Dispersion and Conservation Laws for Finite-Amplitude Rossby Waves in Slowly Varying Parallel Shear Flows

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1. Background

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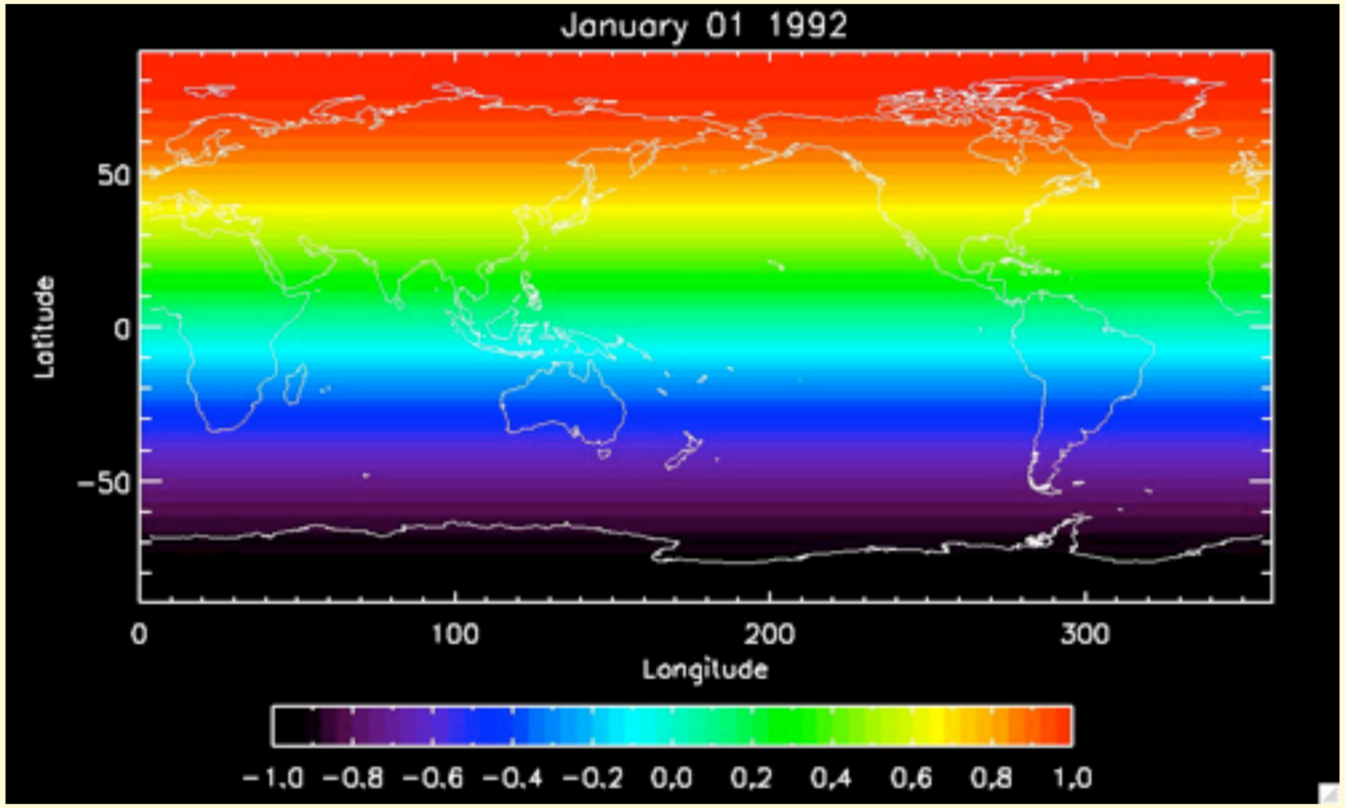
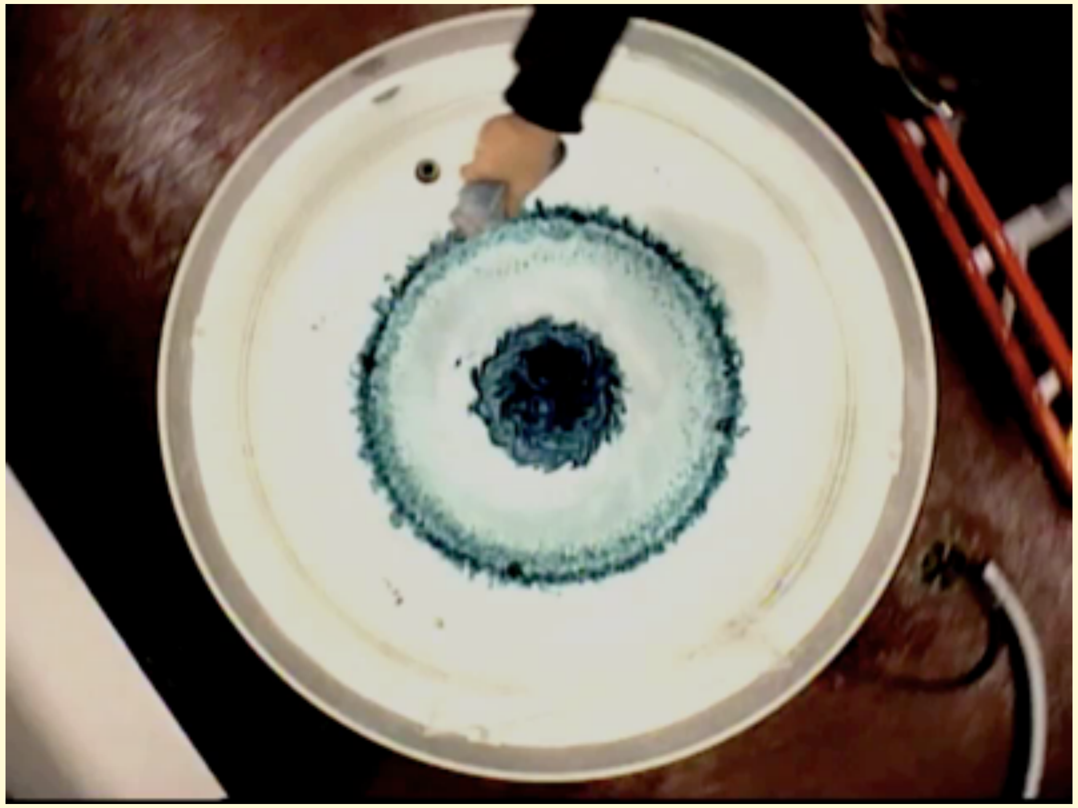


