Hybrid Euler-Lagrange methods in fluid mechanics; glm versus GLM

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Outline

1. INTRODUCTION

Length scales

In large Reynolds number turbulence, motion occurs on a wide range of length scales varying from the large size L of the system down to the very short length viscous length scale l_ν ($\ll L$). Only on that latter length l_ν is viscous dissipation important.

For buoyancy driven MHD systems the problem is complicated by the fact that there are in addition other dissipation lengths such as the thermal and magnetic diffusion length scales l_κ and l_η, which may be of very disparate values depending on the Prandtl numbers ν/κ and ν/η.

When the scale range between L and $l_{\rm max} \equiv \max\,(l_\nu,\ l_\kappa,\ l_\eta)$ is very large, it remains problematic, how to deal with the intermediate length scales l. It is this range, $l_{\rm max} \ll l \ll L$, that has motivates our enquiry and to which we restrict attention.

Lagrangian representation: Advection

In rotating MHD systems, it is well known that the Lagrangian (rather than the Eulerian) representation can often be used very effectively, when $l \gg l_{\max}$.

The idea is most readily appreciated in the context of the advection without diffusion of a passive scalar quantity such as temperature, for which its material derivative vanishes. Then the temperature remains constant following fluid particles.

Likewise in the case of magnetic field in a perfectly conducting fluid, magnetic flux through material surfaces is conserved. By implication the magnetic field at a point moving with the fluid is readily derived in the Lagrangian framework simply by properties of the coordinate transformation relating the current position $\mathbf{x}^{LN}$ of a fluid particle to its original position $\mathbf{x}$.

Lagrangian representation: Momentum

The properties mentioned are kinematic in nature and ultimately provide a useful description of the advected quantities. To determine their temporal evolution, we take advantage of the frozen field results when considering the equation of motion.

The simplest application of the idea is through the investigation of the stability of a static state. Since the pressure gradient in the equation of motion does not transform nicely from a Lagrangian point of view, it is better to consider the equation of motion in its Eulerian form.

The Eulerian perturbation values of frozen quantities like the magnetic field, which appear in the equation of motion, are determined from their Lagrangian description in terms of the small fluid particle displacement $\boldsymbol{\xi}^{LN} = \mathbf{x}^{LN} - \mathbf{x}$. In this way, equations like the temperature and magnetic induction equations are bypassed leaving only an equation governing $\boldsymbol{\xi}^{LN}$.

Hybrid Eulerian–Lagrangian (HEL) representation

In turbulence, the fluid particle displacement $\boldsymbol{\xi}^{LN}$ generally increases indefinitely. Even so the Lagrangian procedure has been adopted and used to obtain Eulerian values (of say the transport of a passive scalar) at quadratic order in the displacement, valid over a limited period of time. Then averaging may be used to determine the evolution of the Eulerian mean quantities, usually under some assumption such as a short correlation time.

Even when $|\boldsymbol{\xi}^{LN}|$ diverges, it may happen that the path displacements $\boldsymbol{\xi}$ ($\neq \boldsymbol{\xi}^{LN}$) of particles from their mean flow trajectories though finite remain of moderate size as exemplified by *wave turbulence riding on a sheared mean flow*. Then the hybrid Eulerian–Lagrangian (HEL) approach, developed by Soward (1972) in the dynamo context and Andrews & McIntyre (1978a) in the atmospheric science context, provides a good way of addressing the evolution of the mean fields correct to $\mathcal{O}(|\boldsymbol{\xi}|^2)$.

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2. TENSOR CALCULUS

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Coordinate transformations

Consider the mapping of a point P: $\mathbf{x}$ to a point P^L: $\mathbf{x}^L \, (= \mathbf{x} + \boldsymbol{\xi})$ through a displacement $\boldsymbol{\xi}(\mathbf{x})$. We write $\mathcal{T}^{*L}(\mathbf{x}) \equiv \mathcal{T}^{*}(\mathbf{x}^L(\mathbf{x}))$, for scalar, vector and tensor quantities $\mathcal{T}^{*}(\mathbf{x})$ evaluated at P^L rather than P. If instead we regard $x_i^L \mapsto x_i$ as a coordinate transformation, we introduce $\mathcal{T}(\mathbf{x})$, which is related to $\mathcal{T}^{*L}(\mathbf{x})$ by

$$\left(\mathcal{T}^{*A\cdots D}_{\ \ E\cdots G}\right)^L = \mathcal{J}^{\mathrm{w}} \left(\nabla^L_E x_e\right) \cdots \left(\nabla^L_G x_g\right) \mathcal{T}^{a\cdots d}_{e\cdots g} \left(\nabla_a x^L_A\right) \cdots \left(\nabla_d x^L_D\right)$$

for mixed tensors of weight w, where $\mathcal{J} = \|\boldsymbol{\nabla}\mathbf{x}^L\|$ is the Jacobian,

$$\nabla^L_j \equiv \partial/\partial x^L_j \,, \qquad\qquad \nabla_i \equiv \partial/\partial x_i \,;$$

we distinguish contravariant (upper index), covariant (lower index) tensors.

Cartesian examples

Regarding x_i as the Cartesian coordinates of a point P and x_i^L as the Cartesian coordinates of a point P^L, we provide illustrations of the general tensor transformations.

For temperature θ^* ($\mathrm{w} = 0$) and density ρ^{*L} ($\mathrm{w} = -1$), we have

$$\theta^{*L} = \theta \qquad \mathcal{J}\rho^{*L} = \rho\,.$$

For velocity $\mathbf{u}^*$ (contravariant, $\mathrm{w} = 0$) and mass flux $\mathbf{m}^*$ (contravariant, $\mathrm{w} = -1$), we have

$$\mathbf{u}^{*L} = \mathbf{u}\cdot\nabla\mathbf{x}^L\,, \qquad \mathcal{J}\mathbf{m}^{*L} = \mathbf{m}\cdot\nabla\mathbf{x}^L\,.$$

NOTE that we will modify this definition for time dependant coordinates $\mathbf{x}^L(\mathbf{x},t)$.

For momentum per unit mass $\mathbf{V}^*$ (covariant, $\mathrm{w} = 0$), we have

$$\left(\mathbf{u}^{*L} = \right)\mathbf{V}^{*L} = (\nabla^L\mathbf{x})\cdot\mathbf{V}\,.$$

Lie derivatives

The Lie derivative is

$$
\begin{aligned}
\left(\mathrm{L}_{\boldsymbol{\eta}} \boldsymbol{\mathcal{T}}\right)^{ab\cdots d}_{ef\cdots g} \equiv (\boldsymbol{\eta}\cdot\boldsymbol{\nabla})\,\mathcal{T}^{ab\cdots d}_{ef\cdots g} - \mathcal{T}^{jb\cdots d}_{ef\cdots g}(\nabla_j\,\eta^a) - \cdots \\
- \mathsf{w}\,(\nabla_j\,\eta^j)\,\mathcal{T}^{ab\cdots d}_{ef\cdots g} + (\nabla_e\,\eta^j)\mathcal{T}^{ab\cdots d}_{jf\cdots g} + \cdots .
\end{aligned}
$$

