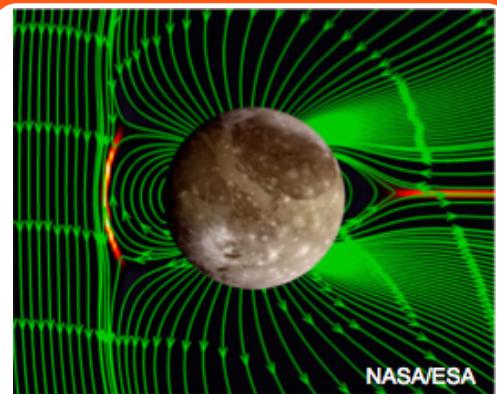


Libration-induced flow in a spherical shell

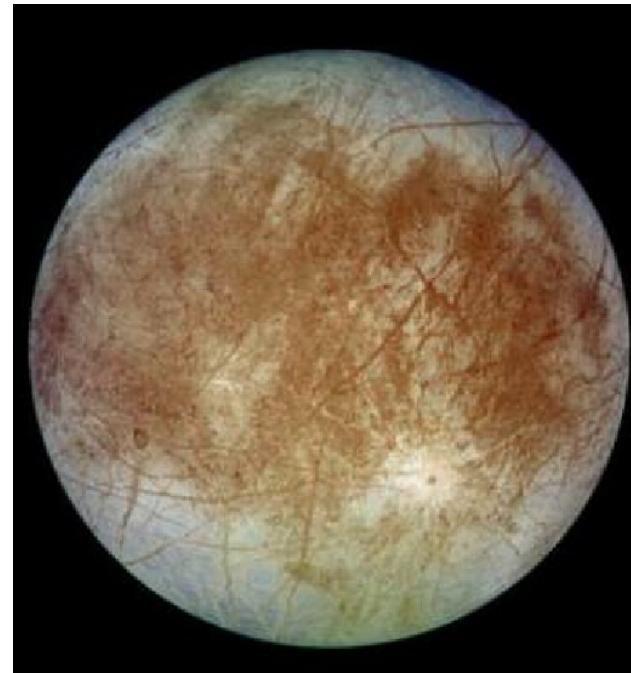
Alban Sauret, Stéphane Le Dizès

IRPHE, CNRS & Aix-Marseille University

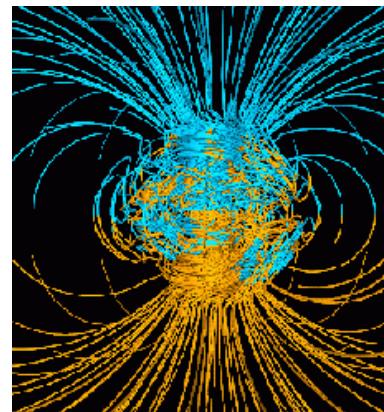
Geo-/Astrophysical context



Magnetic field



Mechanical forcing



Glatzmaier & Roberts,
Nature 377 (1995)

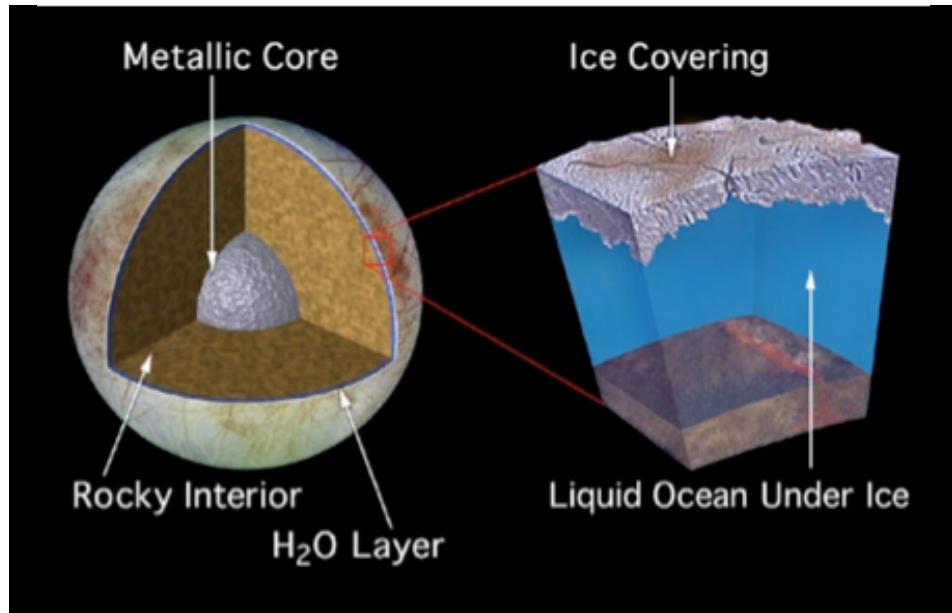


Gravitational effects



Image Credits: NASA/JPL

Applications



Investigation of internal structure
(Europa Jupiter System Mission :
exploration of Jupiter's moons)

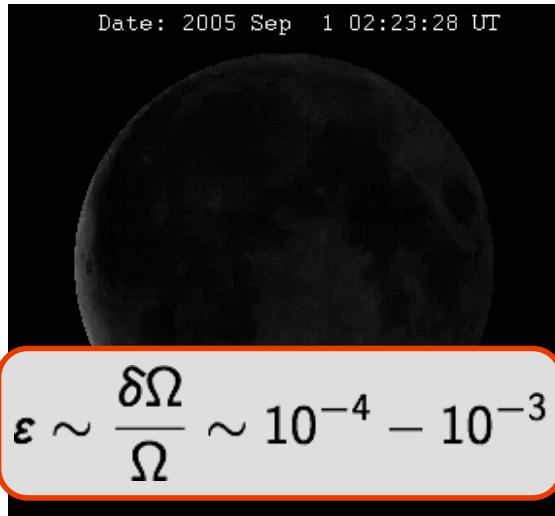
Generation of zonal winds in
atmosphere and core
(alternative mechanism for
patterns, dynamo,...)



Image Credits: NASA/JPL

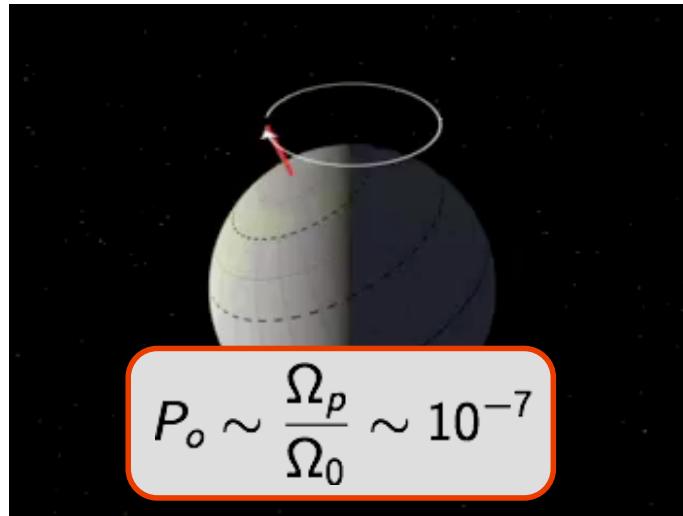
Harmonic Forceings

Animation Credits: NASA/JPL

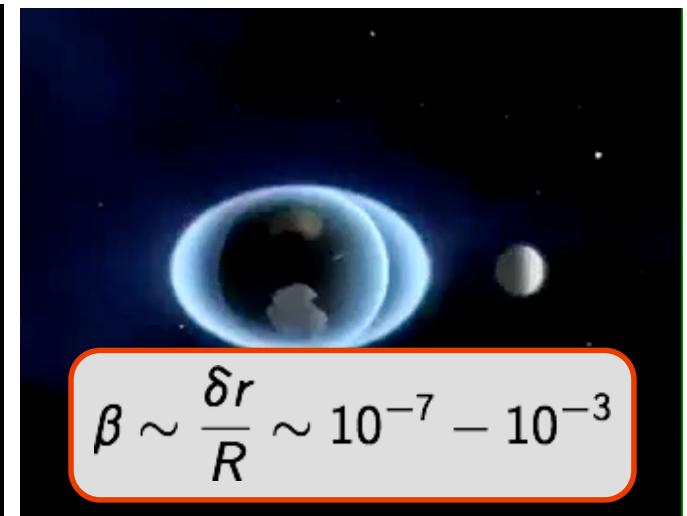


**Longitudinal/latitudinal
libration (m=0)**

$$\Omega(t) = \Omega_0 + \delta\Omega \cos(\omega t)$$



Precession (m=1)



Tides (m=2)

$$R_{ext} = 1 + \frac{\beta}{2} \cos(2\phi)$$

$\Omega(t) = \Omega_0 + \delta\Omega \cos(\omega t)$ Small amplitudes \rightarrow Negligible forcings?

