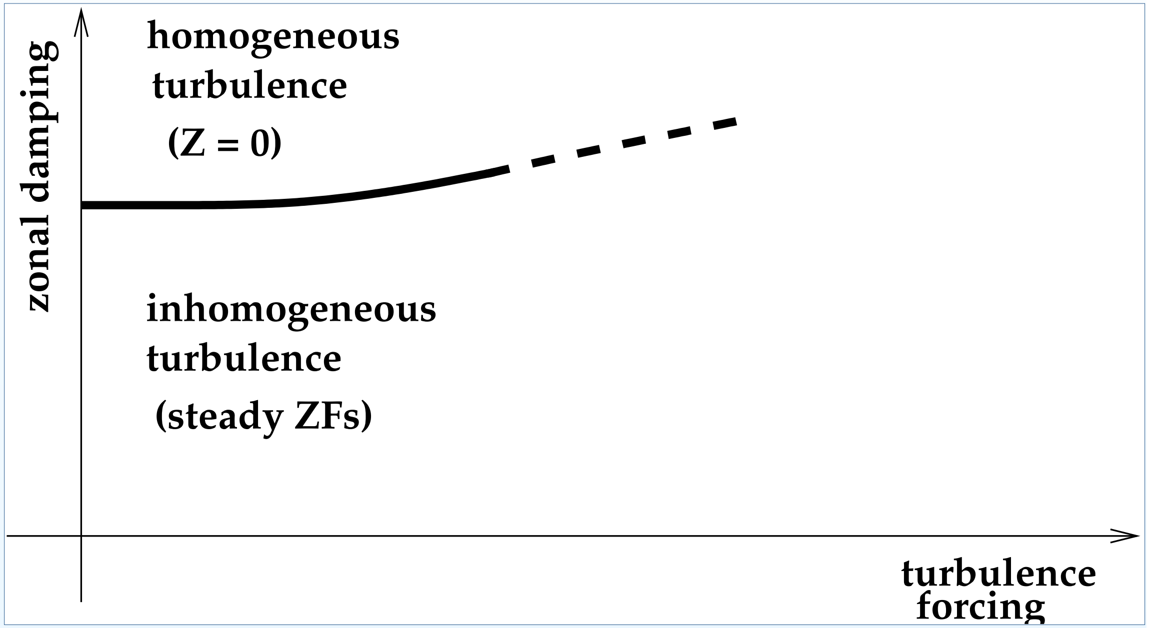


Fig. 10. The simplest bifurcation.

Bifurcation from homogeneous turbulence into steady ZFs is not the whole story.

- The Dimits-shift regime: ZFs, but no or little turbulence.

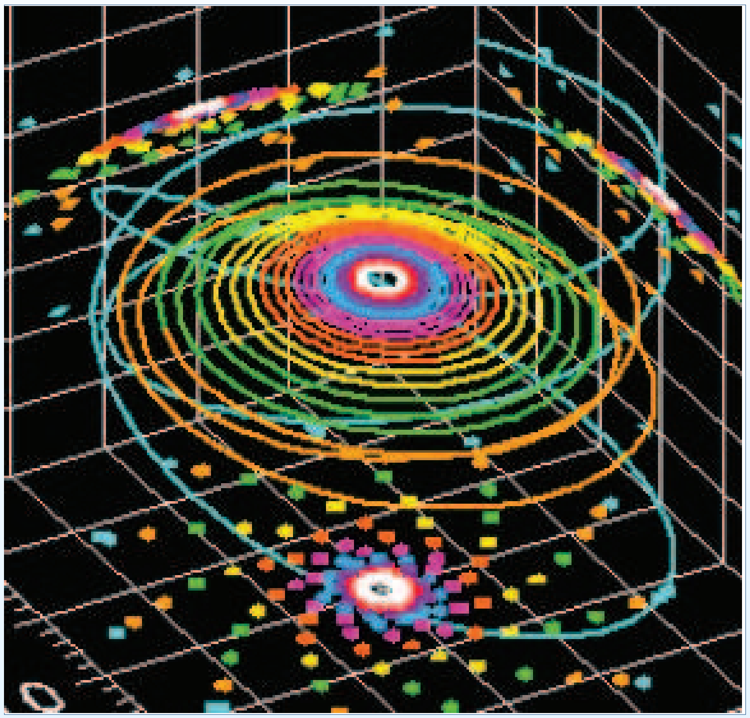


Fig. 10. After an initial burst of turbulence, the system spirals into a fixed point with ZFs but no turbulence.

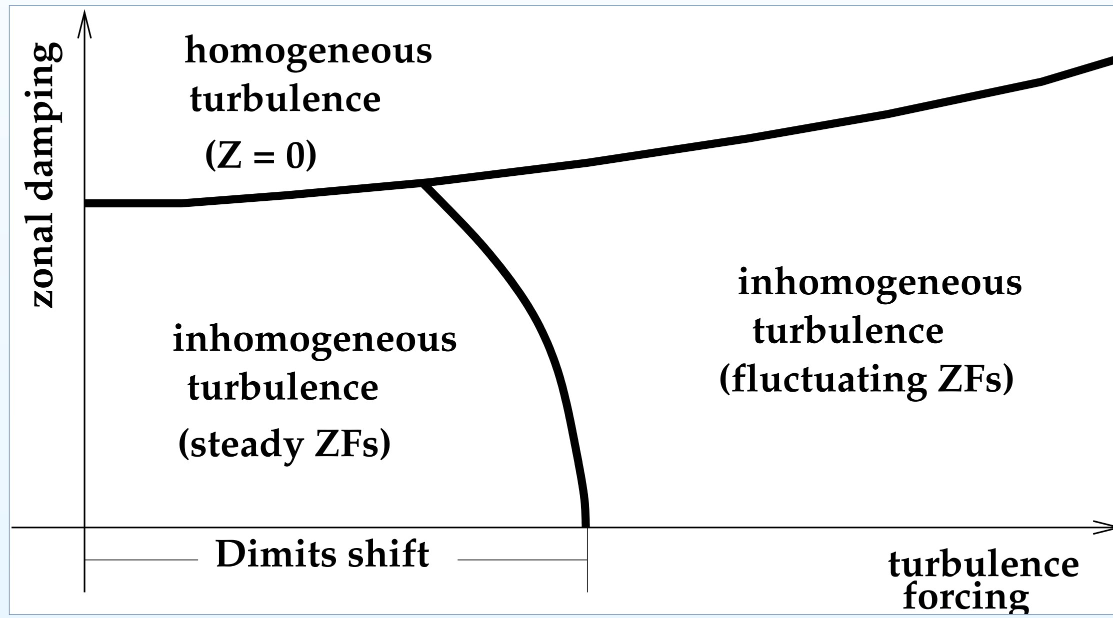


Fig. 11. A conceivable, more complete bifurcation diagram for more complicated models. (No such diagram has been derived from fundamental principles.)

- What about non-steady ZFs?

Summary of Basic Points and Calculations

- Zonostrophic instability:
  - Arises by spontaneous symmetry breaking of a homogeneous turbulent state.
  - Calculated for equivalent barotropic vorticity equation (finite L_d).
  - An example of pattern formation.
  - Modulational instability is a special case of zonostrophic instability.
  - New insights about the relationships between zonal-flow growth, zonostrophic instability, and eddy viscosity.
- Zonostrophic bifurcation:
  - Above the neutral curve, steady equilibria with a continuous range of q exist.
  - Calculated coefficients in the Ginzburg–Landau equation.
  - Numerical calculation of the stability of the bifurcated equilibria farther above the neutral curve.
  - Partial explanation for merging jets.