Relative to Cartesian coordinates some special cases are

$$
\begin{aligned}
\mathrm{L}_{\boldsymbol{\eta}}\theta &= \boldsymbol{\eta}\cdot\boldsymbol{\nabla}\,\theta && \mathsf{w}=0 \\
\mathrm{L}_{\boldsymbol{\eta}}\rho &= \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\eta}) && \mathsf{w}=-1 \\
\mathrm{L}_{\boldsymbol{\eta}}\mathbf{u} &= [\,\boldsymbol{\eta}\,,\,\mathbf{u}\,] \equiv \boldsymbol{\eta}\cdot\boldsymbol{\nabla}\,\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}\,\boldsymbol{\eta} && \text{contra.},\ \mathsf{w}=0 \\
&= -\boldsymbol{\nabla}\times(\boldsymbol{\eta}\times\mathbf{u}) - (\boldsymbol{\nabla}\cdot\boldsymbol{\eta})\,\mathbf{u} + (\boldsymbol{\nabla}\cdot\mathbf{u})\,\boldsymbol{\eta}\,, \\
\mathrm{L}_{\boldsymbol{\eta}}\mathbf{m} &= [\![\,\boldsymbol{\eta}\,,\,\mathbf{m}\,]\!] \equiv [\,\boldsymbol{\eta}\,,\,\mathbf{m}\,] + (\boldsymbol{\nabla}\cdot\boldsymbol{\eta})\,\mathbf{m} && \text{contra.},\ \mathsf{w}=-1 \\
&= -\boldsymbol{\nabla}\times(\boldsymbol{\eta}\times\mathbf{m}\,) + \boldsymbol{\eta}\,(\boldsymbol{\nabla}\cdot\mathbf{m})\,, \\
\mathrm{L}_{\boldsymbol{\eta}}\mathbf{V} &= \{\,\boldsymbol{\eta}\,,\,\mathbf{V}\,\} \equiv \boldsymbol{\eta}\cdot\boldsymbol{\nabla}\,\mathbf{V} + (\boldsymbol{\nabla}\boldsymbol{\eta})\cdot\mathbf{V} && \text{co.},\ \mathsf{w}=0 \\
&= -\,\boldsymbol{\eta}\times(\boldsymbol{\nabla}\times\mathbf{V}\,) + \boldsymbol{\nabla}(\boldsymbol{\eta}\cdot\mathbf{V})\,.
\end{aligned}
$$

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3. EULERIAN APPROACH

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Wave turbulence on shear flows

Braginsky (1964) considered a geodynamo model, which serves to exemplify how the HEL method.

He considered a predominantly mean axisymmetric azimuthal motion, velocity $\overline{\mathbf{u}^*}$, density $\overline{\rho^*}$ of electrically conducting fluid permeated by almost aligned magnetic field $\mathbf{b}^*$, to which we add scalar fields θ^* such as temperature, with mean values $\overline{\mathbf{b}^*}$, $\overline{\theta^*}$. He then envisaged a secondary non-axisymmetric wave motion, velocity $\mathbf{u}^{*\prime}$ riding on that primary flow:

$$
\begin{aligned}
\mathbf{u}^*(\mathbf{x},t) &= \overline{\mathbf{u}^*} + \mathbf{u}^{*\prime}, \qquad & \rho^*(\mathbf{x},t) &= \overline{\rho^*} + \rho^{*\prime}, \\
\mathbf{b}^*(\mathbf{x},t) &= \overline{\mathbf{b}^*} + \mathbf{b}^{*\prime}, \qquad & \theta^*(\mathbf{x},t) &= \overline{\theta^*} + \theta^{*\prime}.
\end{aligned}
$$

The Frieman & Rotenberg approach

Braginsky's Eulerian approach is captured best using a formalism developed by Frieman & Rotenberg (1960), by which the fluctuating velocity is expressed in the exact form

$$\mathbf{u}^{*\prime}(\mathbf{x},t) = \mathrm{D}_t^{FR}\boldsymbol{\zeta} - \boldsymbol{\zeta}\cdot\boldsymbol{\nabla}\,\overline{\mathbf{u}^*} = \partial_t\boldsymbol{\zeta} - \mathrm{L}_{\boldsymbol{\zeta}}\overline{\mathbf{u}^*}\,, \tag{1}$$

$$\mathrm{D}_t^{FR} \equiv \partial_t + \overline{\mathbf{u}^*}\cdot\boldsymbol{\nabla}\,, \qquad \mathrm{L}_{\boldsymbol{\zeta}}\overline{\mathbf{u}^*} = \left[\,\boldsymbol{\zeta}\,,\,\overline{\mathbf{u}^*}\,\right],$$

where $\boldsymbol{\zeta}$ (closely related to $\boldsymbol{\xi}$) has no mean part

$$\overline{\boldsymbol{\zeta}} = \mathbf{0}\,.$$

With diffusion and terms quadratic in the fluctuations neglected, we have the approximate results

$$\theta^{*\prime}(\mathbf{x},t) \approx -\mathrm{L}_{\boldsymbol{\zeta}}\overline{\theta^*} = -\boldsymbol{\zeta}\cdot\boldsymbol{\nabla}\overline{\theta^*}\,, \tag{2a}$$

$$\rho^{*\prime}(\mathbf{x},t) \approx -\mathrm{L}_{\boldsymbol{\zeta}}\overline{\rho^*} = -\boldsymbol{\nabla}\cdot(\overline{\rho^*}\boldsymbol{\zeta})\,, \tag{2b}$$

$$\mathbf{b}^{*\prime}(\mathbf{x},t) \approx -\mathrm{L}_{\boldsymbol{\zeta}}\overline{\mathbf{b}^*} = \boldsymbol{\nabla}\times(\boldsymbol{\zeta}\times\overline{\mathbf{b}^*})\,. \tag{2c}$$

for the fluctuations themselves.

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4. HEL

The HEL construction (or time dependant coordinates)

The Eulerian form (1), (2a-c) contains the rudiments of the HEL construction, in which ζ may be interpreted as the displacement $\boldsymbol{\xi}\,(\approx \zeta)$ that determines the location of the perturbed azimuthal magnetic field line as frozen to the fluid flow, i.e. the magnetic field line at position P: $\mathbf{x}$ following the mean flow is moved to $\mathrm{P}^L : \mathbf{x}^L = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$ at time t.

This viewpoint corresponds to the well known frozen field results

$$\theta^{*L} = \theta\,, \qquad\qquad \mathcal{J}\rho^{*L} = \rho\,,$$
$$\mathcal{J}\mathbf{b}^{*L} = \mathbf{b}\cdot\nabla\mathbf{x}^L = \mathbf{b} + \mathbf{b}\cdot\nabla\boldsymbol{\xi}\,.$$

Here the Jacobian $\mathcal{J}$ has the kinematic property

$$\partial_t\mathcal{J} = \nabla\cdot(\mathcal{J}\mathbf{w})\,; \tag{3a}$$
$$\mathbf{w} = \mathbf{w}^{*L}\cdot(\nabla^L\mathbf{x})\,; \qquad \mathbf{w}^{*L} = \partial_t\mathbf{x}^L = \partial_t\boldsymbol{\xi} \tag{3b,c}$$

is the velocity of the point $\mathrm{P}^L : \mathbf{x}^L(\mathbf{x}, t)$ at fixed $\mathbf{x}$.