But can drive inertial waves & instabilities...

Fluid flows driven by harmonic forcing ?

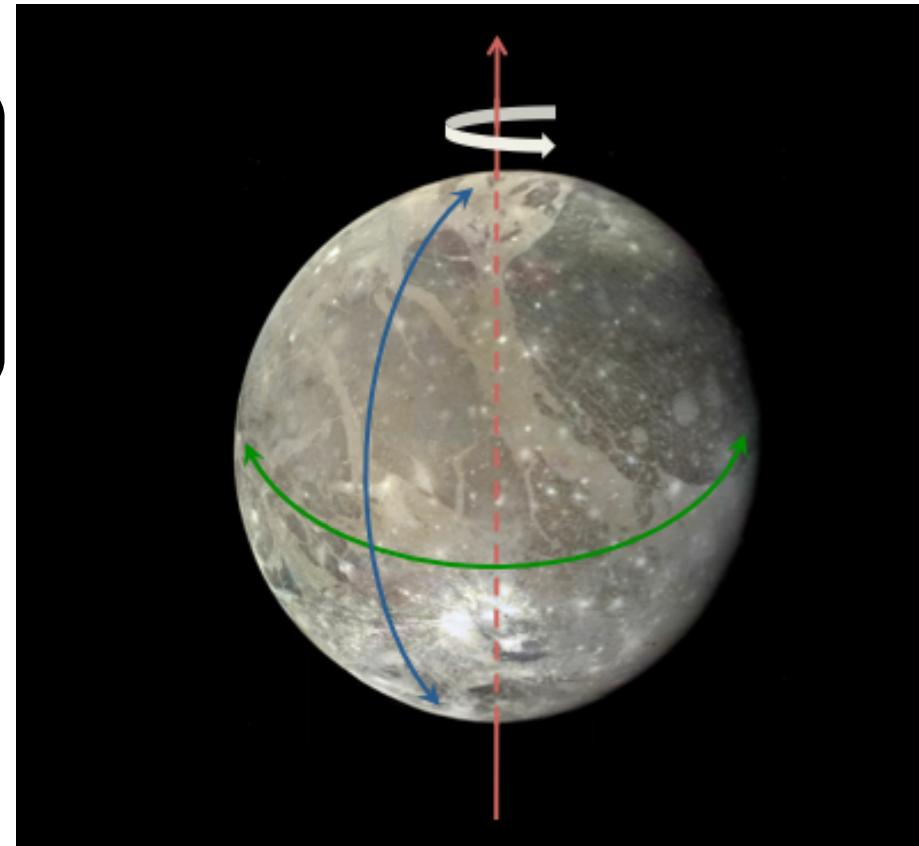
Longitudinal and latitudinal libration

Rotation of a satellite at the mean angular velocity Ω_0

Longitudinal libration: small oscillation of the **rotation rate** of container

$$\Omega(t) = \Omega_0 + \delta\Omega \cos(\omega t)$$

Latitudinal libration: small oscillation of the **rotation axis** of container



Problem Characteristics

Geometry: spherical shell

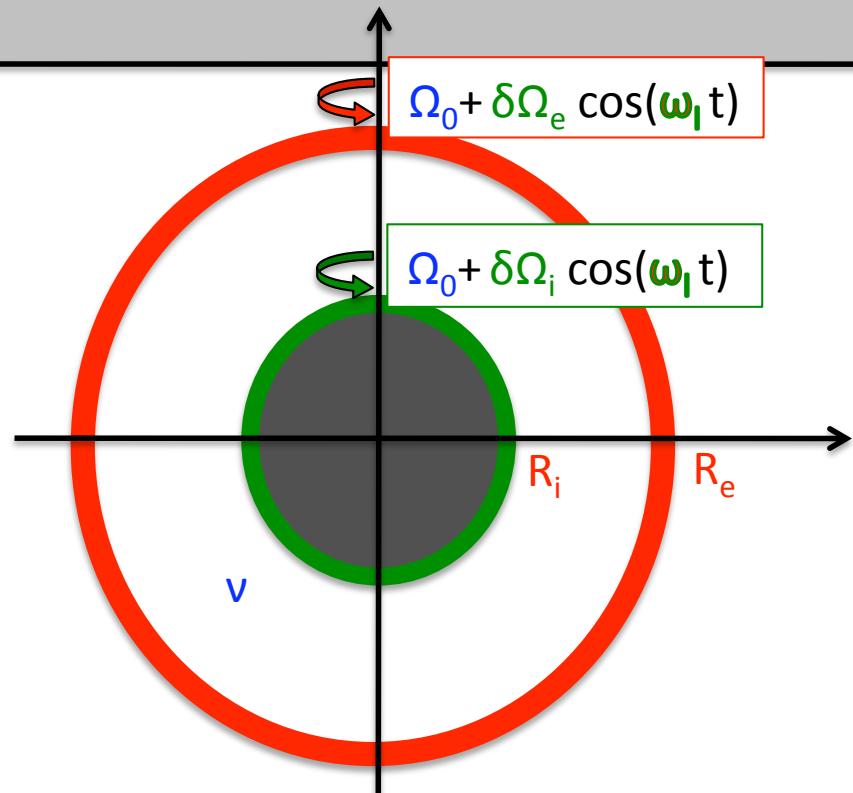
R_e external radius

R_i internal radius

Fluid: rotating, incompressible,
homogeneous, viscous

ν kinematic viscosity

Ω_0 mean rotation rate



Libration

ω_l frequency of libration

$\delta\Omega_e$ amplitude of libration of the outer sphere

$\delta\Omega_i$ amplitude of libration of the inner sphere

Problem Characteristics

Geometry: spherical shell

R_e external radius
 R_i internal radius

$$a = \frac{R_i}{R_e} \quad \text{Core size}$$

**Fluid: rotating, incompressible,
homogeneous, viscous**

ν kinematic viscosity
 Ω_0 mean rotation rate

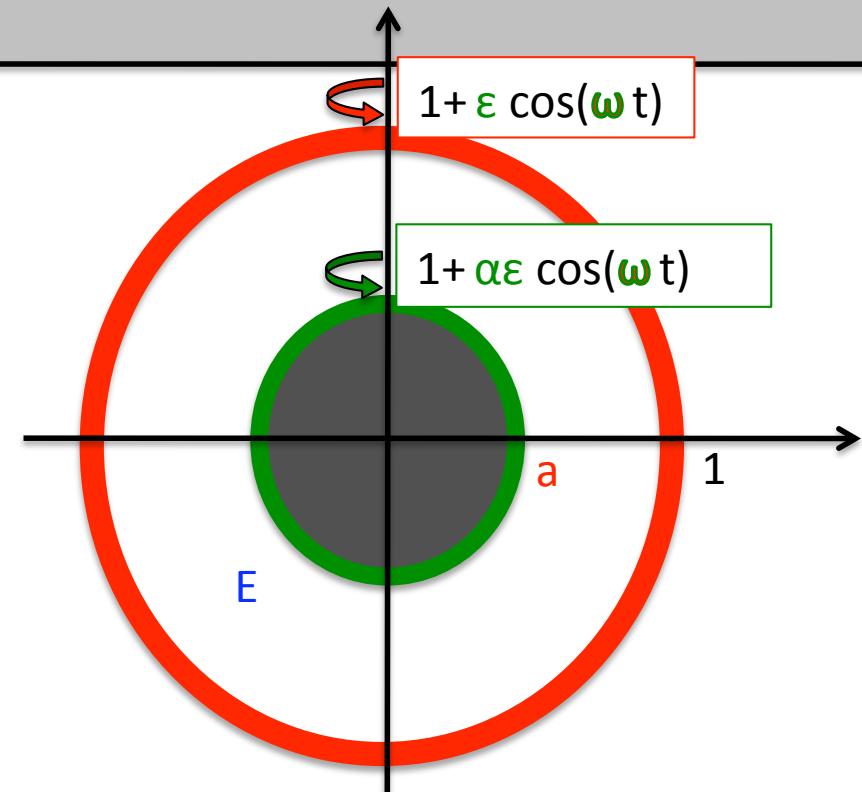
$$E = \frac{\nu}{\Omega_0 R_e^2} \quad \text{Ekman number}$$