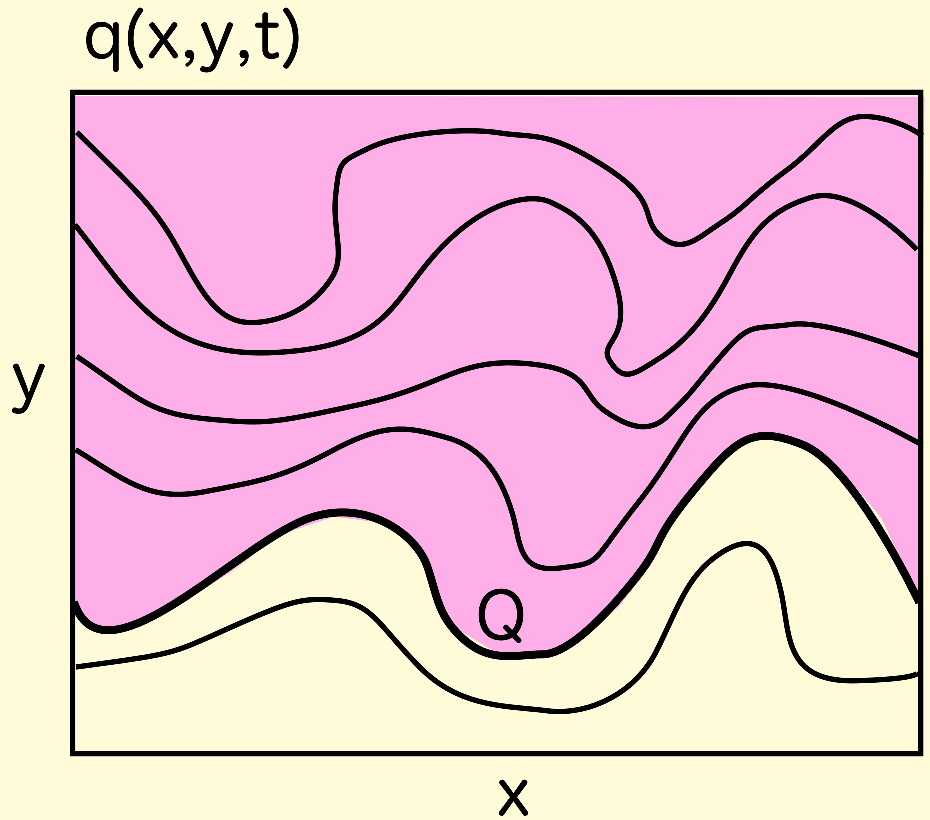


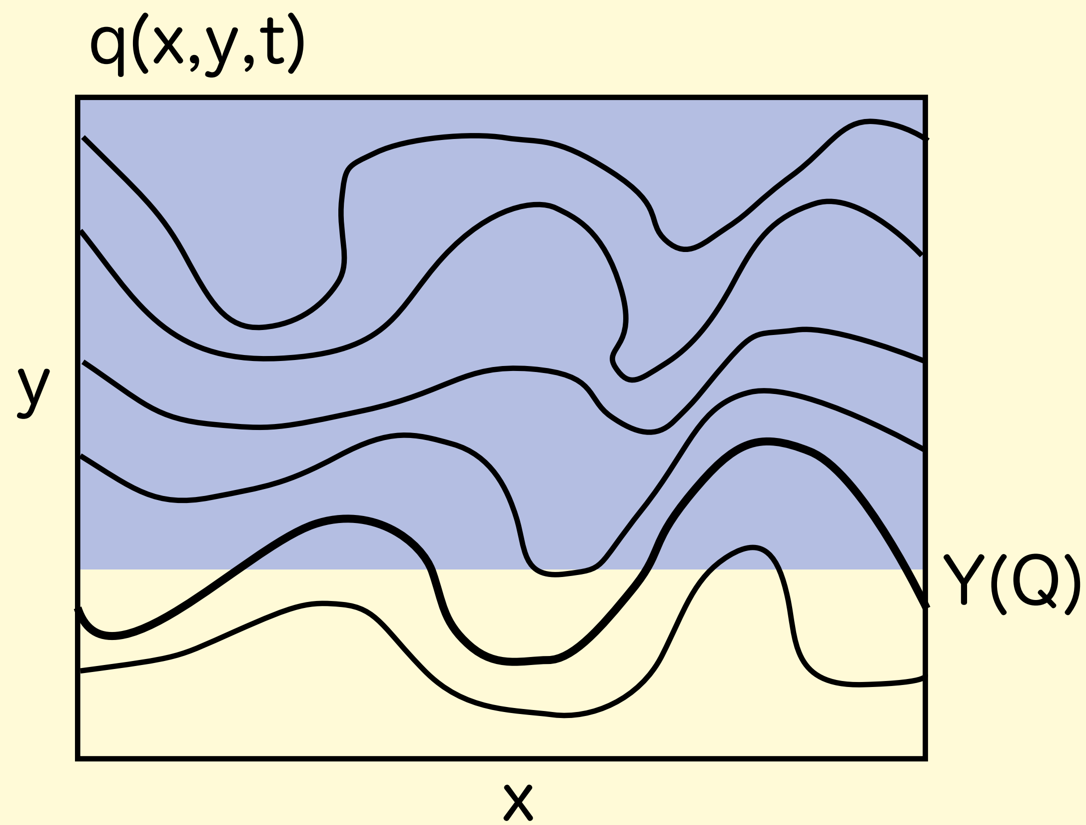
Barotropic Vorticity Equation
(Charney-Hasegawa-Mima equation)

$$\frac{\partial}{\partial t} q + J(\psi, q) = 0, \qquad q = \nabla^2 \psi + f(y)$$

2. Exact conservation laws

Nakamura and Zhu (2010, JAS)
Solomon and Nakamura (2012 JFM)

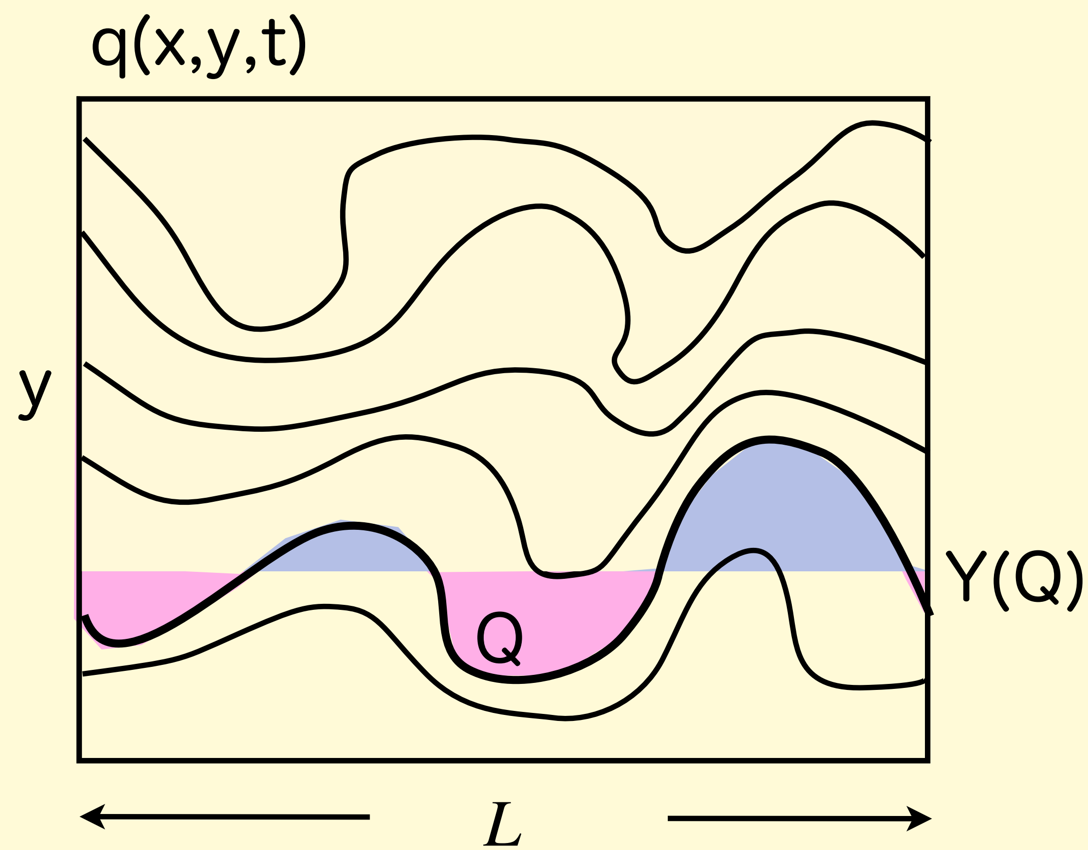




$$C(Q) = \iint_{q \geq Q} q \, dx \, dy \qquad C(Y(Q)) = \iint_{y \geq Y(Q)} q \, dx \, dy$$

2. Exact conservation laws

Nakamura and Zhu (2010, JAS)
Solomon and Nakamura (2012 JFM)



$$A^*(Y,t) \equiv \frac{C(Q)-C(Y(Q))}{L} = L^{-1}\left(\iint_{q\geq Q} q\,dx\,dy - \iint_{y\geq Y(Q)} q\,dx\,dy \right), \qquad A^* \geq 0$$

$$\frac{\partial}{\partial t}A^* = -\overline{v'q'} = \frac{\partial}{\partial y}\overline{u'v'}$$

Finite-amplitude pseudomomentum density

Linear limit: $A^* \to \frac{1}{2}\frac{dQ}{dy}\overline{\eta'^2}$

2. Exact conservation laws

Nakamura and Zhu (2010, JAS)
Solomon and Nakamura (2012 JFM)

$$\frac{\partial}{\partial t}A^* = \frac{\partial}{\partial y}\overline{u'v'}$$

$$\frac{\partial}{\partial t}\overline{u} = -\frac{\partial}{\partial y}\overline{u'v'}$$

$$U_{REF}(y) \equiv \overline{u}(y,t) + A^*(y,t)$$

(nonacceleration theorem)