Transformation of the time derivatives

Temperature:

$$\big(\partial_t \theta^*\big)^L = \partial_t \theta - \mathrm{L}_{\mathbf{w}} \theta \,, \qquad \mathrm{L}_{\mathbf{w}} \theta = \mathbf{w} \cdot \boldsymbol{\nabla} \rho \,,$$

Density:

$$\mathcal{J}\big(\partial_t \rho^*\big)^L = \partial_t \rho - \mathrm{L}_{\mathbf{w}} \rho \,, \qquad \mathrm{L}_{\mathbf{w}} \rho = \boldsymbol{\nabla} \cdot (\rho \mathbf{w}) \,,$$

Magnetic field:

$$\mathcal{J}\big(\partial_t \mathbf{b}^*\big)^L \cdot \boldsymbol{\nabla}^L \mathbf{x} = \partial_t \mathbf{b} - \mathrm{L}_{\mathbf{w}} \mathbf{b} \,, \qquad \mathrm{L}_{\mathbf{w}} \mathbf{b} = [\![\mathbf{w} \,, \mathbf{b}]\!] \,,$$

Momentum:

$$(\boldsymbol{\nabla} \mathbf{x}^L) \cdot \big(\partial_t \mathbf{V}^*\big)^L = \partial_t \mathbf{V} - \mathrm{L}_{\mathbf{w}} \mathbf{V} \,, \qquad \mathrm{L}_{\mathbf{w}} \mathbf{V} = \{ \mathbf{w} \,, \mathbf{V} \} \,,$$

Relative velocity

We transform the relative velocity $\mathbf{v}^{*L} = \mathbf{u}^{*L} - \mathbf{w}^{*L}$ (true velocity $\mathbf{u}^*$ less frame velocity $\mathbf{w}^*$) and write

$$\left.\begin{aligned} \mathbf{u}^{*L} &= \mathbf{v}\cdot\nabla\mathbf{x}^L\,,\\ \mathbf{w}^{*L} &= \mathbf{w}\cdot\nabla\mathbf{x}^L\,,\\ \mathbf{v} &= \mathbf{u}+\mathbf{w} \end{aligned}\right\} \quad\Longleftrightarrow\quad \left\{\begin{aligned} \mathbf{u} &= \mathbf{v}^{*L}\cdot\nabla^L\mathbf{x}\,,\\ \mathbf{w} &= \mathbf{w}^{*L}\cdot\nabla^L\mathbf{x}\,,\\ \mathbf{v}^{*L} &= \mathbf{u}^{*L}-\mathbf{w}^{*L}\,. \end{aligned}\right.$$

In this way,

$$\begin{aligned} \mathbf{u}^{*L} &= \mathbf{w}\cdot\nabla\mathbf{x}^L + \mathbf{u}\cdot\nabla\mathbf{x}^L\\ &= \partial_t\mathbf{x}^L + \mathbf{u}\cdot\nabla\mathbf{x}^L = \mathrm{D}_t\mathbf{x}^L = \mathbf{u} + \mathrm{D}_t\boldsymbol{\xi}\,, \end{aligned}$$

where

$$\mathrm{D}_t \equiv \partial_t + \mathbf{u}\cdot\nabla\,, \qquad\qquad \mathbf{x}^L = \mathbf{x} + \boldsymbol{\xi}$$

should be interpreted as the material derivative associated with HEL advection at P:$\mathbf{x}$ with velocity $\mathbf{u}$.

Ideal fluids

The introduction of the relative velocity leads to the invariant structure of the ideal fluid equations involving advection.

Temperature:

$$\partial_t \theta^* + \mathrm{L}_{\mathbf{u}^*} \theta^* = 0 \qquad \Longrightarrow \qquad \partial_t \theta + \mathrm{L}_{\mathbf{u}} \theta = 0\,.$$

Density:

$$\partial_t \rho^* + \mathrm{L}_{\mathbf{u}^*} \rho^* = 0 \qquad \Longrightarrow \qquad \partial_t \rho + \mathrm{L}_{\mathbf{u}} \rho = 0\,.$$

Magnetic field:

$$\partial_t \mathbf{b}^* + \mathrm{L}_{\mathbf{u}^*} \mathbf{b}^* = \mathbf{0} \qquad \Longrightarrow \qquad \partial_t \mathbf{b} + \mathrm{L}_{\mathbf{u}} \mathbf{b} = \mathbf{0}\,.$$

Momentum:

$$\partial_t \mathbf{V}^* + \mathrm{L}_{\mathbf{u}^*} \mathbf{V}^* \qquad \longrightarrow \qquad \partial_t \mathbf{V} + \mathrm{L}_{\mathbf{u}} \mathbf{V}\,.$$

Solenoidal conditions

To simplify matters we will restrict attention to constant density, ρ_0^*, solenoidal flow (also solenoidal magnetic magnetic field):

$$\nabla\cdot\mathbf{u}^* = 0\,, \qquad \nabla\cdot\mathbf{b}^* = 0\,.$$

Evaluated at P^L, they are

$$(\nabla\cdot\mathbf{u}^*)^L = 0\,, \qquad (\nabla\cdot\mathbf{b}^*)^L = 0\,,$$

which become

$$\nabla\cdot\big(\mathcal{J}(\mathbf{u}+\mathbf{w})\big) = 0\,, \qquad \nabla\cdot\mathbf{b} = 0\,. \qquad \text{(4a,b)}$$

- For the special case of isochoric transformations $\mathcal{J}=1$, (3a) and (4a) determine

$$\nabla\cdot\mathbf{u} = 0\,, \qquad \nabla\cdot\mathbf{w} = 0\,. \qquad \text{(5a,b)}$$

Ideal fluid equations: Eulerian

For constant density ρ_0^*, the ideal fluid equations, relevant to the limit $l_{\max} \ll l \ll L$, are

the heat conduction equation

$$\partial_t \theta^* + \mathbf{u}^* \cdot \boldsymbol{\nabla}\, \theta^* = 0\,, \qquad\qquad \boldsymbol{\nabla} \cdot \mathbf{u}^* = 0\,;$$

the magnetic induction equation

$$\partial_t \mathbf{b}^* = \boldsymbol{\nabla} \times (\mathbf{u}^* \times \mathbf{b}^*)\,, \qquad\qquad \boldsymbol{\nabla} \cdot \mathbf{b}^* = 0\,;$$

Euler's equations equation

$$\big(\partial_t \mathbf{V}^* + \mathrm{L}_{\mathbf{u}^*} \mathbf{V}^* \equiv\big)\, \partial_t \mathbf{V}^* + \mathbf{u}^* \cdot \boldsymbol{\nabla}\, \mathbf{V}^* + \boldsymbol{\nabla}\big(\tfrac{1}{2}|\mathbf{u}^*|^2\big) = -\,\boldsymbol{\nabla}\, \varPi^*\,,$$

with $\qquad \mathbf{V}^* = \mathbf{u}^*\,, \qquad\qquad \varPi^* = (p^*/\rho_0^*) - \tfrac{1}{2}|\mathbf{v}^*|^2\,.$

Ideal fluid equations: HEL

Each Eulerian equation evaluated at P^L transform as follows:

(heat conduction equation)^L

$$D_t\theta \;=\; 0\,, \qquad\qquad \boldsymbol{\nabla}\cdot(\mathbf{u}+\mathbf{w}) \;=\; 0\,;$$

$\mathcal{J}$(magnetic induction equation)$^L\cdot(\boldsymbol{\nabla}^L\mathbf{x})$

$$\partial_t\mathbf{b} \;=\; \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{b})\,, \qquad\qquad \boldsymbol{\nabla}\cdot\mathbf{b} \;=\; 0\,;$$

$(\boldsymbol{\nabla}\mathbf{x}^L)\cdot$(Euler's equations equation)^L

$$\big(\partial_t\mathbf{V}+\mathrm{L}_{\mathbf{u}}\mathbf{V}\;\equiv\big)\,\partial_t\mathbf{V}\;+\;\mathbf{u}\cdot\boldsymbol{\nabla}\,\mathbf{V}+\boldsymbol{\nabla}\big(\tfrac{1}{2}|\mathbf{u}|^2\big)\;=\;-\,\boldsymbol{\nabla}\varPi\,,$$

where $\quad \mathbf{V} \;=\; (\boldsymbol{\nabla}\mathbf{x}^L)\,\cdot\,\mathbf{u}^{*L}\,, \qquad \varPi \;=\; \varPi^{*L}\,.$

Together with $\quad \partial_t\mathcal{J} = \boldsymbol{\nabla}\cdot(\mathcal{J}\mathbf{w}), \quad$ these are the hybrid Eulerian-Lagrangian (HEL) equations governing the HEL variables $\mathbf{u}(\mathbf{x},t)$, $\mathbf{w}(\mathbf{x},t)$, $\mathbf{V}(\mathbf{x},t)$, $\mathbf{b}(\mathbf{x},t)$ and $\theta(\mathbf{x},t)$ (see Soward 1972, Andrews & McIntyre 1978 but also Roberts & Soward 2006).