Libration

ω_l frequency of libration

$\delta\Omega_e$ amplitude of libration of the outer sphere

$\delta\Omega_i$ amplitude of libration of the inner sphere



$$\omega = \frac{\omega_l}{\Omega_0}$$

Frequency

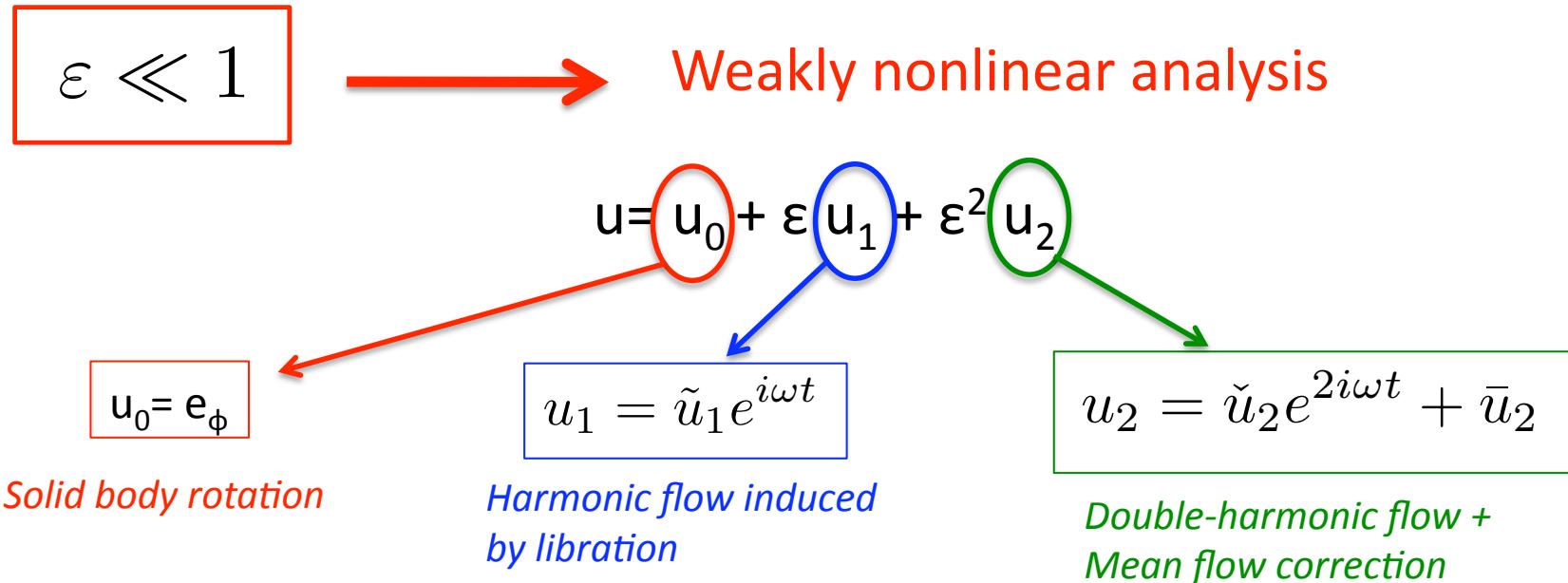
$$\varepsilon = \frac{\delta\Omega_e}{\Omega_0}$$

$$\alpha = \frac{\delta\Omega_i}{\delta\Omega_e}$$

Amplitude

Navier-Stokes equations No-slip boundary conditions

$0 \leq a < 1$	Any core size (a=0: no core)
α real	Inner core librates or not
$\varepsilon \ll 1$	Small libration amplitude
$E \ll 1$	Small Ekman number
$\omega > 2$	Frequency outside the range of inertial waves



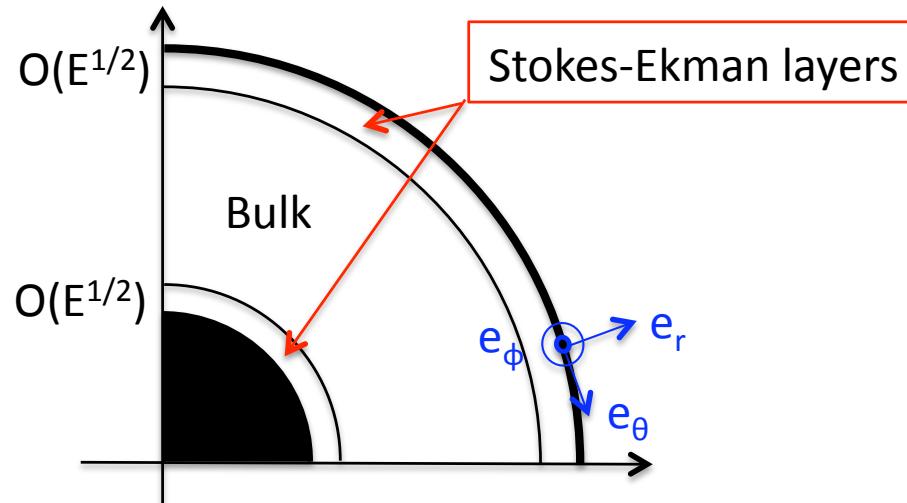
$$E \ll 1$$

Boundary layer analysis

Harmonic flow structure

Order ϵ

$$u_1 = \tilde{u}_1 e^{i\omega t}$$



In the Stokes-Ekman layers:

$$\tilde{u}_1 \sim \frac{\sin \theta}{4} [e^{-\lambda_+ \tilde{r}} + e^{-\lambda_- \tilde{r}}] e_\phi$$

(Outer layer)

$$\tilde{r} = \frac{1 - r}{\sqrt{E}}$$

$$\lambda_{\pm} = (1 + i) \sqrt{\frac{\omega}{2} \pm \cos \theta}$$

In the bulk:

$$\tilde{u}_1 = O(\sqrt{E})$$

Condition of validity:

$$\omega > 2$$

Second harmonic and mean flow correction

Order ϵ^2

$$u_2 = \check{u}_2 e^{2i\omega t} + \bar{u}_2$$

Generated by the non-linear interactions

In the Stokes-Ekman layers: NL= O(1)

$$\check{u}_2 = O(1) \quad \check{u}_2(BL \rightarrow Bulk) \rightarrow 0$$

$$\bar{u}_2 = O(1) \quad \bar{u}_2(BL \rightarrow Bulk) \rightarrow U_{2e}(\theta)$$

In the bulk: NL= O(E)

Nonlinear Corrections are forced by the boundaries and dominated by the mean flow correction.

$$u_2 \sim \bar{u}_2 = O(1)$$

Taylor-Proudman theorem:

$$\bar{u}_2 = u_{\phi_2}(\rho) \mathbf{e}_\phi + u_{z_2}(\rho) \mathbf{e}_z$$

Mean flow correction

Order ϵ^2

$$u_{\phi_2} = \frac{\chi_2}{\rho} \quad u_{z_2} = \frac{\sqrt{E}}{\rho} \frac{\partial \Psi_2}{\partial \rho}$$