U_{REF} is steady even when $\overline{u}$ and A^* are not

We will use U_{REF} as a steady reference state for finite-amplitude eddies (ignoring dissipation)

2. Exact conservation laws

$$\frac{\partial}{\partial t}A^* - \frac{\partial}{\partial y}\overline{u'v'} = 0$$

$$\frac{\partial}{\partial t}U_{REF} = 0$$

$$E^*(y,t) \equiv \frac{1}{2}\overline{\left((\overline{u}+u')^2 + v'^2\right)} - \frac{1}{2}U_{REF}^2 = \overline{e} - U_{REF}A^* + \frac{A^{*2}}{2}$$

$$e \equiv \frac{u'^2 + v'^2}{2}$$

Finite-amplitude pseudoenergy density

$$\frac{\partial E^*}{\partial t} + \frac{\partial}{\partial y}\left(\overline{u}\,\overline{u'v'} + \frac{\overline{v'p'}}{\rho_0} + \overline{v'e}\right) = 0$$

2. Exact conservation laws

$$\frac{\partial A^*}{\partial t} - \frac{\partial}{\partial y}\overline{u'v'} = 0$$

$$U_{REF} = \bar{u} + A^*$$

pseudomomentum density

$$\frac{\partial E^*}{\partial t} + \frac{\partial}{\partial y}\left(\bar{u}\,\overline{u'v'} + \frac{\overline{v'p'}}{\rho_0} + \overline{v'e}\right) = 0$$

pseudoenergy density

- True for eddies with arbitrary amplitude, whether wave-like or turbulence
- A^* and E^* easily calculable from data
- Goal: find the dispersion relation for finite-amplitude Rossby wave that is compatible with these conservation laws (so we can determine effective phase speed from observation)

3. Finding the dispersion relation

Variational principle

Whitham 1965, 1974
Bretherton and Garrett 1968
Seliger and Whitham 1968
Salmon 1998

$$\frac{\partial}{\partial t}q' + \bar{u}\frac{\partial q'}{\partial x} + \beta_{eff}\frac{\partial \psi'}{\partial x} = 0$$

Lagrangian (displacement) information
Relabeling symmetry, etc

$$\delta \iiint L\, dxdydt = 0$$

phase (or zonal) average

$$\delta \iiint \bar{L}(\theta_t, \theta_x, \theta_y, a)\, dxdydt = 0$$

$\delta\theta$

$$\frac{\partial}{\partial t}A + \frac{\partial}{\partial y}\left(c_{gy}A\right) = 0$$

$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial y}\left(c_{gy}E\right) = 0$$

δa

$$c = \bar{u} - \frac{\beta_{eff}}{k^2 + l^2}$$

3. Finding the dispersion relation

Variational principle

$$\frac{\partial}{\partial t}q + J(\psi, q) = 0$$

↕ Lagrangian (displacement) information
Relabeling symmetry, etc

$$\delta \iiint L\, dxdydt = 0$$

↓ phase (or zonal) average

$$\delta \iiint \overline{L}(\theta_t, \theta_x, \theta_y, a)\, dxdydt = 0$$

$\delta\theta$ → ?

δa → ?

3. Finding the dispersion relation

Variational principle

$$\frac{\partial}{\partial t} q + J(\psi, q) = 0$$

Lagrangian (displacement) information
Relabeling symmetry, etc

$$\delta \iiint L \, dxdydt = 0$$

phase (or zonal) average

$$\delta \iiint \overline{L}(\theta_t, \theta_x, \theta_y, a)\, dxdydt = 0$$

$\delta\theta$

$$\frac{\partial}{\partial t} A^* + \frac{\partial}{\partial y} F_A = 0$$

$$\frac{\partial}{\partial t} E^* + \frac{\partial}{\partial y} F_E = 0$$

δa

?

3. Finding the dispersion relation

Variational principle

Challenges:

(i) separation of phase and amplitude

(ii) effects of wave-mean flow interaction

$$\delta \iiint \overline{L}(\theta_t, \theta_x, \theta_y, a)\, dxdydt = 0$$

$\delta\theta$

δa

$$\frac{\partial}{\partial t} A^* + \frac{\partial}{\partial y} F_A = 0$$

$$\frac{\partial}{\partial t} E^* + \frac{\partial}{\partial y} F_E = 0$$

?

4. Variational principle

Waveform (relative to the zonal mean, not the reference state)

$$\psi = \bar{\psi}(Y,T) + \psi' \qquad \psi' = \psi'(\theta) \quad \text{periodic}$$

<u>not</u> $\psi = \psi_{REF}(Y) + \psi'$

$$(X,Y,T) \equiv \varepsilon(x,y,t), \quad \Theta \equiv \varepsilon\theta$$

$$k \equiv \theta_x = \Theta_X \qquad l \equiv \theta_y = \Theta_Y, \qquad \omega \equiv -\theta_t = -\Theta_T$$

Seek solution in which (k,l,ω) is X-independent

$$k_Y = l_X = 0, \; k_T = -\omega_X = 0$$

$$k: \text{constant}, \qquad l = l(Y,T), \qquad \omega = \omega(Y,T)$$

4. Variational principle

Perturbation Lagrangian density
(relative to the reference state, not the zonal mean)

$$L \equiv \mathscr{L} - \mathscr{L}_{REF}$$

zonal average

$$\overline{L} = \overline{L}(\Theta_T, \Theta_X, \Theta_Y, \alpha, Y)$$

But we don't know what α is

$$\delta \iiint \overline{L}\, dX\, dY\, dT = 0$$

4. Variational principle

$$\delta\Theta : \quad \frac{\partial}{\partial T}\left(\Theta_X \frac{\partial \bar{L}}{\partial \Theta_T}\right) + \frac{\partial}{\partial Y}\left(\Theta_X \frac{\partial \bar{L}}{\partial \Theta_Y}\right) = 0$$

$$\frac{\partial}{\partial T}\left(\Theta_T \frac{\partial \bar{L}}{\partial \Theta_T} - \bar{L}\right) + \frac{\partial}{\partial Y}\left(\Theta_T \frac{\partial \bar{L}}{\partial \Theta_Y}\right) = 0$$

$$\frac{\partial A^*}{\partial T} - \frac{\partial}{\partial Y}\overline{u'v'} = 0$$

$$\frac{\partial E^*}{\partial T} + \frac{\partial}{\partial Y}\left(\bar{u}\,\overline{u'v'} + \frac{\overline{v'p'}}{\rho_0} + \overline{v'e}\right) = 0$$

This suggests

$$A^* = \Theta_X \frac{\partial \bar{L}}{\partial \Theta_T} = -k\frac{\partial \bar{L}}{\partial \omega} = -\frac{\partial \bar{L}}{\partial c}, \qquad c = \frac{\omega}{k}$$

$$E^* = \Theta_T \frac{\partial \bar{L}}{\partial \Theta_T} - \bar{L} = c\frac{\partial \bar{L}}{\partial c} - \bar{L} = -cA^* - \bar{L}$$

we must also satisfy…

$$-\overline{u'v'} = \Theta_X \frac{\partial \bar{L}}{\partial \Theta_Y} = k\frac{\partial \bar{L}}{\partial l}, \qquad \bar{u}\,\overline{u'v'} + \frac{\overline{v'p'}}{\rho_0} + \overline{v'e} = \Theta_T \frac{\partial \bar{L}}{\partial \Theta_Y} = -ck\frac{\partial \bar{L}}{\partial l}$$

4. Variational principle

Determining a

$$A^* = -\frac{\partial \bar{L}}{\partial c}, \qquad E^* = c\frac{\partial \bar{L}}{\partial c} - \bar{L} = -cA^* - \bar{L}$$