The Coriolis acceleration $\quad \mathcal{C}^* = 2\boldsymbol{\Omega} \times \mathbf{u}^*$

Though the sum of the Coriolis and centrifugal accelerations transform as a general tensor (covariant vector, $w = 0$) neither does individually! Bearing in mind that limitation, we consider

$$\begin{aligned} \mathcal{C} &= (\nabla \mathbf{x}^L) \cdot \mathcal{C}^{*L} = (\nabla \mathbf{x}^L) \cdot 2(\boldsymbol{\Omega} \times \mathbf{u}^*)^L \\ &= \partial_t \mathcal{R} + (2\boldsymbol{\Omega} + \nabla \times \mathcal{R}) \times \mathbf{u} - \nabla\big((\partial_t \boldsymbol{\xi}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi})\big), \end{aligned}$$

where

$$\begin{aligned} \mathcal{R} &\equiv (\mathbf{I} + \nabla \mathbf{x}^L) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi}) \\ &= 2\boldsymbol{\Omega} \times \boldsymbol{\xi} + (\nabla \boldsymbol{\xi}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi}) \end{aligned}$$

is NOT a covariant vector ($w = 0$), and in which

$$\mathbf{x}^L = \mathbf{x} + \boldsymbol{\xi}\,.$$

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5. ETHEL

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The ETHEL construction

The HEL approach has the unfortunate feature that

- the HEL variables $\mathcal{T}(\mathbf{x}, t)$ define conditions at P^L
- rather than P, where the Eulerian variables $\mathcal{T}^*(\mathbf{x}, t)$ apply.

This shortcoming motivated Holm (2002) to search for an Eulerian development which incorporated features of the HEL theory.

To address the matter, an Euler transformed HEL (ETHEL) method was proposed in Soward & Roberts (2010) building on Moffatt (1986). The strategy is to relate the HEL form, which is a function of the Eulerian form $\mathcal{T}^{*L}(\mathbf{x}, t) = \mathcal{T}^*(\mathbf{x}^L(\mathbf{x}, t), t)$ at P^L, to $\mathcal{T}^*(\mathbf{x}, t)$ at P via a Taylor expansion based on $\mathbf{x}^L = \mathbf{x} + \boldsymbol{\xi}$.

The direct expansion in $\boldsymbol{\xi}$ becomes unwieldy and cumbersome at the $\mathcal{O}(\xi^2)$ level needed to investigate the rôle of the fluctuations

$$\mathcal{T}^{*\prime} = \mathcal{T}^* - \overline{\mathcal{T}^*}$$

on the mean values $\overline{\mathcal{T}^*}$.

Towards Lie dragging

To simplify matters, Soward & Roberts (2010) employed a technique introduced by Moffatt (1986), whereby the mapping $\mathbf{x} \mapsto \mathbf{x}^L$ is achieved by dragging P to P^L by a 'fictitious steady flow' $\boldsymbol{\eta}^*(\mathbf{x}, t)$ in a unit of 'fictitious time' ($0 \le \tau \le 1$; say). From it we may define the contravariant vector $\boldsymbol{\eta}(\mathbf{x}, t)$ ($\mathrm{w} = 0$) implicitly by

$$\boldsymbol{\eta}^{*L} = \boldsymbol{\eta} \cdot \boldsymbol{\nabla} \mathbf{x}^L$$

with the remarkable (to me!) property $\qquad \boldsymbol{\eta}^* = \boldsymbol{\eta}\,. \qquad .$

Essentially the fictitious movement of $\mathrm{P}(\tau = 0)$ to $\mathrm{P}^L(\tau = 1)$ is achieved via intermediate points P^ℓ:$\mathbf{x}^\ell(\tau)$. Values linked to P^L are obtained by evaluating the Maclaurin series in τ at $\tau = 1$. So, e.g.,

$$\begin{aligned}
\boldsymbol{\xi}(\mathbf{x}, t) &= \boldsymbol{\eta} + \tfrac{1}{2}\,\boldsymbol{\eta} \cdot \boldsymbol{\nabla} \boldsymbol{\eta} + \cdots\,,\\
\mathcal{J}(\mathbf{x}, t) - 1 &= \boldsymbol{\nabla} \cdot \boldsymbol{\eta} + \tfrac{1}{2}\,\boldsymbol{\nabla} \cdot \big((\boldsymbol{\nabla} \cdot \boldsymbol{\eta})\,\boldsymbol{\eta}\big) + \cdots\,,\\
\mathbf{w}^*(\mathbf{x}, t) &= \partial_t \boldsymbol{\eta} - \tfrac{1}{2}\big(\boldsymbol{\eta} \cdot \boldsymbol{\nabla}(\partial_t \boldsymbol{\eta}) - (\partial_t \boldsymbol{\eta}) \cdot \boldsymbol{\nabla} \boldsymbol{\eta}\big) - \cdots\,.
\end{aligned}$$

Lie dragging

These special Maclaurin series in τ provide the basis of a more systematic Lie derivative Taylor series expansion of tensors $\mathcal{T}(\mathbf{x},t)$, which may be identified with the technique of 'Lie dragging' :

$$\mathcal{T} \,=\, \mathcal{T}^* \,+\, \mathrm{L}_{\boldsymbol{\eta}}\, \mathcal{T}^* \,+\, \tfrac{1}{2}\big(\mathrm{L}_{\boldsymbol{\eta}}\big)^2\, \mathcal{T}^* \,+\, \cdots \,,$$

in terms of $\mathcal{T}^*(\mathbf{x},t)$ with inverse

$$\mathcal{T}^* \,=\, \mathcal{T} \,-\, \mathrm{L}_{\boldsymbol{\eta}}\, \mathcal{T} \,+\, \tfrac{1}{2}\big(\mathrm{L}_{\boldsymbol{\eta}}\big)^2\, \mathcal{T} \,-\, \cdots \,.$$