Outer layer matching condition

$$\chi_2(\rho) = -2 [1 - \rho^2]^{1/4} \Psi_2(\rho) + \mathcal{F}(\rho; \omega)$$

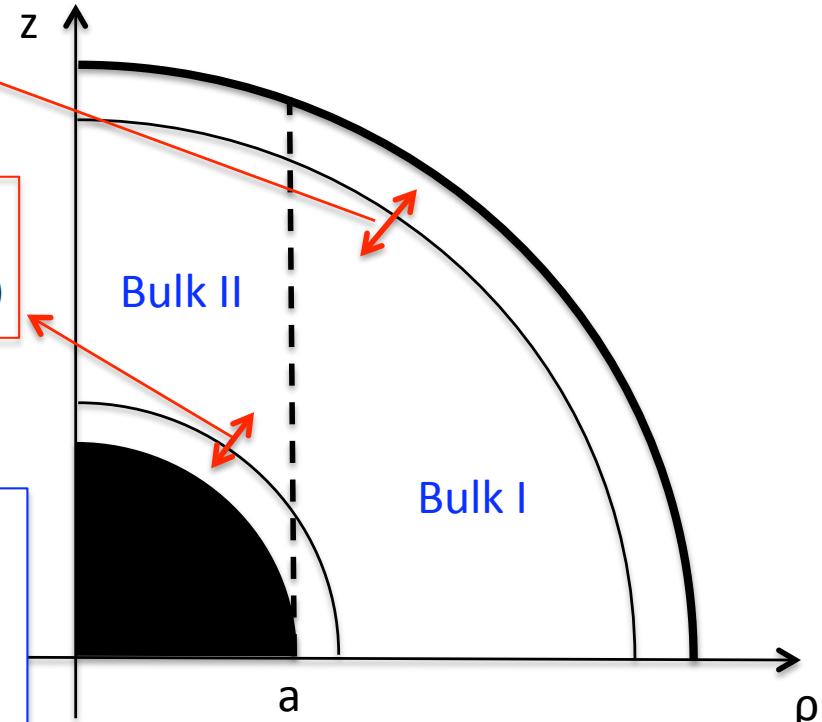
Inner layer matching condition

$$\chi_2(\rho) = 2 [1 - (\rho/a)^2]^{1/4} \Psi_2(\rho) + \alpha^2 a^2 \mathcal{F}(\rho/a; \omega)$$

Bulk II Solution

$$\Psi_2(\rho) = \frac{\mathcal{F}(\rho; \omega) - \alpha^2 a^2 \mathcal{F}(\rho/a; \omega)}{2(1 - \rho^2)^{1/4} + 2[1 - (\rho/a)^2]^{1/4}}$$

$$\chi_2(\rho) = \frac{\alpha^2 a^2 (1 - \rho^2)^{1/4} \mathcal{F}(\rho/a; \omega) + [1 - (\rho/a)^2]^{1/4} \mathcal{F}(\rho; \omega)}{(1 - \rho^2)^{1/4} + [1 - (\rho/a)^2]^{1/4}}$$



Bulk I $\Psi_2(\rho) = 0$
Solution $\chi_2(\rho) = \mathcal{F}(\rho; \omega)$

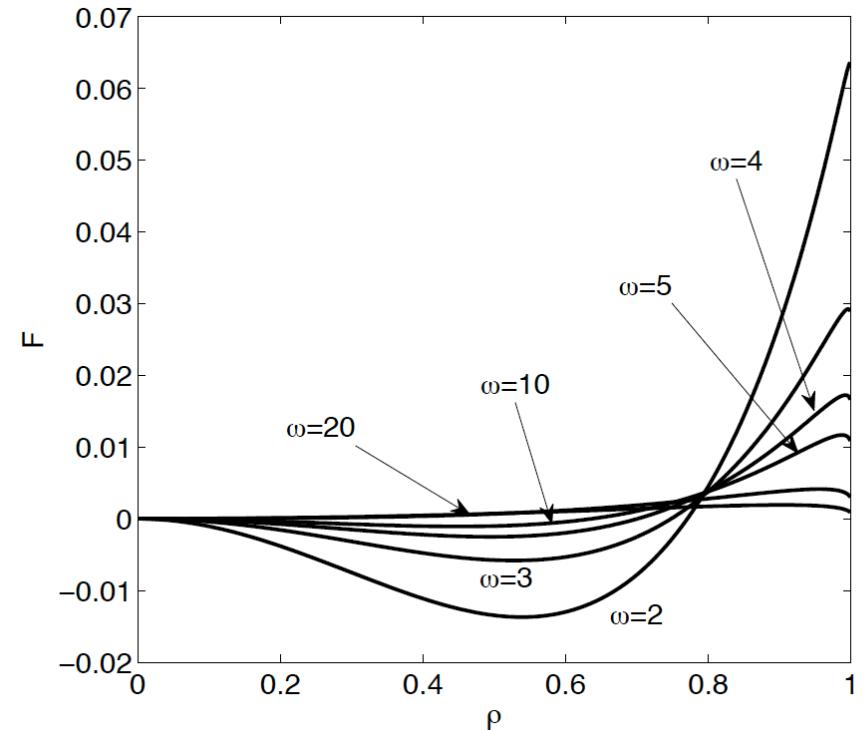
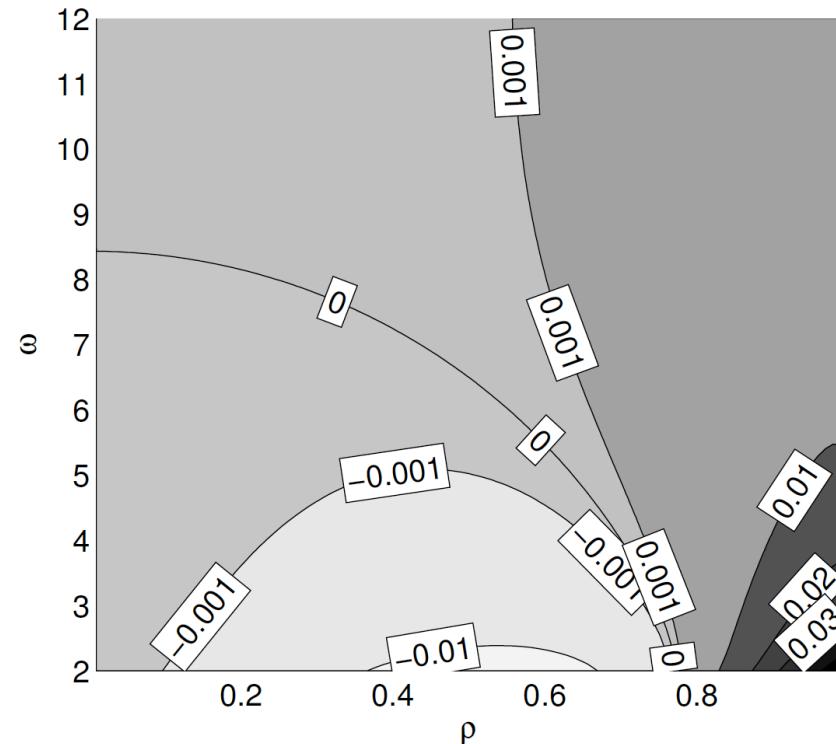
The function $\mathcal{F}(\rho; \omega)$