Let $\bar{L} = \bar{L}(c, l, \alpha, Y)$ and $\partial \bar{L}/\partial \alpha = 0$ $\quad (c = -\Theta_T / k,\ \ l = \Theta_Y)$

$$\delta \bar{L} = \frac{\partial \bar{L}}{\partial c}\delta c + \frac{\partial \bar{L}}{\partial l}\delta l + \frac{\partial \bar{L}}{\partial Y}\delta Y = \delta\left(c\frac{\partial \bar{L}}{\partial c}\right) - c\delta\left(\frac{\partial \bar{L}}{\partial c}\right) + \frac{\partial \bar{L}}{\partial l}\delta l + \frac{\partial \bar{L}}{\partial Y}\delta Y$$

$$0 = \delta\left(c\frac{\partial \bar{L}}{\partial c} - \bar{L}\right) + c\delta A^* + \frac{\partial \bar{L}}{\partial l}\delta l + \frac{\partial \bar{L}}{\partial Y}\delta Y$$
$$= \delta E^* + c\delta A^* + \frac{\partial \bar{L}}{\partial l}\delta l + \frac{\partial \bar{L}}{\partial Y}\delta Y$$

$$E^* = E^*(A^*, l, Y; c), \quad A^* = A^*\left(l, E^*, Y; c\right)$$
$$\Rightarrow \bar{L} = -c\,A^* - E^*$$
$$= \bar{L}\left(c, l, E^*, Y\right) \quad \text{i.e.} \quad \alpha = E^*$$

4. Variational principle

$$\bar{L}(c,l,E^*,Y) = -c\,A^*(c,l,E^*,Y) - E^*$$

$\delta\theta$:

$$\frac{\partial}{\partial T}\left(-\frac{\partial \bar{L}}{\partial c}\bigg|_{l,E^*,Y}\right) + \frac{\partial}{\partial Y}\left(k\frac{\partial \bar{L}}{\partial l}\bigg|_{c,E^*,Y}\right) = \frac{\partial}{\partial T}A^* + \frac{\partial}{\partial Y}\left(\frac{\partial \bar{L}/\partial l}{\partial \bar{L}/\partial \omega}k\frac{\partial \bar{L}}{\partial \omega}\right)$$

$$= \frac{\partial A^*}{\partial T} + \frac{\partial}{\partial Y}\left(c_{gA}A^*\right) = 0, \qquad c_{gA} \equiv -\frac{\partial \bar{L}/\partial l\,\big|_{c,E^*,Y}}{\partial \bar{L}/\partial \omega\,\big|_{l,E^*,Y}}.$$

δE^* :

$$\frac{\partial \bar{L}}{\partial E^*}\bigg|_{c,l,Y} = -c\frac{\partial A^*}{\partial E^*}\bigg|_{c,l,Y} - 1 = 0 \quad \Rightarrow \quad c(Y,T) = -\frac{\partial E^*}{\partial A^*}\bigg|_{l,y}$$

$$E^*(Y,T) = \bar{e} - U_{REF}A^* + \frac{A^{*2}}{2}, \quad U_{REF} = \bar{u} + A^* \quad \Rightarrow \quad \bar{u}(Y,T) - c(Y,T) = \frac{\partial \bar{e}}{\partial A^*}\bigg|_{l,Y}$$

5. Dispersion relation (at each latitude and time)

$$\bar{L}(c,l,E^*,y) = -c\,A^* - E^*$$

$$c(Y,T) = -\left.\frac{\partial E^*}{\partial A^*}\right|_{l,Y}$$

$$\bar{u}(Y,T) - c(Y,T) = \left.\frac{\partial \bar{e}}{\partial A^*}\right|_{l,Y}$$

$$= -\left.\frac{\partial E^*/\partial T}{\partial A^*/\partial T}\right|_{l,Y}$$ only if l is time invariant (or c is Y-independent)

$$E^*(Y,T) = -\int c\,dA^* \equiv -c^* A^*$$

$$c^*(Y,T) = -\frac{E^*}{A^*}$$

$$c^* \equiv \frac{\int c\,dA^*}{\int dA^*} = \frac{\int c\,dA^*}{A^*}$$

$$\bar{u} - c^* = \frac{\bar{e}}{A^*} - \frac{A^*}{2}$$

effective (pseudomomentum-weighted mean) phase speed

5. Dispersion relation (at each latitude and time)

$$\overline{L}(c,l,E^*,Y) = -c\,A^* - E^* = -(c - c^*)A^*$$

$$c^*(Y,T) = -\frac{E^*}{A^*}$$

$$\overline{u}(Y,T) - c^*(Y,T) = \frac{\overline{e}}{A^*} - \frac{A^*}{2}$$

$$\Rightarrow \quad \overline{e} - \left(U_{REF} - c^*\right)A^* + \frac{A^{*2}}{2} = 0$$

Linear limit: $c^* \to c, \quad \overline{L} \to 0, \quad U_{REF} \to \overline{u}, \quad \overline{e} >> A^{*2}/2$

$$\overline{u} - c = \frac{\overline{e}}{A^*} = \frac{\beta_{eff}}{k^2 + l^2}, \qquad A^* = \frac{\overline{e}}{\overline{u} - c}$$

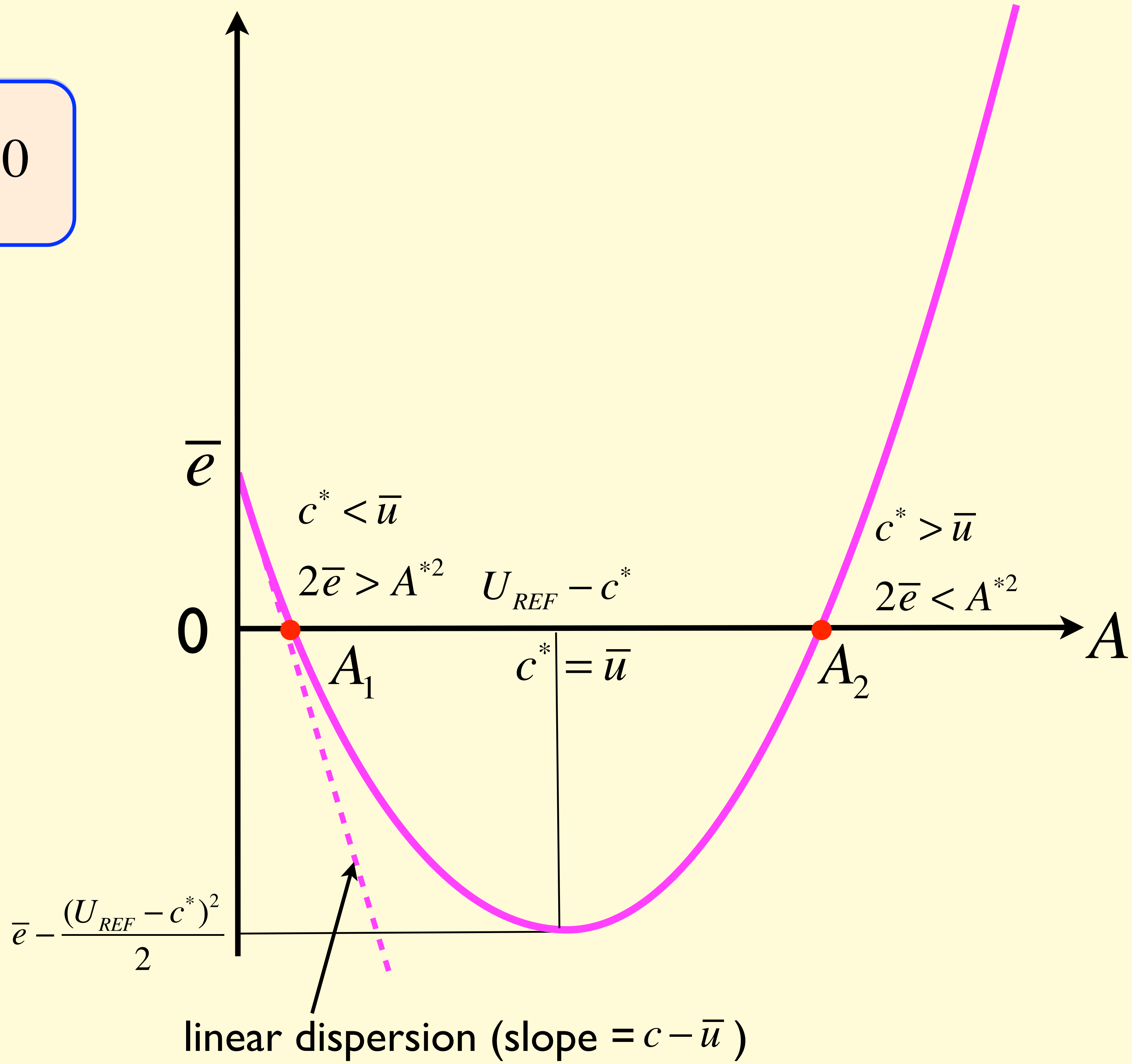
5. Dispersion relation (at each latitude and time)