Some explicit forms are

$$\begin{aligned}
\theta^* \;&= \theta \,-\, \mathrm{L}_{\boldsymbol{\eta}}\,\theta \,+\, \tfrac{1}{2}\big(\mathrm{L}_{\boldsymbol{\eta}}\big)^2\,\theta \,-\, \cdots \,, \qquad\qquad \text{(also} \quad \mathit{\Pi}^*\text{)},\\
\mathbf{b}^* \;&= \mathbf{b} \,-\, \mathrm{L}_{\boldsymbol{\eta}}\,\mathbf{b} \,+\, \tfrac{1}{2}\big(\mathrm{L}_{\boldsymbol{\eta}}\big)^2\,\mathbf{b} \,-\, \cdots \,,\\
\mathbf{u}^* \;&= \mathbf{V} \,-\, \mathrm{L}_{\boldsymbol{\eta}}\,\mathbf{V} \,+\, \tfrac{1}{2}\big(\mathrm{L}_{\boldsymbol{\eta}}\big)^2\,\mathbf{V} \,-\, \cdots \,,\\
\mathbf{u}^* - \mathbf{w}^* \;&= \mathbf{u} \,-\, \mathrm{L}_{\boldsymbol{\eta}}\,\mathbf{u} \,+\, \tfrac{1}{2}\big(\mathrm{L}_{\boldsymbol{\eta}}\big)^2\,\mathbf{u} \,-\, \cdots \,,\\
\mathbf{u}^* \;&= \mathbf{u} - (-\partial_t\boldsymbol{\eta} + \mathrm{L}_{\boldsymbol{\eta}}\mathbf{u}) + \tfrac{1}{2}\mathrm{L}_{\boldsymbol{\eta}}(-\partial_t\boldsymbol{\eta} + \mathrm{L}_{\boldsymbol{\eta}}\mathbf{u}) - \cdots \,.
\end{aligned}$$

The case $\mathcal{J} = 1$

The value of **w** is linked by (3a) to $\mathcal{J}$, which we may define at our convenience. We choose the value $\mathcal{J} = 1$ because of the obvious simplifications. It was made in Soward (1972) but not in the other pioneering study of Andrews & McIntyre (1978), who were concerned with compressible flows for which allowing for the possibility $\mathcal{J} \neq 1$ is quite natural. That is why we have so far not adopted the restriction $\mathcal{J} = 1$ on admissible $\boldsymbol{\xi}$.

Note that

$$1 = \mathcal{J}(\mathbf{x}, t) = 1 + \nabla \cdot \boldsymbol{\eta} + \tfrac{1}{2} \nabla \cdot \big((\nabla \cdot \boldsymbol{\eta}) \, \boldsymbol{\eta} \big) + \cdots$$

is then solved neatly by

$$\nabla \cdot \boldsymbol{\eta} = 0 \, .$$

The Coriolis contribution $\mathcal{R}$ to the momentum

Remember that the Coriolis acceleration $\mathcal{C}^* = 2\boldsymbol{\Omega} \times \mathbf{u}^*$ transforms to

$$\mathcal{C} = \partial_t \mathcal{R} + (2\boldsymbol{\Omega} + \nabla \times \mathcal{R}) \times \mathbf{u} - \nabla\big((\partial_t \boldsymbol{\xi}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi})\big),$$

so that

$$\begin{aligned} \mathcal{R} &= 2\boldsymbol{\Omega} \times \boldsymbol{\xi} + (\nabla \boldsymbol{\xi}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi}) \\ &= \boldsymbol{\Omega} \times \Big(2\boldsymbol{\xi} + \big((\nabla \cdot \boldsymbol{\xi})\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi}\big)\Big) - (\boldsymbol{\Omega} \cdot \nabla \boldsymbol{\xi}) \times \boldsymbol{\xi} \end{aligned}$$

may be regarded as an additional contribution to the momentum.

For $\xi \ll 1$, we have $\boldsymbol{\eta} = \boldsymbol{\xi} - \frac{1}{2}\boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi} + \cdots$. When $\nabla \cdot \boldsymbol{\eta} = 0$, we have $\nabla \cdot \xi = \mathcal{O}(\xi^2)$ as well. Then $\mathcal{R}$ reduces to

$$\mathcal{R} = 2\boldsymbol{\Omega} \times \boldsymbol{\eta} - (\boldsymbol{\Omega} \cdot \nabla \boldsymbol{\eta}) \times \boldsymbol{\eta} + \cdots .$$

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6. MEAN FIELD EQUATIONS

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Eulerian averages

The averages of the ideal fluid equations are

$$\partial_t \overline{\mathbf{u}^*} + \overline{\mathbf{u}^*} \cdot \nabla \overline{\mathbf{u}^*} + \nabla \cdot \left(\overline{\mathbf{u}^{*\prime} \mathbf{u}^{*\prime}} \right) = -\nabla \overline{p^*} ,$$

$$\partial_t \overline{\mathbf{b}^*} = \nabla \times (\overline{\mathbf{u}^*} \times \overline{\mathbf{b}^*}) + \nabla \times \left(\overline{\mathbf{u}^{*\prime} \times \mathbf{b}^{*\prime}} \right) ,$$

$$\partial_t \overline{\theta^*} + \overline{\mathbf{u}^*} \cdot \nabla \overline{\theta^*} + \nabla \cdot \left(\overline{\mathbf{u}^{*\prime} \theta^{*\prime}} \right) = 0 ,$$

where $\nabla \cdot \overline{\mathbf{u}^*} = \nabla \cdot \overline{\mathbf{b}^*} = 0$ and the fluctuating fields satisfy the fluctuating parts of the ideal fluid equations. The fundamental difficulty that must be faced is the evaluation of the mean Reynolds stress $\overline{\mathbf{u}^{*\prime}\mathbf{u}^{*\prime}}$, the mean electromotive force $\overline{\mathbf{u}^{*\prime} \times \mathbf{b}^{*\prime}}$ and the mean heat flux $\overline{\mathbf{u}^{*\prime}\theta^{*\prime}}$, the determination of which lies at the heart of all closure theories of turbulence.

HEL mean

The key assumption (Andrews & McIntyre 1978), is that

$$\mathbf{u} = \overline{\mathbf{u}}\,,$$

the consequences of which are the 'raison d'être' for the use of the complicated HEL formulation. This assumption is achieved on requiring that the fluctuating part of the motion is encompassed completely by the Lagrangian displacement $\boldsymbol{\xi}$. Then the mean of the ideal HEL equations are simply

$$\partial_t\overline{\mathbf{V}} + \mathbf{u}\cdot\boldsymbol{\nabla}\,\overline{\mathbf{V}} + (\boldsymbol{\nabla}\mathbf{u})\cdot\overline{\mathbf{V}} = -\boldsymbol{\nabla}\overline{\mathit{\Pi}}\,, \tag{6a}$$

$$\partial_t\overline{\mathbf{b}} = \boldsymbol{\nabla}\times(\mathbf{u}\times\overline{\mathbf{b}})\,, \qquad \mathrm{D}_t\overline{\theta} = 0\,. \tag{6b,c}$$

With dissipation ignored, (6b,c) and $\mathbf{u} = \overline{\mathbf{u}}$ imply

$$\mathbf{b} = \overline{\mathbf{b}}\,, \qquad \theta = \overline{\theta} \tag{7a,b}$$

too.

Of course, we have only identified the bare bones here, as other physical processes, particularly diffusion, need to be included. Then the assumed consequence (7a,b) needs to be reassessed.

Kelvin's and Alfvén's theorems

An interesting feature of the mean HEL ideal momentum equation is the appearance of a momentum vector $\rho_0\overline{\mathbf{V}}$ different to $\rho_0\overline{\mathbf{u}}$.

The mean HEL ideal momentum an magnetic induction equations possess the useful conservation properties properties

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}}\overline{\mathbf{V}}\cdot\mathrm{d}\mathbf{x} = 0\,, \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}}\overline{\mathbf{b}}\cdot\mathrm{d}\mathbf{S} = \mathbf{0}\,,$$

namely Kelvin's circulation and Alfvén's frozen flux theorems, for a circuit $\mathcal{C}(t)$ and a surface $\mathcal{S}(t)$ each composed of points, whose HEL coordinates $\mathbf{x}$ move with velocity $\mathbf{u}(\mathbf{x},t)$.