$$\begin{aligned}
\mathcal{F}(\sin \theta; \omega) = & - \left[\frac{(s_+^3 + s_-^3) \sin^4 \theta}{32 s_+^3 s_-^3} + \frac{(s_- - s_+) \cos \theta \sin^2 \theta}{8 s_+ s_-} \right] \left[\frac{s_+ - \sqrt{\cos \theta}}{s_+^2 - \cos \theta} - \frac{s_- - \sqrt{\cos \theta}}{s_-^2 + \cos \theta} \right] \\
& + \frac{\sin^2 \theta (s_+ + \sqrt{\cos \theta})}{32 s_+^3 (\cos^2 \theta + 4 s_+^4)} [2 s_+^2 - 2 s_+ \sqrt{\cos \theta} + \cos \theta] [\sin^2 \theta + 4 s_+^2 \cos \theta] \\
& + \frac{\sin^2 \theta (s_- + \sqrt{\cos \theta})}{32 s_-^3 (\cos^2 \theta + 4 s_-^4)} [2 s_-^2 - 2 s_- \sqrt{\cos \theta} + \cos \theta] [\sin^2 \theta - 4 s_-^2 \cos \theta] \\
& - \frac{\sin^2 \theta}{32 s_-^3 s_+^3 (s_-^2 + s_+^2) [s_-^2 + s_+^2 + 2 s_+ \sqrt{\cos \theta} + \cos \theta]^2} \\
& \left[(s_-^2 + s_+^2) [s_-^6 + 7 s_+^2 s_-^4 + 8 s_-^7 s_+^3 - 7 s_-^2 s_+^4 + 16 s_-^5 s_+^5 \right. \\
& - s_+^6 + 4 s_+^3 s_-^3 (-3 + 2 s_+^4)] + [4 s_-^6 s_+ - s_-^4 s_+^3 - 16 s_-^2 s_+^5 - 3 s_+^7 \\
& + 4 s_-^5 s_+^2 (-3 + 16 s_+^4) + s_+^4 s_-^3 (-19 + 32 s_+^4) + s_-^7 (-1 + 32 s_+^4)] \sqrt{\cos \theta} \\
& + [4 s_-^8 s_+^2 - 3 s_+^6 + s_-^6 (1 - 20 s_+^4) + s_-^2 s_+^4 (-11 + 4 s_+^4) \\
& + 4 s_+^3 s_-^3 (-3 + 16 s_+^4) + s_-^5 (-4 s_+ + 64 s_+^5) + s_-^4 (s_+^2 - 20 s_+^6)] \cos \theta \\
& - [4 s_-^7 s_+^2 + 4 s_-^6 s_+^3 + s_+^5 + 56 s_-^4 s_+^5 + s_-^3 s_+^2 (3 - 76 s_+^4) \\
& + s_-^5 (1 - 8 s_+^4) + 3 s_+^3 s_-^2 (1 - 4 s_+^4)] \cos^{3/2} \theta \\
& + [-s_-^8 - 4 s_-^6 s_+^2 - 4 s_+^3 s_-^5 - 48 s_-^4 s_+^4 + 60 s_+^5 s_-^3 + 20 s_-^2 s_+^6 + s_+^8] \cos^2 \theta \\
& + [s_-^7 - 4 s_-^6 s_+ + 8 s_-^5 s_+^2 - 11 s_-^4 s_+^3 + 31 s_+^4 s_-^3 + 20 s_-^2 s_+^5 + 3 s_+^7] \cos^{5/2} \\
& + [-s_-^6 + 4 s_-^5 s_+ - s_+^2 s_-^4 + 12 s_+^3 s_-^3 + 11 s_-^2 s_+^4 + 3 s_+^6] \cos^3 \theta \\
& + (s_-^5 + 3 s_-^3 s_+^2 + 3 s_+^3 s_-^2 + s_+^5) \cos^{7/2} \theta \Big] \\
& + \frac{\sin^2 \theta}{4}, \tag{0.1}
\end{aligned}$$

with

$$\begin{aligned}
s_+ &= \sqrt{\frac{\omega}{2} + \cos \theta} \\
s_- &= \sqrt{\frac{\omega}{2} - \cos \theta}
\end{aligned}$$

The function $\mathcal{F}(\rho; \omega)$



Mean flow correction structure

Order ε^2

Discontinuity at $\rho=a$

Bulk II Solution near $\rho=a$

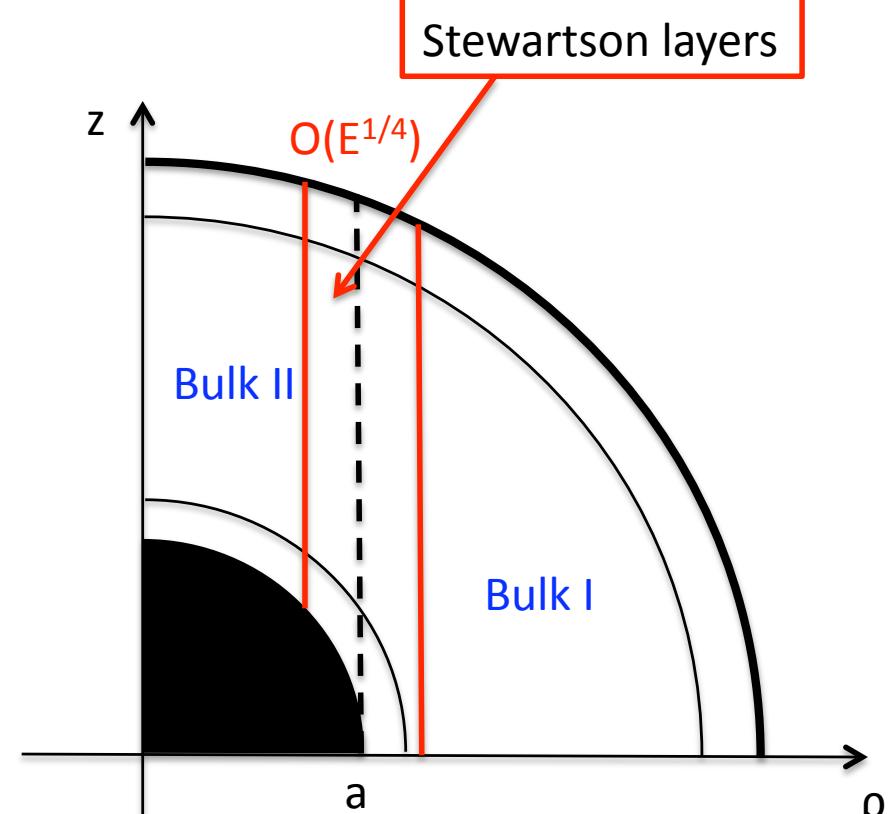
$$\bar{u}_2 \sim \frac{\alpha^2 a}{4\omega^2} \mathbf{e}_\phi$$

Bulk I Solution near $\rho=a$

$$\bar{u}_2 \sim \frac{\mathcal{F}(a; \omega)}{a} \mathbf{e}_\phi$$

Angular velocity jump

$$\delta\Omega_2 = \frac{\mathcal{F}(a; \omega)}{a^2} - \frac{\alpha^2}{4\omega^2}$$



External Outer Layer $E^{1/4}$: Smoothing of azimuthal velocity

$$u_{\phi_2} \sim \frac{\mathcal{F}(a; \omega)}{a} - a\delta\Omega_2 e^{-\tilde{\rho}}$$

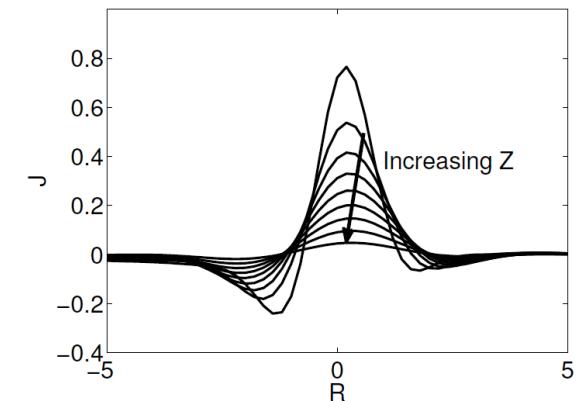
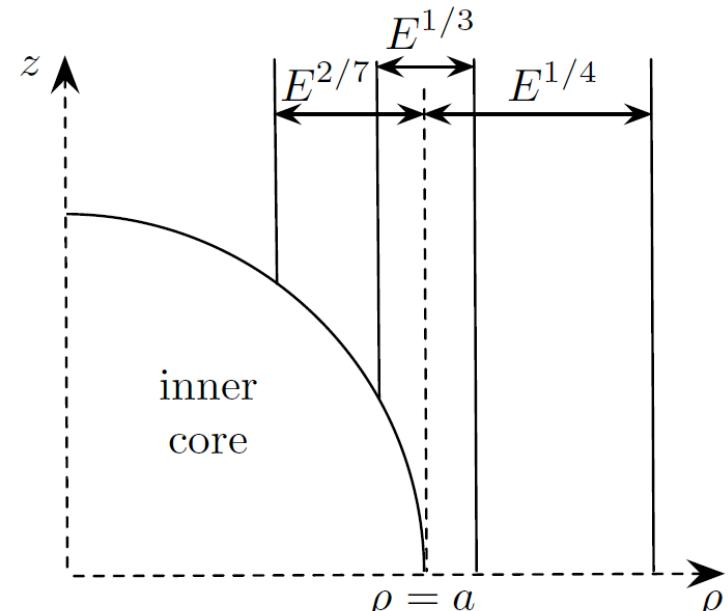
$$\tilde{\rho} = (\rho - a) / \left[(1 - a^2)^{3/8} E^{1/4} \right]$$