$$\bar{e} - \left(U_{REF} - c^*\right)A^* + \frac{A^{*2}}{2} = 0$$

$$\bar{u} - c^* = \frac{\bar{e}}{A^*} - \frac{A^*}{2}$$

$$U_{REF} - c^* = \frac{\bar{e}}{A^*} + \frac{A^*}{2} > 0$$

$$(U_{REF} - c^*)^2 \geq 2\bar{e}$$



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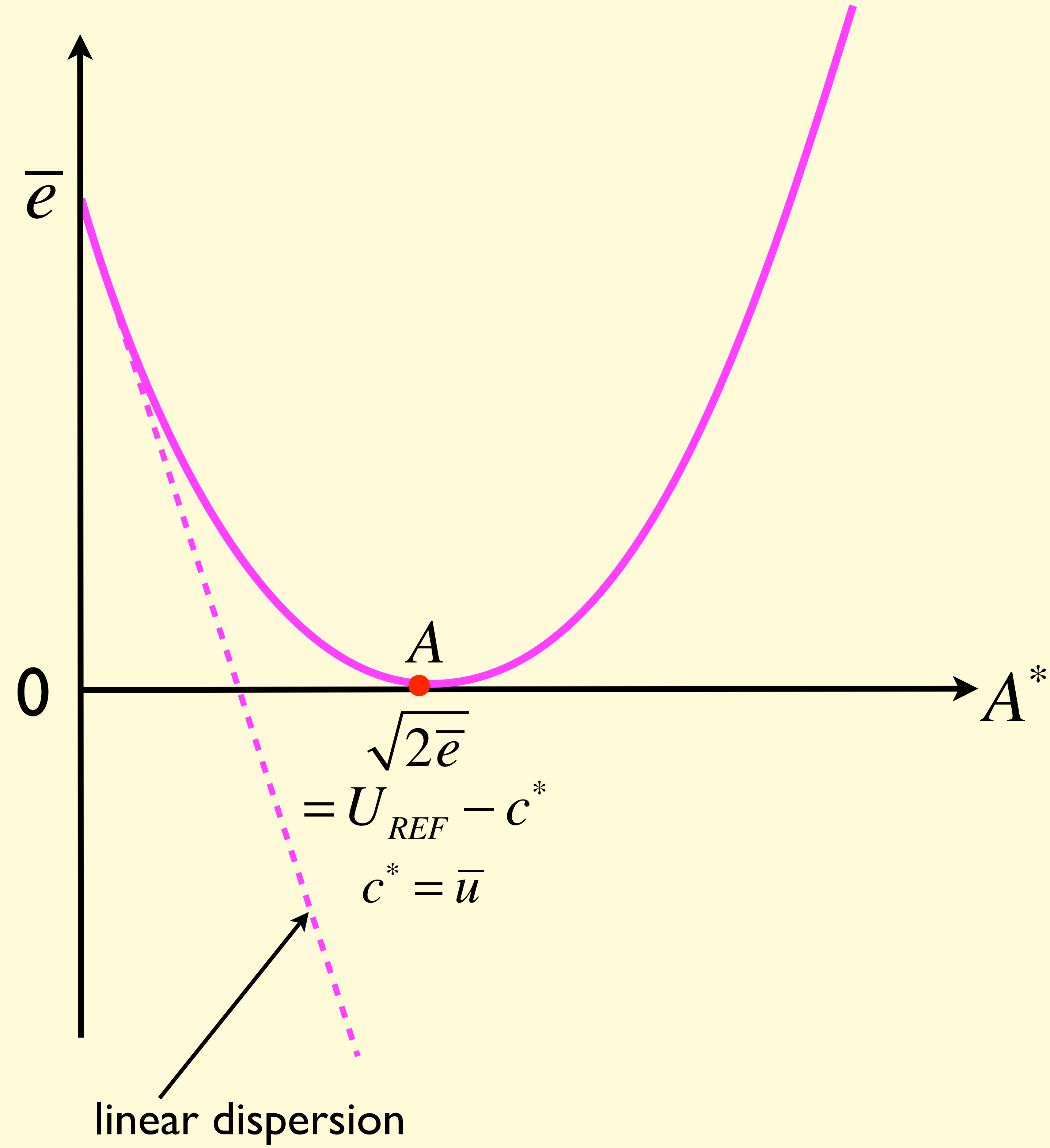
$$\overline{e} - \left(U_{REF} - c^*\right)A^* + \frac{A^{*2}}{2} = 0$$

$$\overline{u} - c^* = \frac{\overline{e}}{A^*} - \frac{A^*}{2}$$

$$U_{REF} - c^* = \frac{\overline{e}}{A^*} + \frac{A^*}{2} > 0$$

$$(U_{REF} - c^*)^2 = 2\overline{e}$$

$\overline{u} - c^* = 0$ at $A^* = \sqrt{2\overline{e}}$



5. Dispersion relation (at each latitude and time)

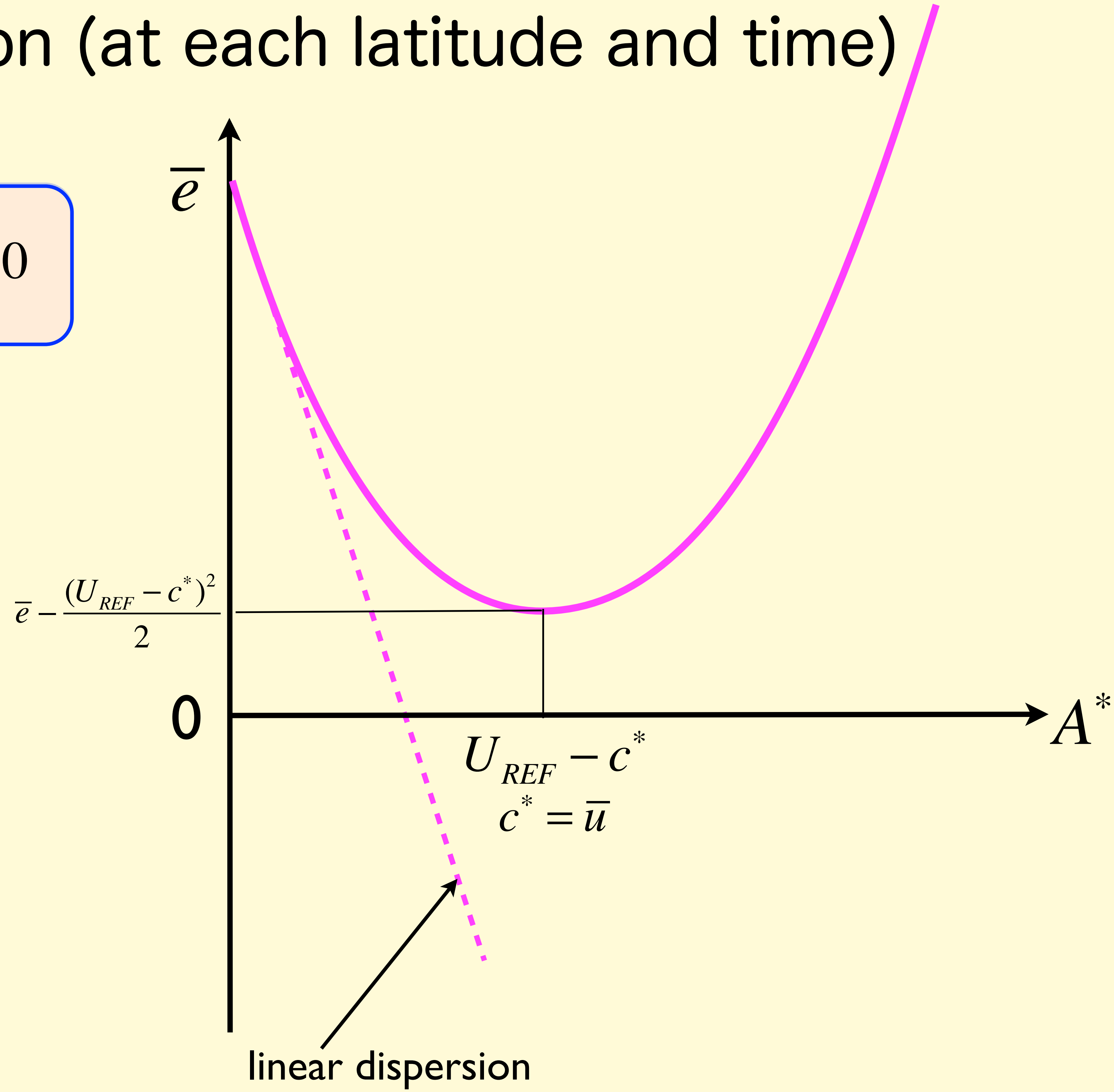
$$\overline{e} - \left(U_{REF} - c^*\right)A^* + \frac{A^{*2}}{2} = 0$$