HEL – Generalized Lagrangian Mean (GLM): $\overline{\boldsymbol{\xi}} = \mathbf{0}$

To make any progress, assumptions need to be made about the displacement field $\boldsymbol{\xi}$. Andrews & McIntyre (1978a) developed their Generalized Lagrangian Mean (GLM) approach under the natural assumption

$$\overline{\boldsymbol{\xi}} = \mathbf{0} \qquad \Longrightarrow \qquad \overline{\mathbf{u}^{*L}} = \mathbf{u}\,,$$

which says that $\mathbf{u}$ is the average of the fluid velocity $\mathbf{u}^{*L}$ at the moving HEL position $\mathbf{x}^L$. In the context of waves riding on a mean flow, they identified the difference

$$\mathbf{u} - \overline{\mathbf{V}} = -\overline{(\boldsymbol{\nabla}\boldsymbol{\xi}) \cdot \mathbf{V}^{*L}}$$

which together with the Coriolis contribution

$$-\overline{\boldsymbol{\mathcal{R}}} = -\overline{(\boldsymbol{\nabla}\boldsymbol{\xi}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi})}$$

constitutes the pseudo- or wave-momentum $\mathbf{p}$ per unit mass.

ETHEL – generalized lagrangian mean (glm): $\overline{\boldsymbol{\eta}} = \mathbf{0}$

Following an idea of Holm (2002), we develop a generalized lagrangian mean (glm) approach under the ETHEL assumption

$$\overline{\boldsymbol{\eta}} = \mathbf{0} \qquad \Longrightarrow \qquad \overline{\mathbf{u}^{*L}} = \mathbf{u} + \mathrm{D}_t \overline{\boldsymbol{\xi}} \qquad \text{(8a,b)}$$

so that $\mathbf{u}$ is no longer the Lagrangian mean velocity.

In consequence, there is a new pseudo-momentum:

$$\mathbf{p} \equiv \mathbf{u} - \overline{\mathbf{V}} - \overline{\boldsymbol{\mathcal{R}}} = -\mathrm{D}_t \overline{\boldsymbol{\xi}} - \overline{(\boldsymbol{\nabla}\boldsymbol{\xi}) \cdot \mathbf{V}^{*L}} - \overline{(\boldsymbol{\nabla}\boldsymbol{\xi}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi})} \,.$$

Rememember that $\mathbf{V}^* = \mathbf{u}^*$ and

$$\boldsymbol{\xi} = \boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\eta} \cdot \boldsymbol{\nabla}\boldsymbol{\eta} + \tfrac{1}{3!}(\boldsymbol{\eta} \cdot \boldsymbol{\nabla})^2 \boldsymbol{\eta} + \ldots \ ,$$

Consideration of the $\boldsymbol{\xi}$- expansion makes it clear that $\boldsymbol{\xi}$ has both a mean and fluctuating part, $\overline{\boldsymbol{\xi}} + \boldsymbol{\xi}'$, where specifically, for $\xi \ll 1$,

$$\overline{\boldsymbol{\xi}} \approx \boldsymbol{\nabla} \cdot \mathsf{K} \,, \qquad \text{with} \qquad \mathsf{K} = \tfrac{1}{2}\overline{\boldsymbol{\eta}\boldsymbol{\eta}} \,.$$

Constant density ρ_0^*

An unfortunate consequence of the

$$\textbf{'GLM' assumption} \qquad \overline{\boldsymbol{\xi}} = \mathbf{0},$$

in our incompressible context, is that it is impossible to demand the vanishing of both $\nabla \cdot \boldsymbol{\xi}$ and $\nabla \cdot \mathbf{u}$ simultaneously:

$$\nabla \cdot \boldsymbol{\xi} \;=\; 0$$

$$\Longrightarrow \qquad \mathcal{J} \neq 1 \qquad \text{and} \qquad \nabla \cdot \mathbf{u} \neq 0$$

in general (the divergence effect, e.g. McIntyre 1988).

Remember the Coriolis contribution

$$-\overline{\boldsymbol{\mathcal{R}}} \;=\; -\overline{(\nabla \boldsymbol{\xi}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi})}$$

to the pseudo-momentum.

The alternative

$$\textbf{‘glm’ assumption,} \qquad \overline{\boldsymbol{\eta}} = \mathbf{0},$$

does not suffer this shortcoming and we may demand

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{\eta} &= 0 \\ \Longrightarrow \qquad \mathcal{J} = 1 \qquad &\text{and} \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0\,. \end{aligned}$$

Consideration of the $\boldsymbol{\xi}$- expansion

$$\boldsymbol{\xi} = \boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}\boldsymbol{\eta} + \tfrac{1}{3!}(\boldsymbol{\eta}\cdot\boldsymbol{\nabla})^2\boldsymbol{\eta} + \ldots \quad ,$$

makes it clear that, for for $\xi \ll 1$,

$$\overline{\boldsymbol{\xi}} \approx \boldsymbol{\nabla}\cdot\mathsf{K}\,, \qquad \text{with} \qquad \mathsf{K} = \tfrac{1}{2}\,\overline{\boldsymbol{\eta}\boldsymbol{\eta}}\,.$$

Also the glm Coriolis pseudo-momentum contribution reduces to

$$-\boldsymbol{\mathcal{R}} = \overline{(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\boldsymbol{\eta})\times\boldsymbol{\eta}} + \cdots\,.$$

Eulerian $\longleftrightarrow$ HEL $\longleftrightarrow$ ETHEL

Our overall strategy is to regard $\mathbf{x}$ as the independent variable.

- HEL variables $\boldsymbol{\mathcal{T}}$ concern conditions at $\mathrm{P}^L : \mathbf{x}^L(\mathbf{x}, t)$.
  $\Longrightarrow$ GLM $\overline{\boldsymbol{\mathcal{T}}}$ also averages conditions at the moving point P^L.
- The essential ETHEL construction is

$$\boldsymbol{\mathcal{T}}^* = \mathrm{e}^{-\mathrm{L}_{\boldsymbol{\eta}}}\,\boldsymbol{\mathcal{T}} \qquad \Longleftrightarrow \qquad \boldsymbol{\mathcal{T}} = \mathrm{e}^{\mathrm{L}_{\boldsymbol{\eta}}}\,\boldsymbol{\mathcal{T}}^*\,, \tag{9a,b}$$

where $\qquad \mathrm{e}^{\mathrm{L}_{\boldsymbol{\eta}}} = \sum_0^\infty (1/n!)(\mathrm{L}_{\boldsymbol{\eta}})^n.$

- The glm representation

$$\overline{\boldsymbol{\mathcal{T}}} = \overline{\mathrm{e}^{\mathrm{L}_{\boldsymbol{\eta}}}\,\boldsymbol{\mathcal{T}}^*} = \overline{\boldsymbol{\mathcal{T}}^*} + \overline{\mathrm{L}_{\boldsymbol{\eta}}\,\boldsymbol{\mathcal{T}}^{*\prime}} + \tfrac{1}{2}\,\overline{(\mathrm{L}_{\boldsymbol{\eta}})^2}\;\overline{\boldsymbol{\mathcal{T}}^*} + \mathcal{O}(\eta^3)$$

is the GLM $\overline{\boldsymbol{\mathcal{T}}}$ expanded in terms of Eulerian averages: $\overline{\mathrm{L}_{\boldsymbol{\eta}}} = 0$, in the sense that $\boldsymbol{\eta}$ is regarded like $\boldsymbol{\mathcal{T}}^*$ as a property at P: $\mathbf{x}$.