Internal Outer Layer $E^{2/7}$: Smoothing of axial vorticity

$$\omega_z^{\max} = E^{-1/4} \frac{\delta\Omega_2 a}{(1 - a^2)^{3/8}}$$

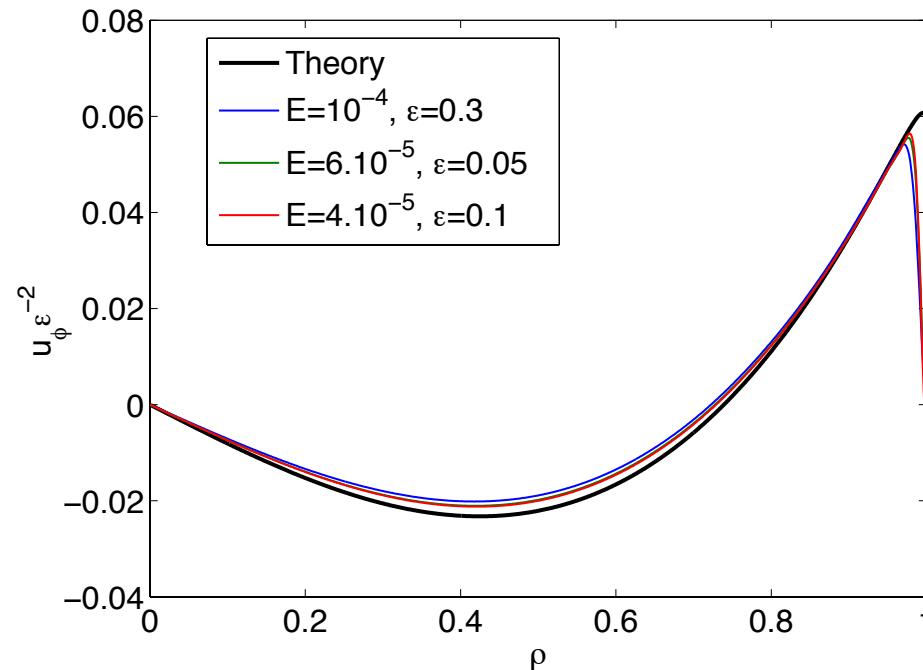
Internal layer $E^{1/3}$: Smoothing of axial velocity

$$u_{z_2} \sim E^{5/42} \frac{\alpha_o \delta\Omega_2}{a^{81/28} (1 - a^2)^{25/84}} J \left(\frac{z}{\sqrt{1 - a^2}}, \frac{\rho - a}{E^{1/3} (1 - a^2)^{1/6}} \right)$$

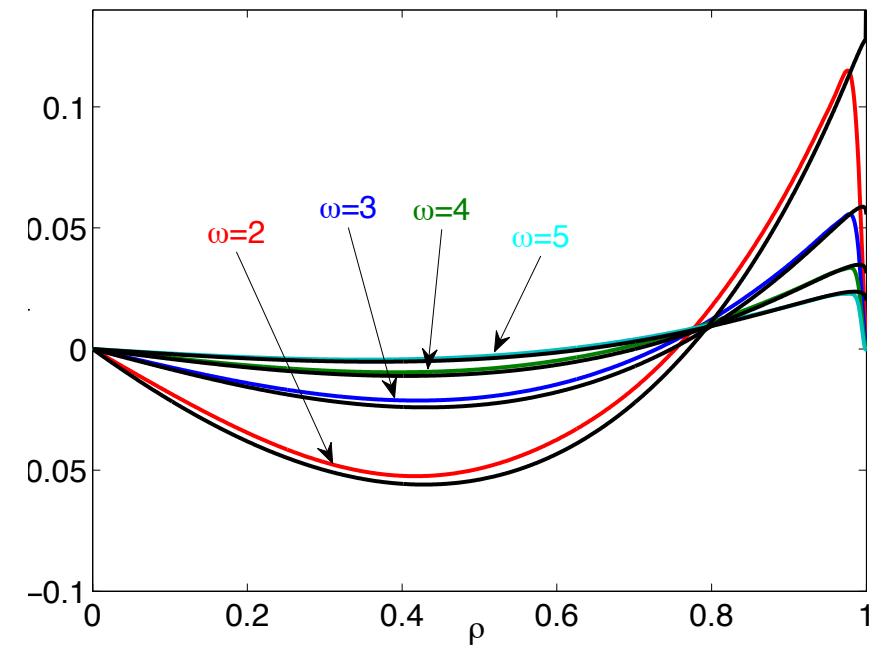


Libration in a sphere ($a=0$)

Azimuthal velocity of mean flow correction



$\omega=3$



$E=5 \cdot 10^{-5}; \varepsilon=0.01$

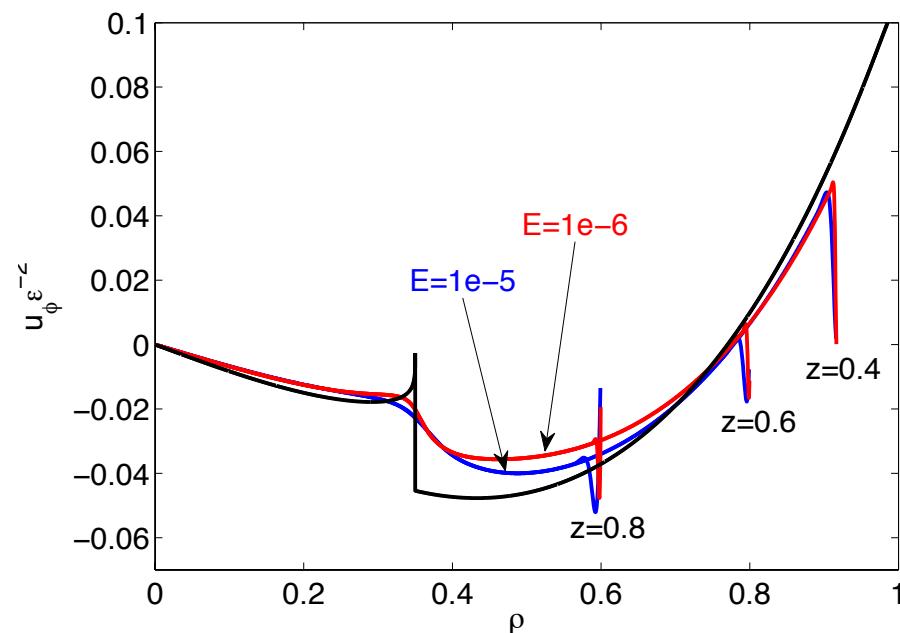
Comparison with numerical results

Libration in a spherical shell ($a=0.35$)