$$\overline{u} - c^* = \frac{\overline{e}}{A^*} - \frac{A^*}{2}$$

$$U_{REF} - c^* = \frac{\overline{e}}{A^*} + \frac{A^*}{2} > 0$$

$$(U_{REF} - c^*)^2 < 2\overline{e}$$

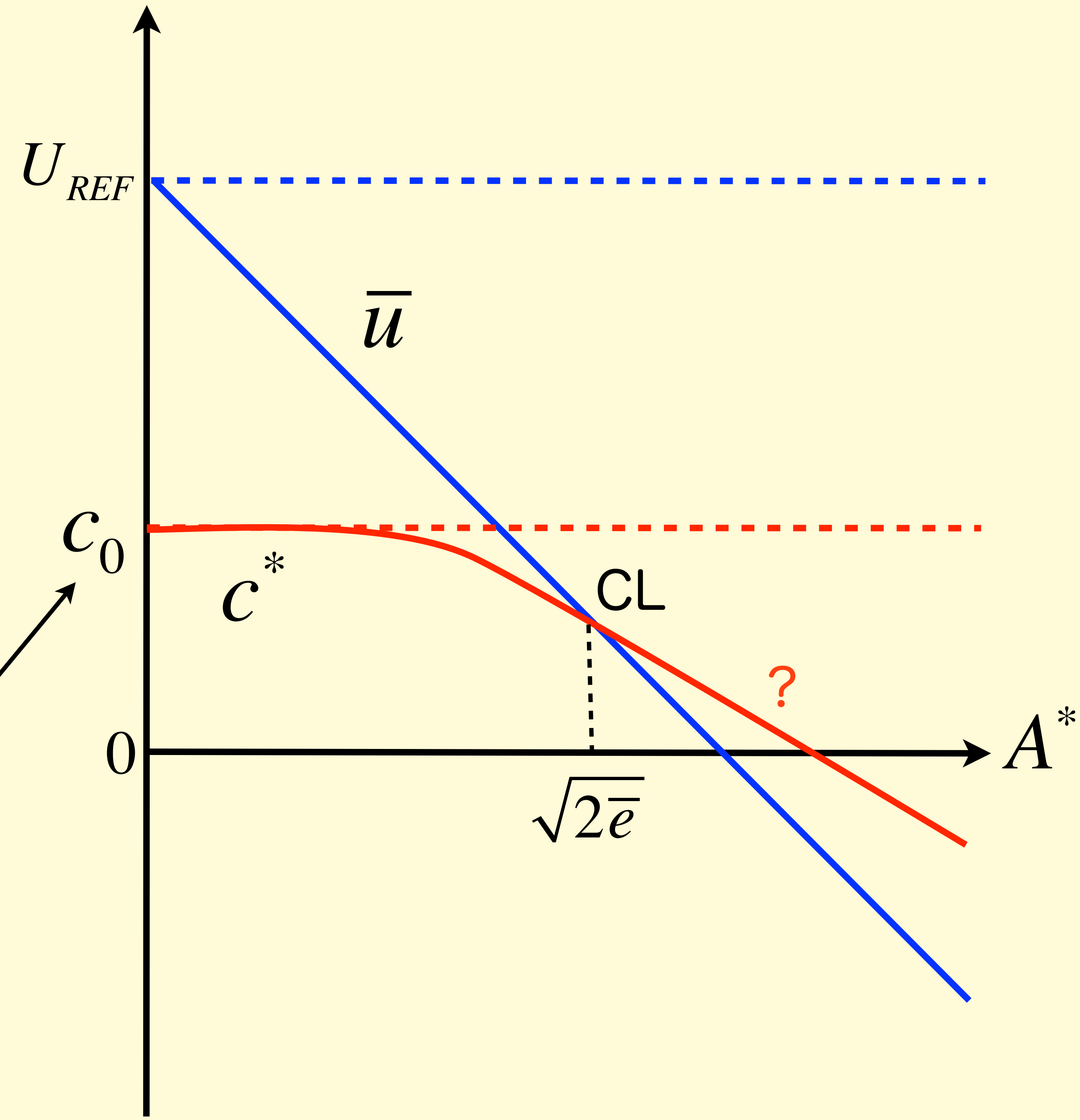
No real solution:
slowly varying waveform and
conservation laws not compatible -
wave breaking?