Remark. With difficulty, equations governing the glm expansions can be derived directly from the original Eulerian equations for

$$\overline{\boldsymbol{\mathcal{T}}^*} \qquad \text{and} \qquad \boldsymbol{\mathcal{T}}^{*\prime}.$$

ETHEL mean

The advantage of working with the GLM equations is clear for those quantities that have the property

$$\overline{\boldsymbol{\mathcal{T}}} = \boldsymbol{\mathcal{T}} \qquad \text{i.e.} \qquad \boldsymbol{\mathcal{T}}' \;=\; \mathbf{0}\,.$$

In that case the ETHEL mean and fluctuating values

$$\begin{aligned} \overline{\boldsymbol{\mathcal{T}}^{*}} \;&=\; \overline{\mathrm{e}^{-\mathrm{L}_{\boldsymbol{\eta}}}\,\boldsymbol{\mathcal{T}}} \;=\; \boldsymbol{\mathcal{T}} \;+\; \tfrac{1}{2}\;\overline{(\mathrm{L}_{\boldsymbol{\eta}})^{2}}\;\boldsymbol{\mathcal{T}} \;+\; \mathcal{O}(\eta^{3})\,,\\ \boldsymbol{\mathcal{T}}^{*\prime} \;&=\; \big(\mathrm{e}^{-\mathrm{L}_{\boldsymbol{\eta}}}\,\boldsymbol{\mathcal{T}}\big)' = \,-\,\mathrm{L}_{\boldsymbol{\eta}}\boldsymbol{\mathcal{T}} \;+\; \mathcal{O}(\eta^{2}) \end{aligned} \tag{10}$$

are readily obtained from (9a).

(10) is in accord with the Frieman & Rotenberg (1960) results (2a-c):

$$\boldsymbol{\mathcal{T}}^{*\prime} \;=\; -\,\mathrm{L}_{\boldsymbol{\zeta}}\boldsymbol{\mathcal{T}} \;+\; \mathcal{O}(\zeta^{2}),$$

where

$$\boldsymbol{\zeta} \;\approx\; \boldsymbol{\eta}\,, \qquad\qquad \overline{\boldsymbol{\zeta}} = \overline{\boldsymbol{\eta}} = \mathbf{0}$$

Implications of $\mathbf{u} = \overline{\mathbf{u}}$

The assumption $\mathbf{u} = \overline{\mathbf{u}}$ and the consequences $\mathbf{b} = \overline{\mathbf{b}}$ and $\theta = \overline{\theta}$ lead to simple ETHEL means exemplified by θ with Lie derivative $\mathrm{L}_{\boldsymbol{\eta}} = \boldsymbol{\eta} \cdot \boldsymbol{\nabla}$:

$$\overline{\theta^*} \approx \theta + \boldsymbol{\nabla} \cdot \big(\mathsf{K} \cdot \boldsymbol{\nabla}\, \theta\big)\,, \tag{11a}$$

$$\theta \approx \overline{\theta^*} + \boldsymbol{\nabla} \cdot \big(\mathsf{K} \cdot \boldsymbol{\nabla}\, \overline{\theta^*} + \overline{\boldsymbol{\eta} \cdot \boldsymbol{\nabla}\, \theta^{*\prime}}\big)\,; \tag{11b}$$

$\mathsf{K} = \frac{1}{2}\,\overline{\boldsymbol{\eta}\boldsymbol{\eta}}$. (11a) has the advantage over (11b) in as much as the term $\overline{\boldsymbol{\eta} \cdot \boldsymbol{\nabla}\, \theta^{*\prime}}$ is not involved; though, of course, (11a,b) are equivalent because from (10) we have

$$\theta^{*\prime} \approx -\boldsymbol{\eta} \cdot \boldsymbol{\nabla}\, \theta \approx -\boldsymbol{\eta} \cdot \boldsymbol{\nabla}\, \overline{\theta^*}\,.$$

Without the HEL construction the ETHEL consequence (11a) would not have been quite so obvious.

Outline

UNIVERSITY OF EXETER

7. MAGNETIC INDUCTION

Newcastle University

Finite magnetic diffusivity $\mathfrak{K}$

The magnetic induction equation,

$$\partial_t \mathbf{b}^* = \boldsymbol{\nabla} \times (\mathbf{u}^* \times \mathbf{b}^* - \mathbf{J}^*), \qquad \mathbf{J}^* = \mathfrak{K}\, \boldsymbol{\nabla} \times \mathbf{b}^*,$$

transforms to

$$\partial_t \mathbf{b} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{b} - \mathbf{J}),$$

$$\mathfrak{K}^{-1}\mathbf{J} = (\boldsymbol{\nabla}\mathbf{x}^L) \cdot (\boldsymbol{\nabla} \times \mathbf{b}^*)^L = (\mathbf{g} \cdot \boldsymbol{\nabla}) \times \mathbf{b} - \boldsymbol{\alpha} \cdot \mathbf{b},$$

in which $\mathbf{g}$ is the contravariant metric tensor and $\boldsymbol{\alpha}$ is akin to the Christoffel symbol:

$$\begin{aligned} g_{ij} &= (\nabla^L_k x_i)(\nabla^L_k x_j), \\ \alpha_{ij} &= (\nabla^L_m x_k)\big(\varepsilon_{klj}\nabla_i - \varepsilon_{ilj}\nabla_k\big)(\nabla^L_m x_l) \\ &= \tfrac{1}{2}(\nabla^L_m x_k)\big(\varepsilon_{kli}\nabla_j + \varepsilon_{klj}\nabla_i\big)(\nabla^L_m x_l) + \tfrac{1}{2}\,\varepsilon_{ijk}\nabla_l\, g_{lk}. \end{aligned}$$

Braginsky's nearly symmetric dynamo

Relative to cylindrical polar coordinates $(s,\ \phi,\ z)$ we write

$$\mathbf{u}^* = \overline{u^*_\phi}\,\widehat{\boldsymbol{\phi}} + \mathbf{u}^{*\prime} + \overline{\mathbf{u}^*_M}\,, \qquad \overline{\mathbf{u}^*_M} = \boldsymbol{\nabla}\times\left(\psi^*\,\widehat{\boldsymbol{\phi}}\right),$$
$$\mathbf{b}^* = \overline{b^*_\phi}\,\widehat{\boldsymbol{\phi}} + \mathbf{b}^{*\prime} + \overline{\mathbf{b}^*_M}\,, \qquad \overline{\mathbf{b}^*_M} = \boldsymbol{\nabla}\times\left(a^*\,\widehat{\boldsymbol{\phi}}\right),$$

where $\overline{\cdots}$ denotes the azimuthal average $\left((2\pi)^{-1}\int_0^{2\pi}\cdots\,\mathrm{d}\phi\right)$, with the ordering

$$\mathbf{u}^{*\prime} = \mathcal{O}\left(Rm^{-1/2}\,\overline{u^*_\phi}\right), \qquad \overline{\mathbf{u}^*_M} = \mathcal{O}\left(Rm^{-1}\,\overline{u^*_\phi}\right),$$
$$\mathbf{b}^{*\prime} = \mathcal{O}\left(Rm^{-1/2}\,\overline{b^*_\phi}\right), \qquad \overline{\mathbf{b}^*_M} = \mathcal{O}\left(Rm^{-1}\,\overline{b^*_\phi}\right),$$

where, for typical azimuthal velocity $\mathcal{U}$ $(u^*_\phi = \mathcal{O}(\mathcal{U}))$ and length $\mathcal{L}$,