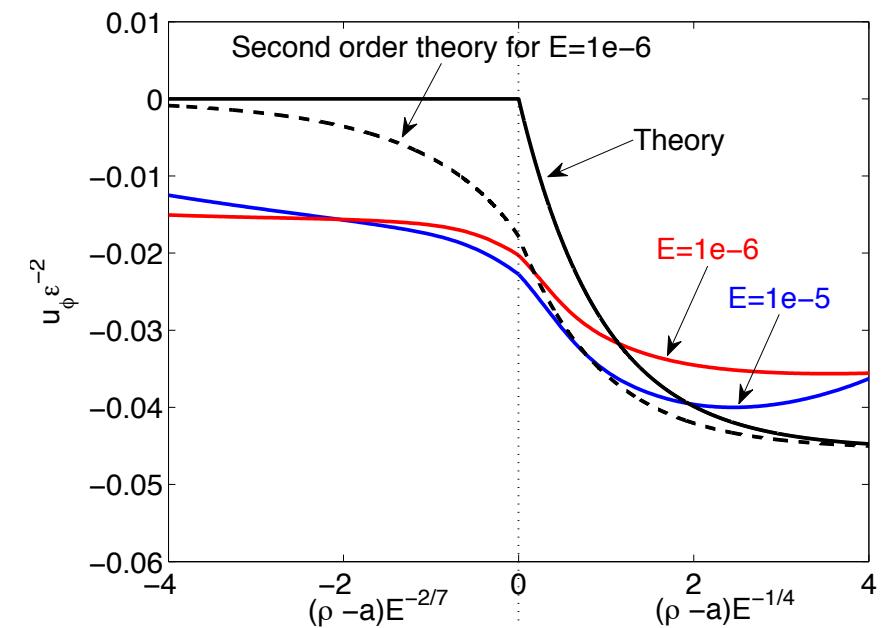
Calkins et al 2010

Azimuthal velocity of mean flow correction

Bulk



Stewartson layers



$$\varepsilon=2.2e-3; \omega=2.2; \alpha=0; a=0.35$$

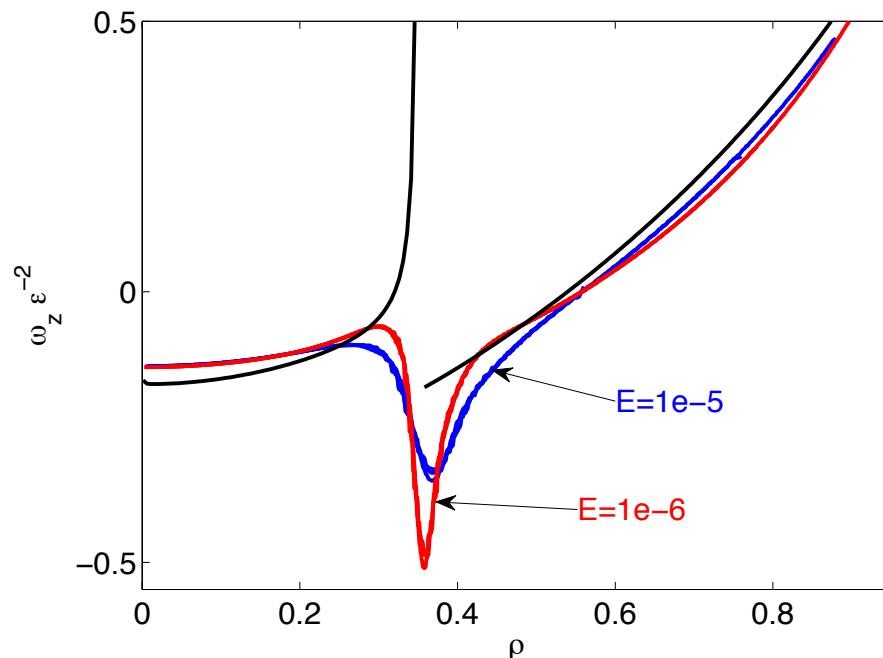
Comparison with numerical results

Libration in a spherical shell ($a=0.35$)