5. Dispersion relation (at each latitude and time)

$$c^* = U_{REF} - \left(\frac{\overline{e}}{A^*} + \frac{A^*}{2} \right)$$

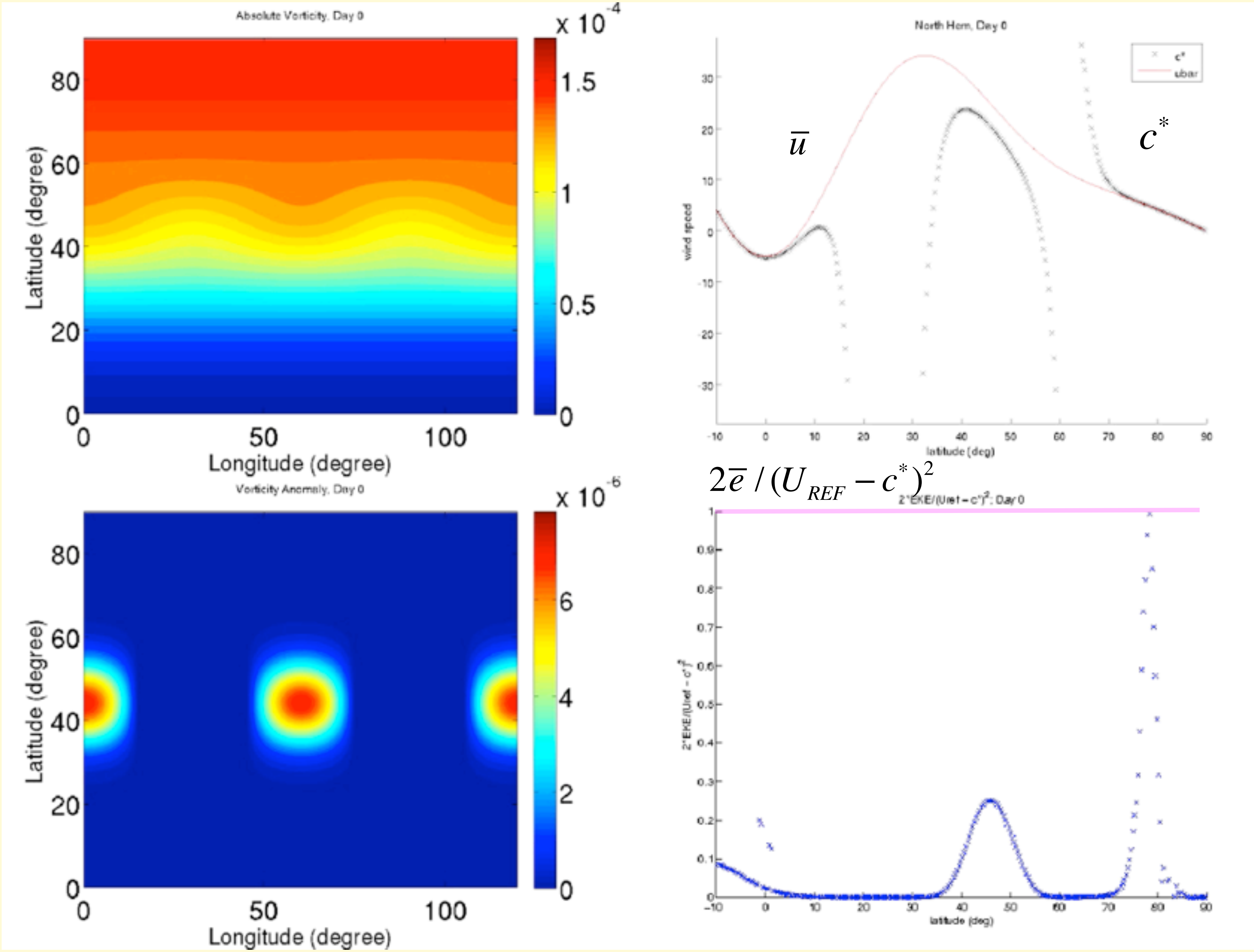
$$\overline{u} = U_{REF} - A^*$$



$$c_0 = U_{REF} - \lim_{A^* \to 0} \frac{\overline{e}}{A^*}$$

6. Numerical simulations

click on images for links to .mov files



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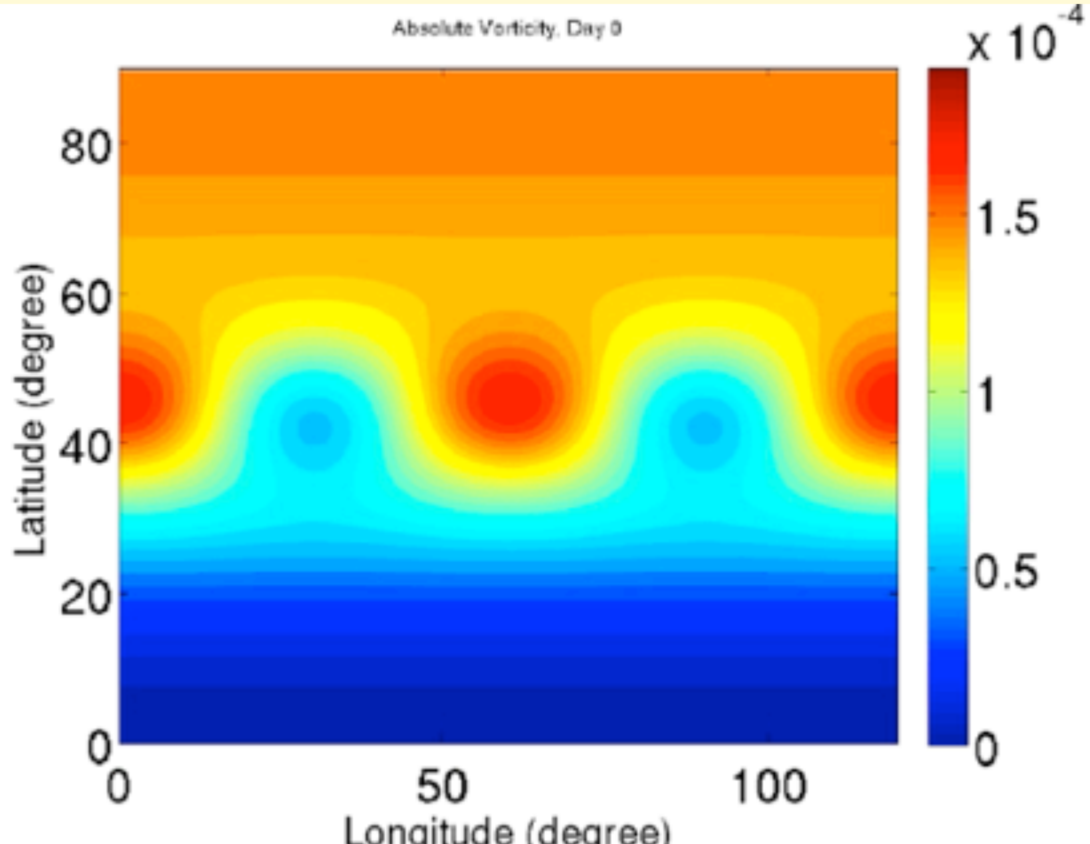
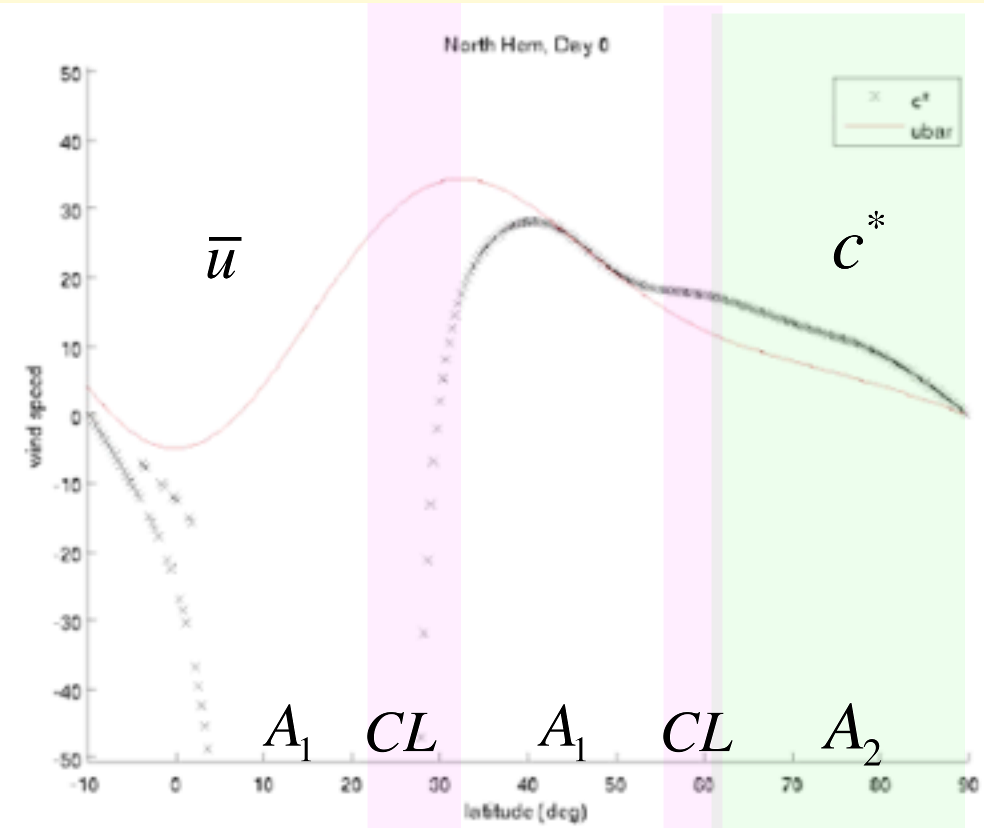
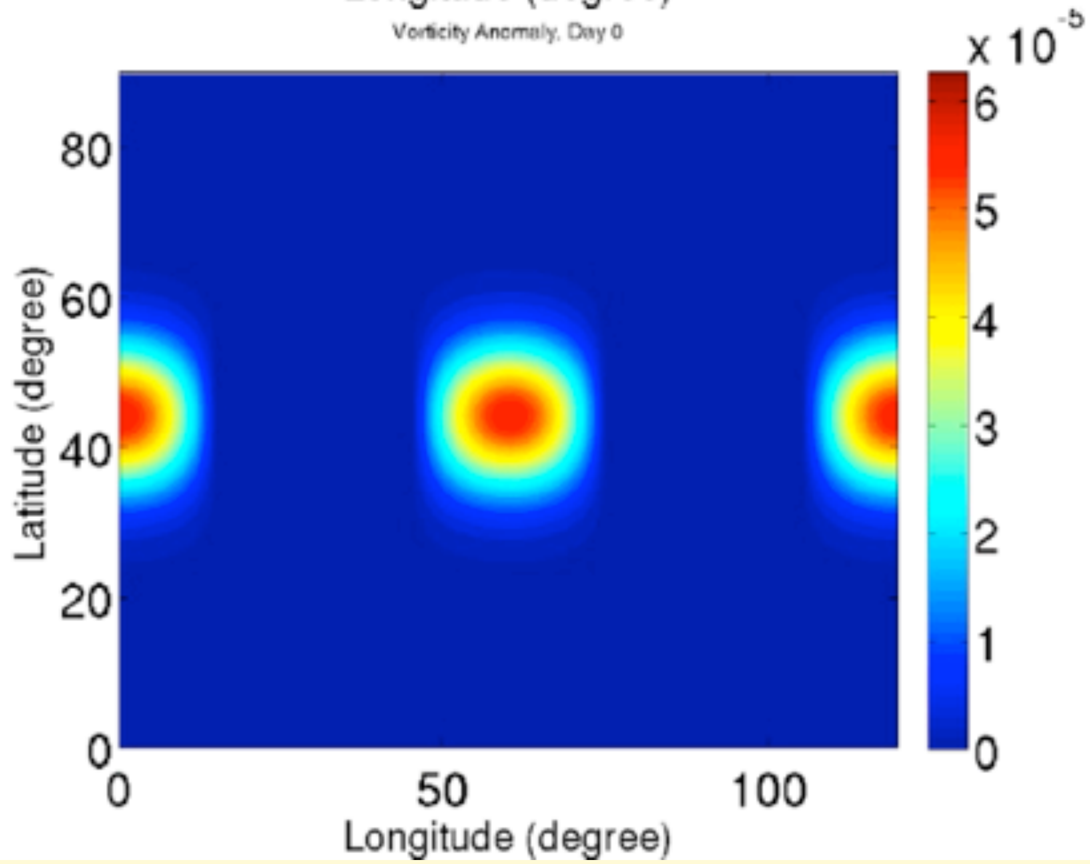
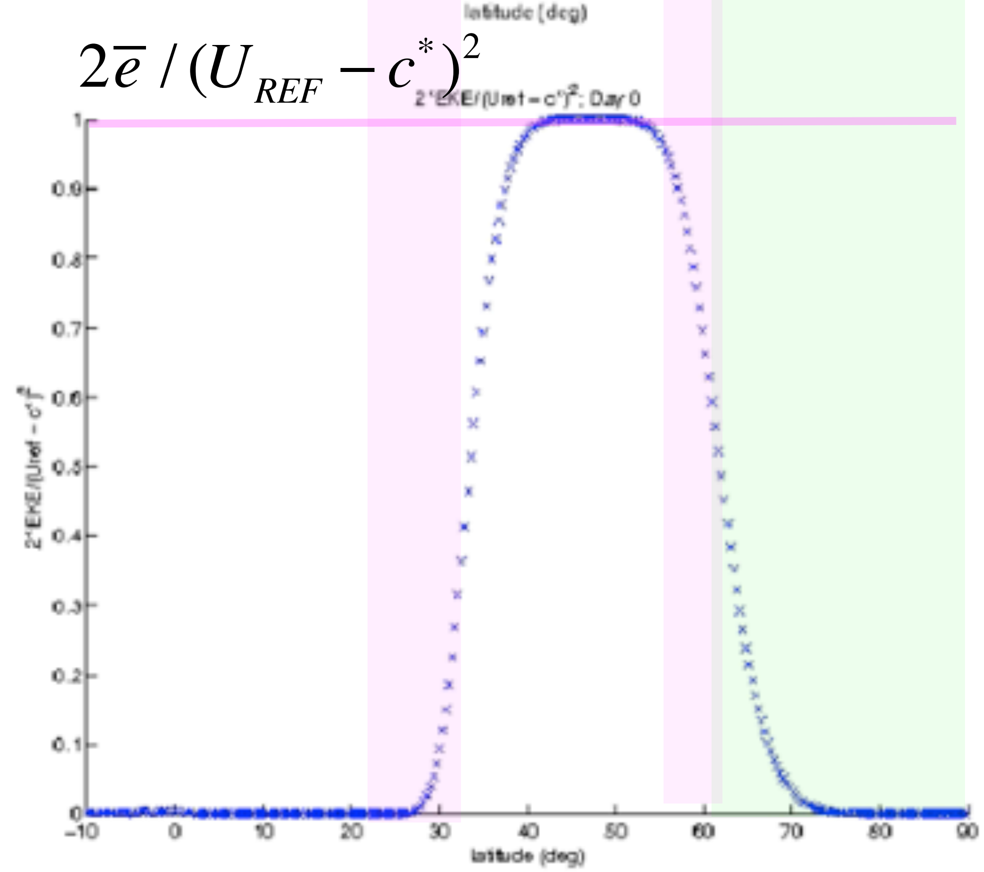