$$Rm = \mathcal{L}\mathcal{U}/\mathfrak{K} \gg 1\,.$$

Mean Eulerian form

The mean field equations are

$$\partial_t a^* + s^{-1}\overline{\mathbf{u}_M^*} \cdot \boldsymbol{\nabla}(s a^*) = \left(\overline{\mathbf{u}^{*\prime} \times \mathbf{b}^{*\prime}}\right)_\phi + \mathfrak{K}\,\Delta_1 a^* \,,$$

$$\partial_t \overline{b_\phi^*} + s\overline{\mathbf{u}_M^*} \cdot \boldsymbol{\nabla}(s^{-1}\overline{b_\phi^*}) = s\overline{\mathbf{b}_M^*} \cdot \boldsymbol{\nabla}(s^{-1}\overline{u_\phi^*}) + \left(\boldsymbol{\nabla} \times \left(\overline{\mathbf{u}^{*\prime} \times \mathbf{b}^{*\prime}}\right)_M\right)_\phi + \mathfrak{K}\,\Delta_1 \overline{b_\phi^*} \,,$$

where

$$\Delta_1 \equiv \Delta - s^{-2} \,.$$

Recall that $a^* = \mathcal{O}(\mathcal{L}\, Rm^{-1}\, \overline{b_\phi^*})$ and

$$\overline{\mathbf{u}^{*\prime} \times \mathbf{b}^{*\prime}} = \mathcal{O}\left(Rm^{-1}\, \overline{\mathbf{u}_\phi^*}\ \overline{b_\phi^*} \right) = \mathcal{O}\left(\left|\overline{\mathbf{u}_M^*}\right|\ \overline{b_\phi^*} \right) = \mathcal{O}\left(\mathcal{L}^{-1}\, Rm \left|\overline{\mathbf{u}_M^*}\right| a^* \right)$$

which is unfortunately rather large!

Braginsky dynamo in GLM form

Relative to cylindrical polar coordinates $(s,\, \phi,\, z)$ we write

$$\mathbf{u} \;=\; u_\phi\, \widehat{\boldsymbol{\phi}} + \mathbf{u}_M\,, \qquad \mathbf{u}_M \;=\; \boldsymbol{\nabla} \times \left(\psi\, \widehat{\boldsymbol{\phi}}\right),$$

$$\mathbf{b} \;=\; \overline{b_\phi}\, \widehat{\boldsymbol{\phi}} + \mathbf{b}' + \overline{\mathbf{b}_M}\,, \qquad \overline{\mathbf{b}_M} \;=\; \boldsymbol{\nabla} \times \left(a\, \widehat{\boldsymbol{\phi}}\right)$$

with the ordering

$$\mathbf{u}' \;=\; \mathbf{0}, \qquad \mathbf{u}_M \;=\; \mathcal{O}\big(Rm^{-1}\, u_\phi\big),$$

$$\mathbf{b}' \;=\; \mathcal{O}\big(Rm^{-3/2}\, \overline{b_\phi}\big), \qquad \overline{\mathbf{b}_M} \;=\; \mathcal{O}\big(Rm^{-1}\, \overline{b_\phi}\big),$$

with

$$\eta \;=\; \mathcal{O}\big(\mathcal{L}\, Rm^{-1/2}\big).$$

Neglect $\mathbf{b}'$

With $\mathbf{b}'$ neglected, the approximate GLM equations are

$$\partial_t a + s^{-1}\mathbf{u}_M \cdot \boldsymbol{\nabla}(sa) \approx \mathfrak{K}\left(s^{-1}\aleph\,\overline{b_\phi} + \Delta_1 a\right), \tag{12}$$

$$\partial_t \overline{b_\phi} + s\mathbf{u}_M \cdot \boldsymbol{\nabla}(s^{-1}\overline{b_\phi}) \approx s\overline{\mathbf{b}}_M \cdot \boldsymbol{\nabla}(s^{-1}u_\phi) + \mathfrak{K}\,\Delta_1 \overline{b^*_\phi}\,.$$

Here we have noted that $-\mathfrak{K}^{-1} \times$ (LHS of (12)) is

$$\left(\overline{(\boldsymbol{\nabla}\mathbf{x}^L)\cdot(\boldsymbol{\nabla}\times\mathbf{b}^*)^L}\right)_\phi = s^{-1}\left(\overline{(\partial_\phi\mathbf{x}^L)\cdot(\boldsymbol{\nabla}\times\mathbf{b}^*)^L}\right)_\phi$$

$$= (2\pi s)^{-1}\int_0^{2\pi}\left(\boldsymbol{\nabla}^L\times\mathbf{b}^{*L}\right)\cdot \mathrm{d}\mathbf{x}^L(\phi) \qquad \text{i.e. at fixed } (s,\,z)\,.$$

So the contribution from $\mathbf{b}^{*L} \approx s^{-1}\overline{b_\phi}\,\partial_\phi\mathbf{x}^L$ is

$$s^{-1}\left(\overline{(\partial_\phi\mathbf{x}^L)\cdot(\boldsymbol{\nabla}^L\times s^{-1}\overline{b_\phi}\,\partial_\phi\mathbf{x}^L)}\right)_\phi \approx -s^{-1}\,\aleph\,\overline{b_\phi}\,,$$

where

$$\aleph = (2\pi s)^{-1}\int_0^{2\pi}\left(\boldsymbol{\nabla}^L\times(\partial_\phi\mathbf{x}^L)\right)\cdot\mathrm{d}\mathbf{x}^L(\phi)\,.$$

glm form

At leading order the glm and GLM dynamo equations are the same. The ETHEL relationships are

$$\psi \approx \psi^* + \varpi\, u_\phi\,, \qquad u_\phi \approx \overline{u_\phi^*}\,,$$

$$a \approx a^* + \varpi\, \overline{b_\phi}\,, \qquad \overline{b_\phi} \approx \overline{b_\phi^*}\,,$$

where

$$\varpi = -\, s^{-1}\, \overline{\eta_z\, \partial_\phi \eta_s}\,,$$

and

$$\tfrac{1}{2}\aleph = \overline{(\nabla_M \eta_z) \cdot \nabla_M(\partial_\phi \eta_s)} + s^{-2}\, \overline{(\partial_\phi \eta_z)\partial_\phi(\partial_\phi \eta_s - \eta_\phi)}\,.$$

Outline

8. Conclusions

The **Main Objective** has been to develop an Eulerian mean field theory in the presence of large scale shear flows
at large Reynolds numbers.

- It is well known that direct Eulerian averaging of the governing equations is hard to implement in a helpful way (as in the Braginsky dynamo).

- The Lagrangian average applied to the HEL equations leads to the useful GLM method.

- HEL and GLM suffer the usual deficiency of all Lagrangian methods that they are non-local.

- glm recasts GLM in an Eulerian (local) setting. Whence the use of the term Eulerian Transformed HEL (ETHEL).
  In ths way the **Main Objective** is accomplished.

Small versus large amplitude theories

The case $\xi \ll 1$.

The possible advantages of glm are self-evident.

The case $\xi \gg 1$.

The HEL (also ETHEL) method need to face up to difficult problems linked to the non-uniqueness of the choice of $\boldsymbol{\xi}$ (also $\boldsymbol{\eta}$).

- Some of that lack of uniqueness is reduced by the requirement $\overline{\boldsymbol{\xi}} = \mathbf{0}$ (also $\overline{\boldsymbol{\eta}} = \mathbf{0}$).
- Nevertheless, the GLM (also glm) method is clearly still hard to implement except in special cases and is the more likely to be useful as a diagnostic tool.