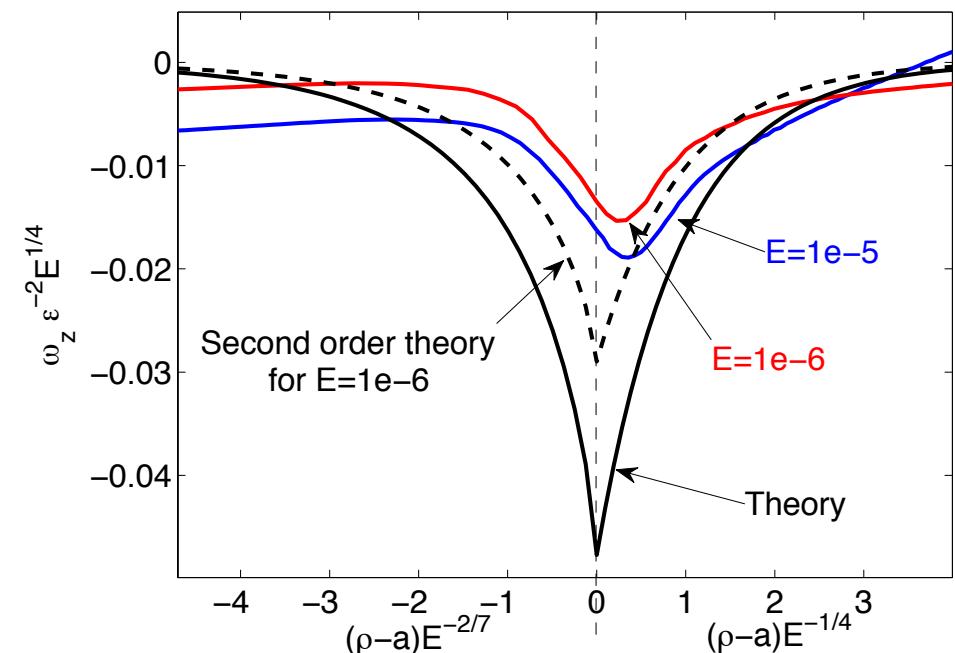
Calkins et al 2010

Axial vorticity of mean flow correction

Bulk



Stewartson layers

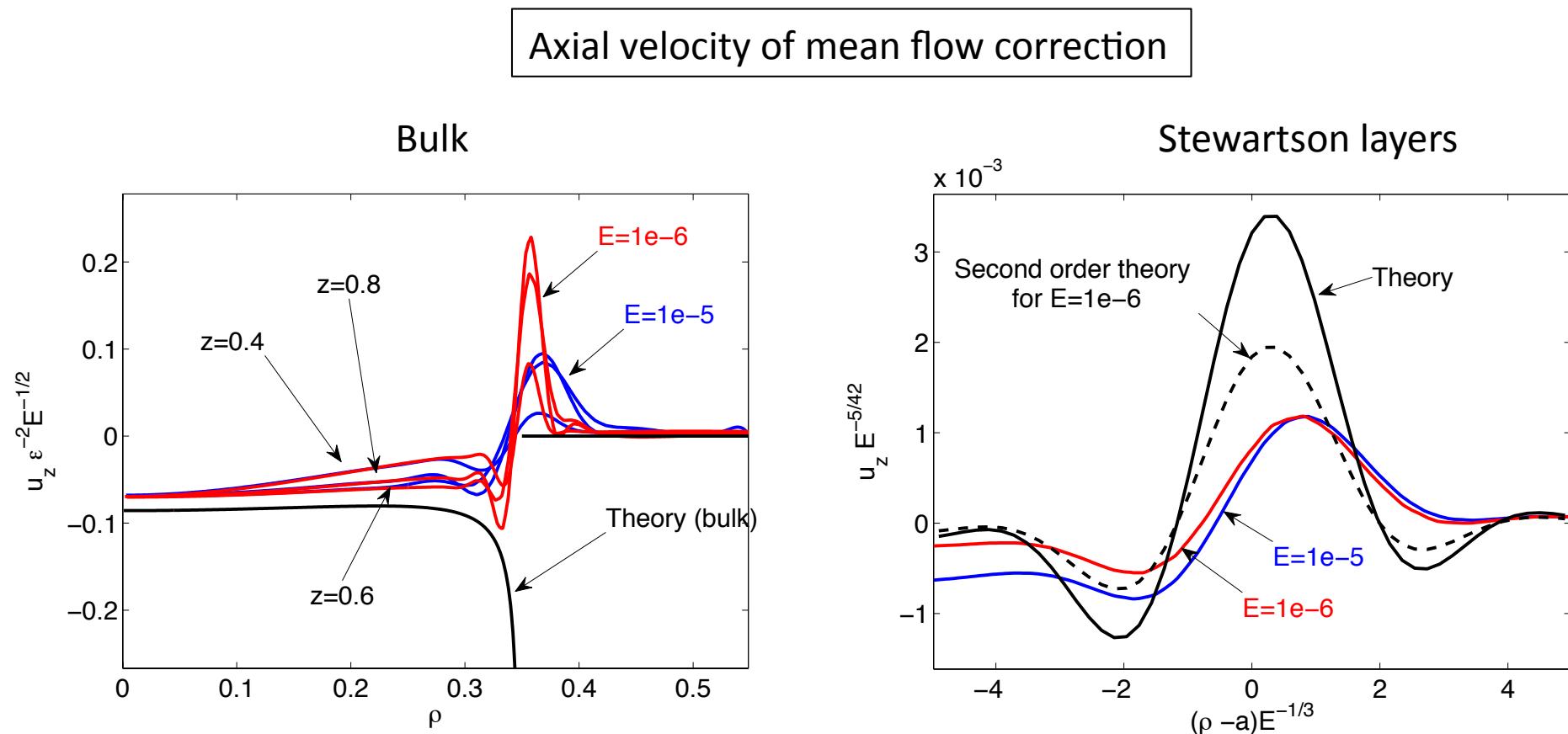


$$\varepsilon=2.2e-3; \omega=2.2; \alpha=0; a=0.35$$

Comparison with numerical results

Libration in a spherical shell ($a=0.35$)

Calkins et al 2010



$$\epsilon=2.2e-3; \omega=2.2; \alpha=0; a=0.35$$

- Exact formulas for the zonal flow in a spherical shell geometry for any core size a , any frequency $\omega > 2$, any relative amplitude α , for small ε and small E .
- Good agreement with numerical results for $E < 10^{-4}$ and $\varepsilon < 0.3$ in a sphere, and for $E < 10^{-6}$ and $\varepsilon < 0.3$ in a spherical shell.
- Effect of a core
 - Different zonal flow within the cylinder tangent to the core
 - Stewartson layers where the azimuthal velocity discontinuity is smoothed with a strong axial vorticity peak $O(E^{-1/4})$, and a weak axial velocity $O(E^{5/24})$



A. Sauret & S. Le Dizès, 2013. *Libration-induced mean flow in a spherical shell*.
J. Fluid Mech. **718**, 181–209

$\omega < 2$

First difficulty: Solve the harmonic structure

- **Singularity at the critical latitude**

(Roberts & Stewartson, 1965; Kerswell, 1995; Kida, 2011...)

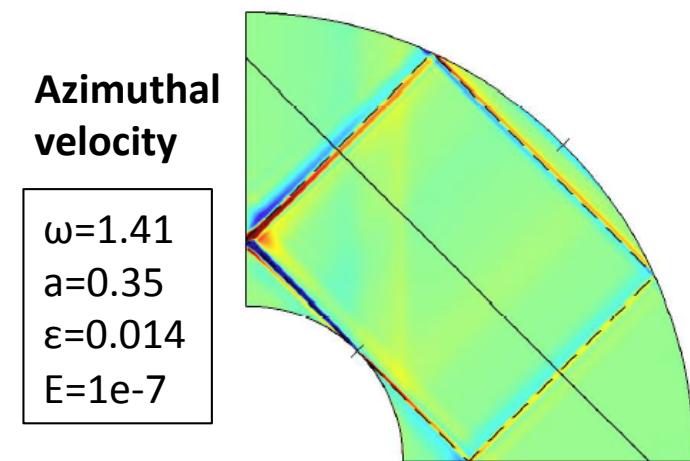
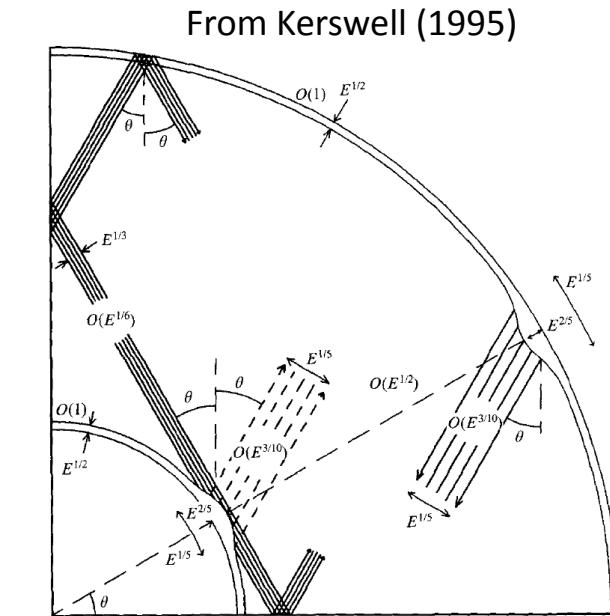
- **Internal shear layers**

(Rieutord et al, 1997, Tilgner, 1999; Noir et al, 2001, Lin et al, 2014...)

- **Resonance with inertial modes?**

- **Amplitude of the response?**

(Aldridge & Toomre, 1969; Zhang et al, 2013...)



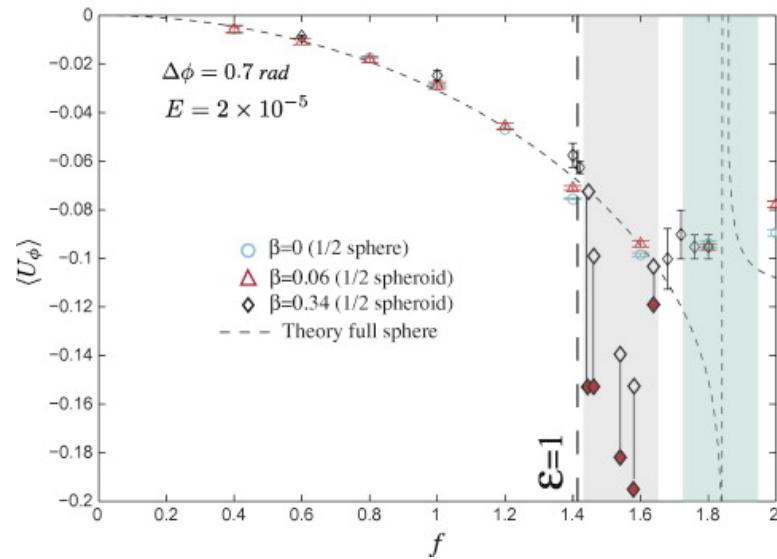
From Lin, Noir, Calkins (2014)

$$\omega < 2$$

Second difficulty: Find the mean flow associated with

- the singularity at the critical latitude
- internal shear layers

Comparison of measurement with theory without taking into account internal shear layers

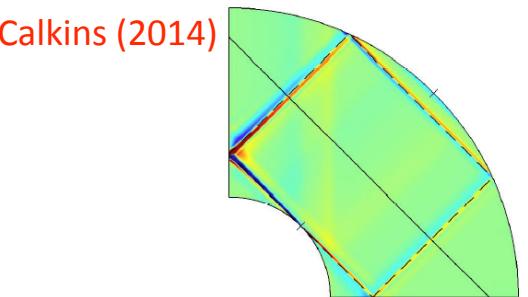


From Noir et al (2012)

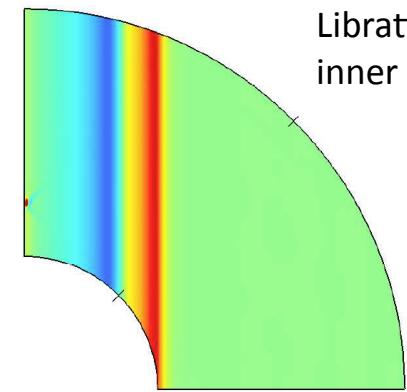
From Lin, Noir, Calkins (2014)

$$\omega = 1.41 \\ E = 1e-7$$

Mean azimuthal flow



Librating inner core



Librating outer core

Thank you for your attention!