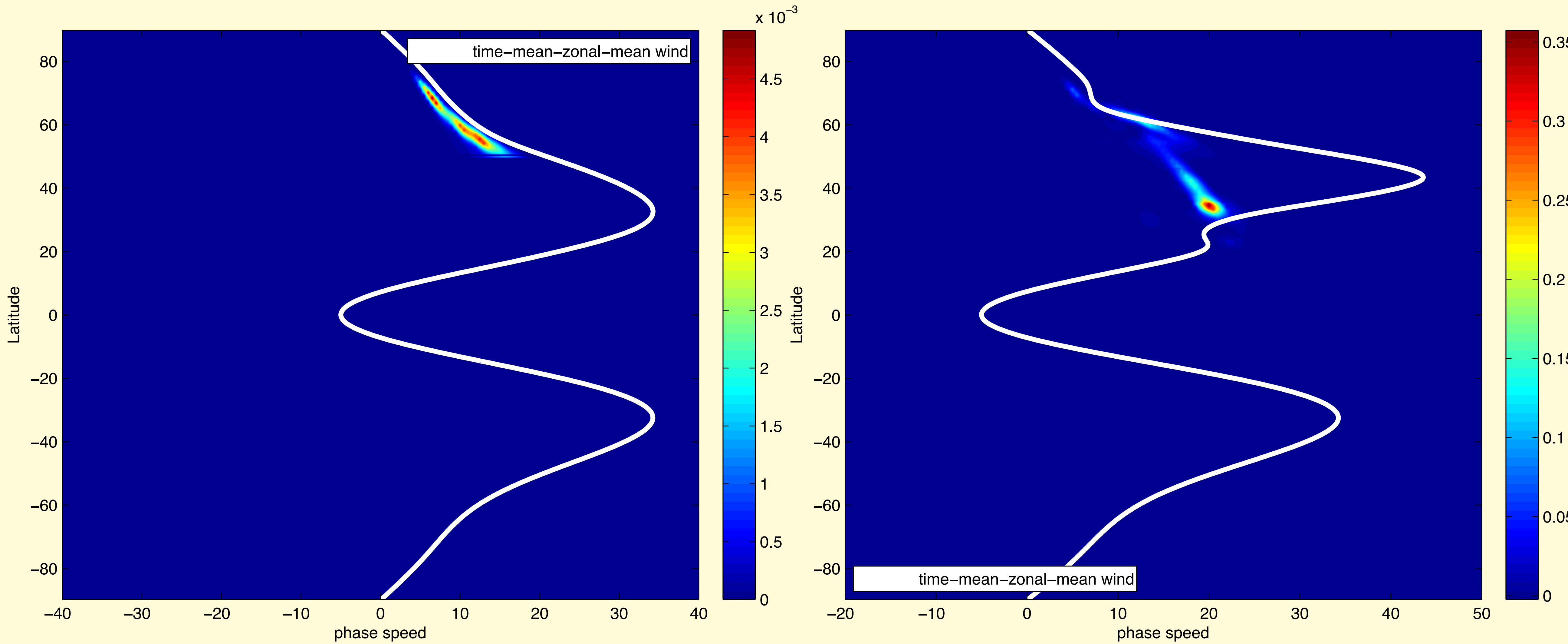






6. Numerical simulations

“Rossby Chromatography” (Hayashi 1971, Randel and Held 1991)



7. Conclusions

- A WKB dispersion relation is derived for finite-amplitude barotropic Rossby waves in slowly varying parallel shear flows
- The dispersion relation is written in terms of (easily calculable) pseudomomentum and pseudoenergy densities and may be used to evaluate the effective zonal phase speed
- The dispersion relation addresses the dependence of effective zonal phase speed on amplitude, and the effects of wave-mean flow interaction
- Finite-amplitude waves can create critical lines by decelerating the mean flow. Critical lines are no longer singularities but more like bifurcations, at which slowly varying waveform ceases to be compatible with conservation laws (onset of wave breaking?)